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LINEARIZATIONS FOR INTERPOLATORY BASES – A COMPARISON: NEW FAMILIES OF LINEARIZATIONS*

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Abstract. One strategy to solve a nonlinear eigenvalue problem $T(\lambda)x = 0$ is to solve a polynomial eigenvalue problem (PEP) $P(\lambda)x = 0$ that approximates the original problem through interpolation. Then, this PEP is usually solved by linearization. Because of the polynomial approximation techniques, in this context, $P(\lambda)$ is expressed in a non-monomial basis. The bases used with most frequency are the Chebyshev basis, the Newton basis and the Lagrange basis. Although, there exist already a number of linearizations available in the literature for matrix polynomials expressed in these bases, new families of linearizations are introduced because they present the following advantages: 1) they are easy to construct from the matrix coefficients of $P(\lambda)$ when this polynomial is expressed in any of those three bases; 2) their block-structure is given explicitly; 3) it is possible to provide equivalent formulations for all three bases which allows a natural framework for comparison. Also, recovery formulas of eigenvectors (when $P(\lambda)$ is regular) and recovery formulas of minimal bases and minimal indices (when $P(\lambda)$ is singular) are provided. The ultimate goal is to use these families to compare the numerical behavior of the linearizations associated to the same basis (to select the best one) and with the linearizations associated to the other two bases, to provide recommendations on what basis to use in each context. This comparison will appear in a subsequent paper.

Key words. Nonlinear eigenvalue problem, Polynomial eigenvalue problem, Linearization, eigenvalue, Eigenvector, Minimal basis, Minimal indices, Chebyshev basis, Newton basis, Lagrange basis, Interpolation.

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1. Introduction. Nonlinear eigenvalue problems of the form

$$T(\lambda)x = 0 \quad \text{and} \quad y^T T(\lambda) = 0, \tag{1.1}$$

where $T : \Omega \subseteq \mathbb{C} \to \mathbb{C}^{n\times n}$ is a regular complex-valued matrix function holomorphic in a complex region Ω, often arise in applications [13]. The scalar $\lambda \in \Omega$ is called an *eigenvalue* of $T(\lambda)$, and x and y are associated *right and left eigenvectors*.

A possible approach for solving the nonlinear eigenvalue problem (1.1) is to replace $T(\lambda)$ with a matrix polynomial approximation $P(\lambda)$ [12, 27, 28]. Such polynomial approximant can be found via interpolation, i.e., for a given set of points $\{x_1, x_2, \ldots, x_{k+1}\} \subset \Omega$, whose elements we call the *nodes*, one replaces T by the

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unique matrix polynomial P of degree at most k satisfying

$$T(x_i) = P(x_i) \quad (i = 1, \dots, k+1). \tag{1.2}$$

This process replaces the nonlinear eigenvalue problem (1.1) by a *polynomial eigenvalue problem* (PEP)

$$P(\lambda)x = 0 \quad \text{and} \quad y^T P(\lambda) = 0. \tag{1.3}$$

If the interpolation error $\max_{\lambda \in \Omega} \|P(\lambda) - T(\lambda)\|_2$ is small, one expects the eigenvalues of $P(\lambda)$ in Ω and their corresponding eigenvectors to be reliable approximations to the eigenvalues and eigenvectors of $T(\lambda)$ in a backward error sense [13].

One of the most popular techniques for solving polynomial eigenvalue problems is *linearization* [19]. A linearization of a matrix polynomial $P(\lambda)$ replaces (1.3) with a (larger) *generalized eigenvalue problem*

$$\lambda Bv = Av \quad \text{and} \quad \lambda w^T B = w^T A \tag{1.4}$$

with the same eigenvalues (and multiplicities) as the original PEP. The linearized eigenvalue problem (1.4) can be solved by using the QZ algorithm (for small/medium sizes) or a Krylov method (for larger sizes) [29].

It is well-known that the linearization transformation is not unique [1, 5, 23]. Common choices are the Frobenius companion linearizations [5], which are based on an expansion of $P(\lambda)$ in the monomial basis

$$P(\lambda) = \sum_{i=0}^{k} P_i \, \lambda^i, \quad P_0, \dots, P_k \in \mathbb{C}^{n \times n}. \tag{1.5}$$

Since polynomial interpolation in the monomial basis can be potentially unstable –due to the ill conditioning of Vandermonde matrices– we will consider instead matrix polynomials of the form

$$P(\lambda) = \sum_{i=0}^{k} P_i \, n_i(\lambda), \quad P_0, \dots, P_k \in \mathbb{C}^{n \times n}, \tag{1.6}$$

where $\{n_i(\lambda)\}_{i=0}^k$ denotes either the Newton, Lagrange or Chebyshev polynomial bases, since these bases are the most common choices for dealing with polynomial interpolants in numerical practice [3, 11, 16].

In the literature, linearizations of a matrix polynomial expressed in either of these bases can be found in [1, 12, 20, 22, 25, 27]. Among these linearizations, those used most often in applications can be considered "equivalent" to the Frobenius linearizations in the monomial case. They are called *colleague linearizations*. Our ultimate goal in a forthcoming paper is to compare the numerical performance (in terms of conditioning and backward errors [17, 18, 26]) of the linearizations of a matrix polynomial expressed in the three bases: Chebyshev, Newton, and Lagrange in the following sense. First, we would like to determine if the colleague linearizations used in practice are the "best" linearizations for a given basis. In order to do this analysis, we need a whole family of linearizations to choose from and compare with. Secondly, once we have chosen the best linearization for each basis, we want to compare the performance of these linearizations for the three given bases in terms of the selection of nodes for interpolation. The relative position of the eigenvalues with respect to the interpolation nodes has an important effect on the numerical behavior of these linearizations.

In order to achieve the ultimate goal mentioned above, in this paper, we present three families of strong linearizations for matrix polynomials expressed in the Chebyshev, Newton, and Lagrange bases,

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respectively. The main reason to construct these families, despite the fact that some families of linearizations already exist for some bases, such as Chebyshev and Newton, is because these available constructions in the literature are implicit (see, for example [22, 24], or [21] for the Bernstein basis) and, thus, not easy to use for the numerical analysis that we intend to do. Moreover, we have used a block minimal basis approach ([8]) for the construction of the linearizations (thus, providing their explicit block-structure) which allows equivalent formulations for the three bases. This makes the numerical analysis and comparison much more straightforward. For completion, we give linearizations for both polynomials that are regular and singular, and also provide recovery formulas for eigenvectors, minimal bases, and minimal indices. The numerical analysis and comparison is postponed to a subsequent paper to limit the length of the paper.

As for the structure of the paper, after some preliminaries (Sections 2.1–2.6), where we introduce the notation used throughout the paper and background knowledge, we present in Section 2.7 the so-called block minimal basis linearizations. This family of linearizations was introduced recently in [8], and will allow us to construct in Sections 3, 4 and 5 linearizations for matrix polynomials expressed in the Newton, Lagrange and Chebyshev bases, respectively. For each of the considered polynomial bases, we introduce an infinite family of linearizations, and for each of these families, we obtain eigenvector formulas, and show how to recover the eigenvectors, minimal indices and minimal bases of the original matrix polynomial from those of any of its linearizations. Our results put into a unified framework some results scattered in the linearization literature [1, 20, 25], and fill some important gaps in the literature regarding eigenvector formulas, recovery procedures for eigenvectors and minimal bases and minimal indices, and explicit constructions.

2. Background and notation. Although most of the definitions and results in this paper hold over a generic field, we focus on the complex numbers.

2.1. Block vectors and the block transpose. A *block vector* is a matrix of the form

$$v = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix} \quad \text{or} \quad v = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix},$$

where the entries V_i are (possibly) matrices of compatible size. We sometimes use $v(i)$ to denote the ith block entry of a block vector v. The *block transpose* operation, denoted by $\mathcal{B}$, is the blockwise transposition, i.e.,

$$\begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}^{\mathcal{B}} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}^{\mathcal{B}} = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}.$$

Note that, in the first case, we are assuming that all the blocks entries have the same number of columns and, in the second case, we are assuming that all the block entries have the same number of rows.

2.2. Matrix polynomials. Let us consider an $m \times n$ matrix polynomial with complex matrix coefficients of the form

$$P(\lambda) = \sum_{i=0}^{k} P_i \, \lambda^i, \quad P_0, \ldots, P_k \in \mathbb{C}^{m \times n}. \tag{2.7}$$

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If P_k is nonzero, we say that $P(\lambda)$ has *degree* k; otherwise, we say that $P(\lambda)$ has grade k. We denote the degree of a matrix polynomial $P(\lambda)$ by $\deg P(\lambda)$. When dealing with interpolation polynomials, the notion of grade is more natural than the notion of degree, since one cannot guarantee a priori a nonzero leading term.

A matrix polynomial of size $n \times 1$ is called a *(column) vector polynomial*.

We say that a matrix polynomial $P(\lambda)$ is *regular* if $m = n$ and $\det(P(\lambda))$ is not identically zero. In other words, a regular matrix polynomial $P(\lambda)$ is an invertible matrix over the field $\mathbb{C}(\lambda)$ of rational functions with complex coefficients. We say that $P(\lambda)$ is *singular* if either $m \neq n$ or $\det(P(\lambda)) \equiv 0$.

We say that the matrix polynomial given in (2.7) is expressed in the monomial basis, since $\{1, \lambda, \ldots, \lambda^k\}$ is a basis of the set of polynomials $C_k[\lambda]$ of degree at most k (that is, of grade k). As explained in the introduction, in interpolation problems, it is more convenient to express a matrix polynomial in other polynomial bases. In the paper, we focus on matrix polynomials expressed either in the Newton, Lagrange or Chebyshev bases. We recall these bases next.

2.3. Polynomial interpolation bases.

2.3.1. Newton interpolation basis. For a given set of nodes $\{x_1, \ldots, x_{k+1}\} \subset \mathbb{C}$, the Newton polynomial $n_i(\lambda)$ is defined as

$$n_i(\lambda) = \prod_{j=1}^{i} (\lambda - x_j) \quad (i = 1, \ldots, k), \tag{2.8}$$

and $n_0(\lambda) = 1$. We notice that the Newton polynomials satisfy the following recurrence relation

$$n_i(\lambda) = (\lambda - x_i) n_{i-1}(\lambda) \quad (i = 1, \ldots, k). \tag{2.9}$$

The interpolation matrix polynomial, i.e., the unique grade-k matrix polynomial $P(\lambda)$ satisfying (1.2), can be written as

$$P(\lambda) = \sum_{i=0}^{k} P_i \, n_i(\lambda) \tag{2.10}$$

where the matrix coefficients $P_i \in \mathbb{C}^{n\times n}$ can be found, for example, by using the method of divided differences. Setting $y_i := T(x_i)$ $(i = 1, \ldots, k+1)$, the divided differences are defined as

$$[y_i] := y_i, \quad [y_i, y_{i+1}, \ldots, y_{i+j}] := \frac{[y_{i+1}, \ldots, y_{i+j}] - [y_i, y_{i+1}, \ldots, y_{i+j-1}]}{x_{i+j} - x_i}.$$

Then, $P_i = [y_1, \ldots, y_{i+1}]$ for $i = 0, 1, \ldots, k$.

2.3.2. Lagrange interpolation basis. For a given set of nodes $\{x_1, x_2, \ldots, x_{k+1}\} \subset \mathbb{C}$, the *Lagrange polynomial* $\ell_i(\lambda)$ is defined as

$$\ell_i(\lambda) := \frac{\prod_{j=1, j\neq i}^{k+1} (\lambda - x_j)}{\prod_{j=1, j\neq i}^{k+1} (x_i - x_j)} \quad (i = 1, \ldots, k+1). \tag{2.11}$$

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The Lagrange polynomial $\ell_i(\lambda)$ has the property

$$\ell_i(x_j) = \begin{cases} 1 & \text{if } j = i, \text{ and} \\ 0 & \text{otherwise} \end{cases} \qquad (i, j = 1, \ldots, k+1).$$

Hence, the unique matrix polynomial $P(\lambda)$ satisfying (1.2) can be written in terms of Lagrange polynomials as

$$P(\lambda) = \sum_{i=1}^{k+1} P_i \, \ell_i(\lambda), \tag{2.12}$$

where $P_i = T(x_i)$ $(i = 1, \ldots, k+1)$.

For our purposes, it will be more convenient to express the Lagrange polynomials in the equivalent modified way

$$\ell_i(\lambda) = \ell(\lambda) \frac{\omega_i}{\lambda - x_i} \quad (i = 1, \ldots, k+1), \tag{2.13}$$

where

$$\ell(\lambda) = \prod_{i=1}^{k+1} (\lambda - x_i) \qquad \text{and} \qquad \omega_i = \frac{1}{\prod\limits_{j \neq i} (x_i - x_j)} \quad (i = 1, \ldots, k+1). \tag{2.14}$$

The quantities ω_i are known as the barycentric weights. Using (2.13), the matrix polynomial $P(\lambda)$ in (2.12) takes the form

$$P(\lambda) = \ell(\lambda) \sum_{i=1}^{k+1} P_i \frac{\omega_i}{\lambda - x_i}, \tag{2.15}$$

which is known as the first barycentric form of (2.12).

2.4. The Chebyshev bases of the first and second kind. The *Chebyshev polynomials of the first kind* $\{T_n(x) : n \in 0 \cup \mathbb{N}\}$ are obtained from the recurrence relation

$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x), \tag{2.16}$$

where $T_0(x) = 1$ and $T_1(x) = x$. The *Chebyshev polynomials of the second kind* $\{U_n(x) : n \in 0 \cup \mathbb{N}\}$ are obtained from the same recurrence relation (2.16) with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$.

Chebyshev polynomials can be used to interpolate nonlinear matrix-valued functions $T : [-1, 1] \to \mathbb{C}^{n \times n}$. Two types of nodes are usually considered: (1) Chebyshev nodes of the first kind

$$x_i = \cos\left(\frac{2i-1}{k+1} \frac{\pi}{2}\right), \quad i \in \{1, 2, \ldots, k+1\},$$

and (2) Chebyshev nodes of the second kind

$$x_i = \cos\left(\frac{i-1}{k} \pi\right), \quad i \in \{1, 2, \ldots k+1\}.$$

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In both cases, the unique grade-k matrix polynomial $P(\lambda)$ satisfying (1.2) can be written in the form

$$P(\lambda) = \sum_{i=0}^{k} P_i \, T_i(\lambda), \tag{2.17}$$

where the matrix coefficients P_i $(i = 0, \dots, k)$ can be efficiently computed by a sequence of inverse discrete cosine transforms of type III or type I, respectively. Details can be found in [2].

Remark 2.1. Although the Chebyshev polynomials are usually considered to be defined on the real interval $[-1, 1]$, there is a generalization of these polynomials in the complex plane: Given a compact set $K \subseteq \mathbb{C}$, the nth Chebyshev polynomial associated with K is defined to be the (unique) monic polynomial which minimizes the supremum norm on K among all monic polynomials of the same degree. However, as far as we know, there is not a formula to compute these polynomials in an arbitrary set K, which is a drawback compared to Newton and Lagrange. Thus, in Section 5, we assume the ordinary Chebyshev polynomials defined on the interval $[-1, 1]$.

The following lemma will be used in future sections.

Lemma 2.2. *[20] The Chebyshev polynomials satisfy the following identities:*

$$\begin{aligned}
T_{r+\ell}(\lambda) &= U_r(\lambda)T_\ell(\lambda) - U_{r-1}(\lambda)T_{\ell-1}(\lambda) \quad (\ell \neq 0),\\
T_{r+\ell+1}(\lambda) &= 2\lambda U_r(\lambda)T_\ell(\lambda) - U_r(\lambda)T_{\ell-1}(\lambda) - U_{r-1}(\lambda)T_\ell(\lambda) \quad (\ell \neq 0),\\
U_{r+\ell}(\lambda) &= U_r(\lambda)U_\ell(\lambda) - U_{r-1}(\lambda)U_{\ell-1}(\lambda),\\
U_{r+\ell+1}(x) &= 2\lambda U_r(\lambda)U_\ell(\lambda) - U_r(\lambda)U_{\ell-1}(\lambda) - U_{r-1}(\lambda)U_\ell(\lambda).
\end{aligned}$$

2.5. Eigenvalues and eigenvectors of regular matrix polynomials. Let $P(\lambda)$ be a regular matrix polynomial of grade k as in (2.7). We say that $\lambda_0 \in \mathbb{C}$ is a *finite eigenvalue* of $P(\lambda)$ if $P(\lambda_0)x = 0$ for some nonzero vector x. The vector x is called a *right eigenvector* of $P(\lambda)$ associated with λ_0. A vector y is said to be a *left eigenvector* of $P(\lambda)$ associated with λ_0 if $y^T P(\lambda_0) = 0$, where y^T denotes the transpose of y. We say that $P(\lambda)$ has an *eigenvalue at infinity* if zero is an eigenvalue of the *k-reversal* $\mathrm{rev}_k P(\lambda)$ of $P(\lambda)$, where

$$\mathrm{rev}_k P(\lambda) = \lambda^k P\left(1/\lambda\right). \tag{2.18}$$

In this case, a right (resp., left) eigenvector of $P(\lambda)$ associated with an infinite eigenvalue is a right (resp., left) eigenvector of $\mathrm{rev}_k P(\lambda)$ associated with 0.

Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ of the same size are said to be *strictly equivalent* if there are invertible matrices U and V such that $Q(\lambda) = UP(\lambda)V$. We recall that two strictly equivalent matrix polynomials have the same finite and infinite eigenvalues with the same algebraic, partial and geometric multiplicities.

In future sections, we will consider eigenvalues at infinity of matrix polynomials expressed in polynomial bases other than the monomial. The following lemma provides the reversal of such a polynomial. We omit the proof since it follows immediately from the definition of reversal.

Lemma 2.3. *Let $P(\lambda) = \sum_{i=0}^{k} P_i \, \phi_i(\lambda)$ be a matrix polynomial of grade k expressed in the polynomial basis $\{\phi_0, \phi_1, \dots, \phi_k\}$. Then,*

$$\mathrm{rev}_k \, P(\lambda) = \sum_{i=0}^{k} P_i \, \mathrm{rev}_k \, \phi_i(\lambda).$$

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In particular, if $\phi_i(\lambda) = \prod_{j=0}^{s}(\lambda - a_j)$*, where* $s \leq k$*, then*

$$\operatorname{rev}_k \phi_i(\lambda) = \lambda^{k-s} \prod_{j=0}^{s}(1 - a_j\lambda).$$

2.6. Singular matrix polynomials and dual minimal bases. If an $m \times n$ matrix polynomial $P(\lambda)$ is singular, then it has non-trivial left and/or right rational null spaces:

$$\mathcal{N}_\ell(P) := \{y(\lambda) \in \mathbb{C}(\lambda)^{m\times 1} : y(\lambda)^T P(\lambda) = 0\}, \quad \text{and}$$
$$\mathcal{N}_r(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : P(\lambda)x(\lambda) = 0\}.$$

Each of these vector spaces contains a basis consisting of vector polynomials [14]. We call a basis consisting of vector polynomials a *polynomial basis*. The *order* of a polynomial basis is the sum of the degrees of its vectors. Among all the polynomial bases we consider those with least order.

Definition 2.4. (Minimal basis) Let $\mathcal{V}$ be a rational subspace of $\mathbb{C}(\lambda)^{n\times 1}$. A *minimal basis* of $\mathcal{V}$ is any polynomial basis of $\mathcal{V}$ with least order among all polynomial bases.

Minimal bases for a rational subspace $\mathcal{V}$ are not unique, but the ordered list of the degrees of the vector polynomials in each of them is the same. These degrees are called *the minimal indices* of $\mathcal{V}$ [14].

Definition 2.5. (Minimal indices of singular matrix polynomials) Let $P(\lambda)$ be an $m \times n$ singular matrix polynomial and let $\{y_1(\lambda)^T, \ldots, y_q(\lambda)^T\}$ and $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ be minimal bases of $\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$, respectively, ordered so that $\deg(y_1(\lambda)) \leq \cdots \leq \deg(y_q(\lambda))$ and $\deg(x_1(\lambda)) \leq \cdots \leq \deg(x_p(\lambda))$. Let $\mu_j = \deg(y_j(\lambda))$ for $j = 1, 2, \ldots, q$, and $\epsilon_j = \deg(x_j(\lambda))$ for $j = 1, 2, \ldots, p$. Then, $\mu_1 \leq \cdots \leq \mu_q$ and $\epsilon_1 \leq \cdots \leq \epsilon_p$ are, respectively, the *left and right minimal indices* of $P(\lambda)$.

Theorem 2.7 provides a useful characterization of minimal bases. To state this result, we need the following definition from [7].

Definition 2.6. Let $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$ be a matrix polynomial with row degrees $d_1, d_2, \ldots, d_m$. The highest row degree coefficient matrix of $P(\lambda)$, denoted by P_h, is the $m \times n$ constant matrix whose jth row is the coefficient of λ^{d_j} in the jth row of $P(\lambda)$ for $j = 1, 2, \ldots, m$. The matrix polynomial $P(\lambda)$ is called row reduced if P_h has full row rank.

Theorem 2.7. [7, Theorem 2.14] *The rows of a matrix polynomial* $P(\lambda)$ *are a* minimal basis of the rational subspace they span if and only if $P(\lambda_0)$ has full row rank for all $\lambda_0 \in \mathbb{C}$ and $P(\lambda_0)$ is row reduced. A matrix polynomial is called *minimal basis* if its rows form a minimal basis of the rational subspace they span.

The linearizations for matrix polynomials that we introduce in the following section use the notion of dual minimal bases [14].

Definition 2.8. (Dual minimal bases) Two matrix polynomials $K(\lambda) \in \mathbb{F}[\lambda]^{m_1\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{m_2\times n}$ are said to be dual minimal bases if $K(\lambda)$ and $D(\lambda)$ are both minimal bases, $m_1+m_2 = n$, and $K(\lambda)D(\lambda)^T = 0$.

2.7. Strong linearizations of matrix polynomials, and block minimal basis pencils. A matrix pencil $L(\lambda)$ is said to be a *linearization* of a matrix polynomial $P(\lambda)$ as in (1.2) if there exist a positive

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integer s and two unimodular matrices (i.e., matrix polynomials whose determinant is a nonzero constant) $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

A linearization $L(\lambda)$ of a grade-k matrix polynomial $P(\lambda)$ is *strong* if $\mathrm{rev}_1 L(\lambda)$ is a linearization of $\mathrm{rev}_k P(\lambda)$ [19].

Remark 2.9. A strong linearization of a matrix polynomial $P(\lambda)$ preserves the finite and infinite eigenvalues of $P(\lambda)$ and their multiplicities, and the dimension of the right and left nullspaces.

Remark 2.10. Any matrix pencil strictly equivalent to a strong linearization of a matrix polynomial $P(\lambda)$ is also a strong linearization of $P(\lambda)$.

One of our main objective in this paper is to find strong linearizations for matrix polynomials of the form

$$P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i(\lambda), \quad P_0, \ldots, P_k \in \mathbb{C}^{m\times n}, \tag{2.19}$$

where $\{\phi_i\}$ denotes either the Newton, Lagrange or Chebyshev bases, that can be easily constructed from the coefficients P_i and the nodes. We will find such linearizations in the family of so-called block minimal basis pencils [8].

Definition 2.11. (Block minimal basis pencils) A matrix pencil

$$L(\lambda) = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right] \tag{2.20}$$

is called a *block minimal basis pencil* if $K_1(\lambda)$ and $K_2(\lambda)$ are both minimal bases. If, in addition, the row degrees of $K_1(\lambda)$ are all equal to 1, the row degrees of $K_2(\lambda)$ are all equal to 1, the row degrees of a minimal basis dual to $K_1(\lambda)$ are all equal and the row degrees of a minimal basis dual to $K_2(\lambda)$ are equal, then $L(\lambda)$ is a *strong block minimal basis pencil*. The submatrix $M(\lambda)$ is called the *body of* $L(\lambda)$.

Theorems 2.12 and 2.13 are two key results on strong block minimal basis pencils. Theorem 2.12 says that every strong block minimal basis pencil is always a strong linearization of a certain matrix polynomial.

Theorem 2.12. [8] *Let $K_1(\lambda)$ and $D_1(\lambda)$, and $K_2(\lambda)$ and $D_2(\lambda)$ be two pairs of dual minimal bases, let $L(\lambda)$ be a strong block minimal basis pencil as in* (2.20)*, and let*

$$Q(\lambda) := D_2(\lambda)M(\lambda)D_1(\lambda)^T. \tag{2.21}$$

Then:

(a) *$L(\lambda)$ is a linearization of $Q(\lambda)$.*
(b) *If $L(\lambda)$ is a strong block minimal basis pencil, then $L(\lambda)$ is a strong linearization of $Q(\lambda)$, considered as a polynomial with grade $1 + \deg(D_1(\lambda)) + \deg(D_2(\lambda))$.*

Theorem 2.13 says essentially two things: 1) given a matrix polynomial $P(\lambda)$, it says that we can always find a pencil $M(\lambda)$ such that the strong block minimal basis pencil (2.20) is a strong linearization of $P(\lambda)$; 2) it provides a characterization of all the pencils $M(\lambda)$ that make the block minimal basis pencil (2.20) a strong linearization of the given polynomial $P(\lambda)$.

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Theorem 2.13. [9] *Let $P(\lambda)$ be an $m\times n$ matrix polynomial, let $K_1(\lambda)$ and $D_1(\lambda)$, and $K_2(\lambda)$ and $D_2(\lambda)$ be two pairs of dual minimal bases such that $D_1(\lambda)$ has n rows, $D_2(\lambda)$ has m rows, and $\deg(P(\lambda)) \le 1+\deg(D_1(\lambda))+\deg(D_2(\lambda))$, and let $L(\lambda)$ be a strong block minimal basis pencil as in (2.20). Then:*

(a) *The linear equation*

$$P(\lambda) = D_2(\lambda)M(\lambda)D_1(\lambda)^T \tag{2.22}$$

is solvable for the matrix pencil $M(\lambda)$.

(b) *If $M_0(\lambda)$ is a solution of (2.22), then any other solution is of the form*

$$M(\lambda) = M_0(\lambda) + AK_1(\lambda) + K_2(\lambda)^T B$$

for some constant matrices A and B.

Remark 2.14. For the linearizations introduced in Sections 3, 4 and 5, we will be able to construct a matrix pencil $M(\lambda)$ satisfying (2.22) directly from the matrix coefficients of the matrix polynomial $P(\lambda)$.

Theorem 2.15 will allow us to prove that the linearizations we introduce in this work are more than strong linearizations, since we will be able to recover minimal indices, minimal bases and left and right eigenvectors of the original matrix polynomial $P(\lambda)$ from those of its linearizations. Due to its technicality, we postpone the proof of Theorem 2.15 to the Appendix.

Theorem 2.15. *Let $P(\lambda)$ be an $m\times n$ matrix polynomial as in (2.7), let $K_1(\lambda)$ and $D_1(\lambda)$, and $K_2(\lambda)$ and $D_2(\lambda)$ be two pairs of dual minimal bases, and let $L(\lambda)$ be a strong block minimal basis pencil as in (2.20) such that*

$$P(\lambda) = D_2(\lambda)M(\lambda)D_1(\lambda)^T.$$

Suppose right- and left-sided factorizations of the form

$$L(\lambda)\begin{bmatrix} D_1(\lambda)^T \\ X(\lambda)\end{bmatrix} = v\otimes P(\lambda) \quad \textit{and} \quad \begin{bmatrix} D_2(\lambda) \; Y(\lambda)^T\end{bmatrix} = w^T\otimes P(\lambda)$$

hold for some matrix polynomials $X(\lambda)$ and $Y(\lambda)$, and for some nonzero vectors $v, w\in\mathbb{C}^k$.

Assume $m=n$ and $P(\lambda)$ is regular. If λ_0 is a finite eigenvalue of $P(\lambda)$ with geometric multiplicity g, then

(a) *$\{x_1,\ldots,x_g\}$ is a basis for $\mathcal{N}_r(P(\lambda_0))$ if and only if $\{v_1,\ldots,v_g\}$ is a basis for $\mathcal{N}_r(L(\lambda_0))$, where $v_i = \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0)\end{bmatrix} x_i$ for $i=1,\ldots,g$.*

(b) *$\{y_1,\ldots,y_g\}$ is a basis for $\mathcal{N}_\ell(P(\lambda_0))$ if and only if $\{w_1,\ldots,w_g\}$ is a basis for $\mathcal{N}_r(L(\lambda_0))$, where $w_i = \begin{bmatrix} D_2(\lambda_0)^T \\ Y(\lambda_0)\end{bmatrix} y_i$ for $i=1,\ldots,g$.*

Assume $P(\lambda)$ is singular. If $\dim \mathcal{N}_r(P(\lambda)) = p$ and $\dim \mathcal{N}_\ell(P(\lambda)) = q$, then

(c) *$\{x_1(\lambda),\ldots,x_p(\lambda)\}$ is a minimal basis for $\mathcal{N}_r(P(\lambda))$ if and only if $\{v_1(\lambda),\ldots,v_p(\lambda)\}$ is a minimal basis for $\mathcal{N}_r(L(\lambda))$, where $v_i = \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda)\end{bmatrix} x_i(\lambda)$ for $i=1,\ldots,p$.*

(d) *$\{y_1(\lambda),\ldots,y_q(\lambda)\}$ is a basis for $\mathcal{N}_\ell(P(\lambda))$ if and only if $\{w_1(\lambda),\ldots,w_q(\lambda)\}$ is a basis for $\mathcal{N}_\ell(L(\lambda))$, where $w_i(\lambda) = \begin{bmatrix} D_2(\lambda)^T \\ Y(\lambda)\end{bmatrix} y_i(\lambda)$ for $i=1,\ldots,q$.*

Moreover, if $0\le\epsilon_1\le\epsilon_2\le\cdots\le\epsilon_p$ are the right minimal indices of $P(\lambda)$, and $0\le\mu_1\le\mu_2\le\cdots\le\mu_q$ are the left minimal indices of $P(\lambda)$, then

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(e) $\epsilon_1 + \deg(D_1(\lambda)) \leq \epsilon_2 + \deg(D_1(\lambda)) \leq \cdots \leq \epsilon_q + \deg(D_1(\lambda))$ *are the right minimal indices of* $L(\lambda)$, *and*

(f) $\mu_1 + \deg(D_2(\lambda)) \leq \mu_2 + \deg(D_2(\lambda)) \leq \cdots \leq \mu_q + \deg(D_2(\lambda))$ *are the left minimal indices of* $L(\lambda)$.

3. Strong linearizations for matrix polynomials in the Newton basis. Let $\{x_1, \dots, x_k\}$ be a set of k distinct nodes, and let $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ be an $m \times n$ matrix polynomial expressed in the Newton basis associated with this set of nodes.

Associated with the set of nodes $\{x_1, x_2, \dots, x_k\}$ we introduce the following polynomials

$$\gamma_j(\lambda) := \lambda - x_j \quad (j = 1, 2, \dots, k) \tag{3.23}$$

and

$$n_i^j(\lambda) := \begin{cases} \prod_{\ell=i}^{j} \gamma_\ell(\lambda) & \text{if } j \geq i, \\ 1 & \text{if } j < i \end{cases} \qquad (i, j = 1, 2, \dots, k). \tag{3.24}$$

Notice that $n_1^j(\lambda)$ is just the jth Newton polynomial $n_j(\lambda)$ for $j = 1, 2, \dots, k$.

Let $0 \leq \mu \leq k-1$ be an integer and let n and m be positive integers. We define the matrix pencils

$$K_1^N(\lambda) := \begin{bmatrix} -I_n & \gamma_{k-1}(\lambda) I_n & & & \\ & -I_n & \gamma_{k-2}(\lambda) I_n & & \\ & & \ddots & \ddots & \\ & & & -I_n & \gamma_{\mu+1}(\lambda) I_n \end{bmatrix} \quad \text{and} \tag{3.25}$$

$$K_2^N(\lambda) := \begin{bmatrix} -I_m & \gamma_{\mu}(\lambda) I_m & & & \\ & -I_m & \gamma_{\mu-1}(\lambda) I_m & & \\ & & \ddots & \ddots & \\ & & & -I_m & \gamma_{1}(\lambda) I_m \end{bmatrix}, \tag{3.26}$$

where the polynomials $\gamma_j(\lambda)$ are defined in (3.23), and where the empty block-entries are assumed to be zero blocks. We note that, if $\mu = 0$ (resp., $\mu = k-1$), the matrix $K_2^N(\lambda)$ (resp., $K_1^N(\lambda)$) is an empty matrix.

LEMMA 3.1. *Let* $\{x_1, \dots, x_k\}$ *be a set of distinct nodes, and let* $0 \leq \mu \leq k-1$ *be an integer. The matrix pencils* $K_1^N(\lambda)$ *and* $K_2^N(\lambda)$ *defined, respectively, in* (3.25) *and* (3.26) *are minimal bases when they are not empty. Moreover, in this case,*

$$D_1^N(\lambda)^T := \begin{bmatrix} n_{\mu+1}^{k-1}(\lambda) I_n \\ n_{\mu+1}^{k-2}(\lambda) I_n \\ \vdots \\ n_{\mu+1}^{\mu+1}(\lambda) I_n \\ I_n \end{bmatrix} \quad \text{and} \quad D_2^N(\lambda)^T := \begin{bmatrix} n_{\mu}(\lambda) I_m \\ n_{\mu-1}(\lambda) I_m \\ \vdots \\ n_1(\lambda) I_m \\ I_m \end{bmatrix}, \tag{3.27}$$

where the $n_i^j(\lambda)$ *polynomials are defined in* (3.24), *are dual minimal bases of* $K_1^N(\lambda)$ *and* $K_2^N(\lambda)$, *respectively.*

Proof. The minimality of $K_1^N(\lambda)$, $K_2^N(\lambda)$, $D_1^N(\lambda)$ and $D_2^N(\lambda)$ follows immediately from the characterization of minimal bases in Theorem 2.7. The duality of the pairs $(K_1^N(\lambda), D_1^N(\lambda))$ and $(K_2^N(\lambda), D_2^N(\lambda))$ can be established by direct matrix multiplication. ∎

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We now consider strong block minimal basis pencils of the form

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2^N(\lambda)^T \\ K_1^N(\lambda) & 0 \end{bmatrix}. \tag{3.28}$$

We will refer to (3.28) as a *Newton pencil*. In Theorem 3.2, we show how to choose the body of a Newton pencil $L(\lambda)$ as in (3.28) so that $L(\lambda)$ is a strong linearization of a prescribed matrix polynomial.

Theorem 3.2. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ be an $m \times n$ matrix polynomial expressed in the Newton basis associated with the nodes $\{x_1, \dots, x_k\}$. Let $0 \le \mu \le k-1$ be an integer, and let*

$$M_\mu^N(\lambda) := \left[\begin{array}{cccc|c} \gamma_k(\lambda)P_k + P_{k-1} & P_{k-2} & \cdots & P_{\mu+1} & P_\mu \\ \hline & & & & P_{\mu-1} \\ & & & & P_{\mu-2} \\ & & 0 & & P_2 \\ & & & & P_1 \\ & & & & P_0 \end{array}\right]. \tag{3.29}$$

Then, the Newton pencil

$$N_P^\mu(\lambda) := \begin{bmatrix} M_\mu^N(\lambda) & K_2^N(\lambda)^T \\ K_1^N(\lambda) & 0 \end{bmatrix} \tag{3.30}$$

is a strong linearization of $P(\lambda)$. We will refer to (3.30) as the colleague Newton pencil *of $P(\lambda)$ associated with μ.*

Proof. By direct matrix multiplication, we have $D_2^N(\lambda)M_\mu^N(\lambda)D_1^N(\lambda)^T = P(\lambda)$. Hence, by Theorem 2.12 together with Lemma 3.1, the colleague Newton pencil $N_P^\mu(\lambda)$ is a strong linearization of $P(\lambda)$. □

Using Theorem 2.13, we can now construct an infinite family of Newton pencils that are strong linearizations of a prescribed $m \times n$ matrix polynomial $P(\lambda)$ expressed in the Newton basis.

Theorem 3.3. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ be an $m \times n$ matrix polynomial expressed in the Newton basis associated with the nodes $\{x_1, \dots, x_k\}$ and let $0 \le \mu \le k-1$ be an integer. Let $M_\mu^N(\lambda)$ be defined as in (3.29), and let A and B be two arbitrary matrices of size $(\mu+1)m \times (k-\mu-1)n$ and $\mu m \times (k-\mu)n$, respectively. Then, the Newton pencil*

$$\mathcal{N}(\lambda) = \begin{bmatrix} M_\mu^N(\lambda) + AK_1^N(\lambda) + K_2^N(\lambda)^T B & K_2^N(\lambda)^T \\ K_1^N(\lambda) & 0 \end{bmatrix}. \tag{3.31}$$

is a strong linearization of $P(\lambda)$. We will refer to (3.31) as a Newton linearization of the matrix polynomial $P(\lambda)$ with parameter μ.

Remark 3.4. Note that every Newton linearization (3.31) of a matrix polynomial $P(\lambda)$ can be factored as

$$\begin{bmatrix} I_{(\mu+1)m} & A \\ 0 & I_{(k-\mu-1)n} \end{bmatrix} \begin{bmatrix} M_\mu^N(\lambda) & K_2^N(\lambda)^T \\ K_1^N(\lambda) & 0 \end{bmatrix} \begin{bmatrix} I_{(k-\mu)n} & 0 \\ B & I_{\mu m} \end{bmatrix}.$$

Hence, for a fixed integer μ, all Newton linearizations of the form (3.31) are strictly equivalent to the colleague Newton pencil (3.30). Notice that, in particular, the matrix A (resp., B) can be chosen to contain a single nonzero block-entry, which can be interpreted as an elementary (e.g. Gaussian) block-row (resp., block-column) operation on the matrix pencil (3.30). Using this idea, we produce some examples of Newton linearizations in Example 3.5.

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EXAMPLE 3.5. Let $P(\lambda)=\sum_{i=0}^{5} P_i\, n_i(\lambda)$ be an $m\times n$ matrix polynomial of degree 5 expressed in the Newton basis. Let $\mu=2$. Then, the Newton colleague linearization of $P(\lambda)$ associated with μ is given by

$$\mathcal{N}_P^2(\lambda)=\left[\begin{array}{ccc|cc} \gamma_5(\lambda)P_5+P_4 & P_3 & P_2 & -I_m & 0 \\ 0 & 0 & P_1 & \gamma_2(\lambda)I_m & -I_m \\ 0 & 0 & P_0 & 0 & \gamma_1(\lambda)I_m \\ \hline -I_n & \gamma_4(\lambda)I_n & 0 & 0 & 0 \\ 0 & -I_n & \gamma_3(\lambda)I_n & 0 & 0 \end{array}\right].$$

By Theorem 3.3, the following Newton pencils are also strong linearizations of $P(\lambda)$. They are obtained from $\mathcal{N}_P^2(\lambda)$ by applying a finite number of elementary block-row or block-column operations. Using the notation in Theorem 3.3, we specify the matrices A and B used to obtain the body of each particular linearization. For lack of space, we omit the dependence in λ of the $\gamma_i(\lambda)$ polynomials.

The following linearization has been obtained from $\mathcal{N}_P^2(\lambda)$ by adding to the first block-row the fifth block-row multiplied by P_3:

$$\mathcal{N}_1(\lambda)=\left[\begin{array}{ccc|cc} \gamma_5P_5+P_4 & 0 & \gamma_3P_3+P_2 & -I_m & 0 \\ 0 & 0 & P_1 & \gamma_2I_m & -I_m \\ 0 & 0 & P_0 & 0 & \gamma_1I_m \\ \hline -I_n & \gamma_4I_n & 0 & 0 & 0 \\ 0 & -I_n & \gamma_3I_n & 0 & 0 \end{array}\right], \quad A=\begin{bmatrix} 0 & P_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B=0.$$

The following linearization has been obtained from $\mathcal{N}_1(\lambda)$ by adding to the first block-row the fourth block-row multiplied by P_4:

$$\mathcal{N}_2(\lambda)=\left[\begin{array}{ccc|cc} \gamma_5P_5 & \gamma_4P_4 & \gamma_3P_3+P_2 & -I_n & 0 \\ 0 & 0 & P_1 & \gamma_2I_n & -I_n \\ 0 & 0 & P_0 & 0 & \gamma_1I_n \\ \hline -I_n & \gamma_4I_n & 0 & 0 & 0 \\ 0 & -I_n & \gamma_3I_n & 0 & 0 \end{array}\right], \quad A=\begin{bmatrix} P_4 & P_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B=0.$$

REMARK 3.6. In the literature, a family of strong linearizations of a matrix polynomial $P(\lambda)$ expressed in the Newton basis can be found in [22]. The pencils in this family receive the name of Newton-Fiedler pencils, since they generalize the family of Fiedler pencils [5]. As the Newton linearizations, Newton-Fiedler pencils can be easily constructed from the coefficients P_i and the nodes $\{x_1,\ldots,x_k\}$. However, one of the drawbacks of the family of Newton-Fiedler pencils is that the Newton-Fiedler pencils are defined implicitly as products of matrices, while the Newton linearizations, being block minimal basis pencils, are given in an explicit way.

In the following two sections, we will show how to recover the eigenvectors, minimal indices and minimal bases of a matrix polynomial from those of its Newton linearizations. We will need the following definition.

DEFINITION 3.7. (Newton-Horner shifts) Given a matrix polynomial $P(\lambda)=\sum_{i=0}^{k} P_i\, n_i(\lambda)$ expressed in the Newton basis associated with nodes $\{x_1,\ldots,x_k\}$, the ith *Newton-Horner shift* of $P(\lambda)$ is given by

$$P^i(\lambda):=P_k\, n_{k+1-i}^{k}(\lambda)+P_{k-1}\, n_{k+1-i}^{k-1}(\lambda)+\cdots+P_{k+1-i}\, n_{k+1-i}^{k+1-i}(\lambda)+P_{k-i},$$

where the $n_i^j(\lambda)$ polynomials are defined in (3.24). In particular, $P^1(\lambda)=P_k\, n_k^k(\lambda)+P_{k-1}$ and $P^k(\lambda)=P(\lambda)$.

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Newton-Horner shifts satisfy the following recurrence relation

$$P^{i+1}(\lambda) = \gamma_{k-i}(\lambda)\, P^{i}(\lambda) + P_{k-i-1} \quad (i = 1, \ldots, k-1), \tag{3.32}$$

where $\gamma_{k-i}(\lambda)$ is as in (3.23).

Theorem 3.8 gives right- and left-sided factorizations of the Newton colleague pencil (3.30).

THEOREM 3.8. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ be an $m \times n$ matrix polynomial expressed in the Newton basis associated with nodes $\{x_1, \ldots, x_k\}$. Let $\mathcal{N}_P^{\mu}(\lambda)$ be the Newton colleague pencil associated with μ, and let $D_1^N(\lambda)$ and $D_2^N(\lambda)$ be the minimal bases in (3.27).*

For $0 < \mu \le k-1$, let

$$H_N^{\mu}(\lambda)^T := \begin{bmatrix} D_1^N(\lambda) & P^{k-\mu}(\lambda)^T & \cdots & P^{k-2}(\lambda)^T & P^{k-1}(\lambda)^T \end{bmatrix}^T,$$

and for $\mu = 0$, let

$$H_N^{\mu}(\lambda)^T := D_1^N(\lambda) = \begin{bmatrix} n_{k-1}(\lambda) I_n & n_{k-2}(\lambda) I_n & \cdots & n_1(\lambda) I_n & I_n \end{bmatrix}.$$

For $0 \le \mu < k-1$, let

$$G_N^{\mu}(\lambda) := \begin{bmatrix} D_2^N(\lambda) & n_{\mu}(\lambda)\, P^1(\lambda) & n_{\mu}(\lambda)\, P^2(\lambda) & \cdots & n_{\mu}(\lambda)\, P^{k-\mu-1}(\lambda) \end{bmatrix},$$

and for $\mu = k-1$, let

$$G_N^{\mu}(\lambda) := D_2^N(\lambda) = \begin{bmatrix} n_{k-1}(\lambda) I_m & n_{k-2}(\lambda) I_m & \cdots & n_1(\lambda) I_m & I_m \end{bmatrix}.$$

Then, the following right- and left-sided factorizations hold

$$\mathcal{N}_P^{\mu}(\lambda) H_N^{\mu}(\lambda) = e_{\mu+1} \otimes P(\lambda) \quad \text{and} \quad G_N^{\mu}(\lambda) \mathcal{N}_P^{\mu}(\lambda) = e_{k-\mu}^T \otimes P(\lambda),$$

where the vector e_i denotes the ith column of the $k \times k$ identity matrix.

Proof. With the help of the recurrence (3.32) and the fact that $n_{i+1}(\lambda) = \gamma_{i+1}(\lambda)\, n_i(\lambda)$, the results can be directly checked by multiplying $\mathcal{N}_P^{\mu}(\lambda) H_N^{\mu}(\lambda)$ and $G_N^{\mu}(\lambda) \mathcal{N}_P^{\mu}(\lambda)$ ∎

3.1. Recovery of eigenvectors from Newton linearizations. Assume that the matrix polynomial $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ is regular. In this section, we provide recovery formulas for the (left and right) eigenvectors of $P(\lambda)$ from those of its Newton linearizations.

We start by giving a close formula for the right and left eigenvectors of the Newton colleague pencil (3.30) associated with its finite eigenvalues.

THEOREM 3.9. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ be an $n \times n$ regular matrix polynomial expressed in the Newton basis associated with nodes $\{x_1, \ldots, x_k\}$. Let λ_0 be a finite eigenvalue of $P(\lambda)$. Let $\mathcal{N}_P^{\mu}(\lambda)$ be the Newton colleague pencil in (3.30) associated with μ. Then, z (resp., ω) is a right (resp., left) eigenvector of $\mathcal{N}_P^{\mu}(\lambda)$ associated with λ_0 if and only if $z = H_N^{\mu}(\lambda_0) x$ (resp., $\omega = G_N^{\mu}(\lambda_0)^T y$), where x (resp., y) is a right (resp., left) eigenvector of $P(\lambda)$ associated with λ_0.*

Proof. If follows immediately from Theorems 2.15 and 3.8. ∎

The next result provides recovery formulas of eigenvectors associated with finite and infinite eigenvalues of a matrix polynomial from those of its Newton linearizations. The eigenvectors of the linearizations are considered block vectors of length k with block-entries of length n.

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Theorem 3.10. (Recovery of eigenvectors from Newton linearizations) *Let $P(\lambda) = \sum_{i=0}^{k} P_i n_i(\lambda)$ be an $n \times n$ regular matrix polynomial expressed in the Newton basis associated with nodes $\{x_1, \ldots, x_k\}$. Let λ_0 be an eigenvalue of $P(\lambda)$. Let $N(\lambda)$ be a Newton linearization of $P(\lambda)$ with parameter μ as in (3.31). Let z and ω be, respectively, a right and a left eigenvector of $N(\lambda)$ associated with λ_0.*

1. *Assume λ_0 is finite. Then,*
   - *$z(k-\mu)$ is a right eigenvector of $P(\lambda)$ associated with λ_0. If, in addition, $\lambda_0 \notin \{x_{\mu+1}, \ldots, x_{k-1}\}$, then the block-entries $z(1), z(2), \ldots z(k-\mu)$ are also right eigenvectors of $P(\lambda)$ associated with λ_0.*
   - *$\omega(\mu+1)$ is a left eigenvector of $P(\lambda)$ associated with λ_0. If, in addition, $\lambda_0 \notin \{x_1, \ldots, x_\mu\}$, then the block-entries $\omega(1), \omega(2), \ldots \omega(\mu+1)$ are left eigenvectors of $P(\lambda)$ associated with λ_0.*
2. *Assume λ_0 is infinite. Then,*
   - *$z(1)$ is a right eigenvector of $P(\lambda)$ associated with λ_0.*
   - *$\omega(1)$ is a left eigenvector of $P(\lambda)$ associated with λ_0.*

Proof. We prove the result for the right eigenvectors. The proof is similar for the left eigenvectors.

We show first that the theorem holds for the Newton colleague pencil $\mathcal{N}_P^\mu(\lambda)$.

Case I: Assume that λ_0 is a finite eigenvalue. By Theorem 3.9, $z = H_N^\mu(\lambda_0)x$ for some eigenvector x of $P(\lambda)$ associated with λ_0. Since the $(k-\mu)$th block-entry of $H_N^\mu(\lambda_0)$ is the identity matrix, we have that $z(k-\mu) = x$ is a right eigenvector of $P(\lambda)$ with eigenvalue λ_0. Further, if $\lambda_0 \notin \{x_{\mu+1}, \ldots, x_{k-1}\}$, then all the block-entries of $H_N^\mu(\lambda_0)x$ in positions $1, 2, \ldots, k-\mu-1$ are nonzero multiples of the vector x. Hence, $z(1), \ldots, z(k-\mu)$ are all eigenvectors of $P(\lambda)$ with eigenvalue λ_0.

Case II: Assume that λ_0 is an infinite eigenvalue. This implies that 0 is an eigenvalue of $\operatorname{rev}_k P(\lambda)$ and $\operatorname{rev}_1 \mathcal{N}_P^\mu(\lambda)$. By Lemma 2.3, we have

$$\operatorname{rev}_k P(\lambda) = \sum_{i=0}^{k} P_i \operatorname{rev}_k n_i(\lambda) = \sum_{i=0}^{k} P_i\, \lambda^{k-i} \widetilde{n_i}(\lambda),$$

where $\widetilde{n_i}(\lambda) = \prod_{j=1}^{i}(1 - x_j\lambda)$. Thus, $\operatorname{rev}_k P(0) = P_k$, which implies that x is a right eigenvector of $P(\lambda)$ with eigenvalue at infinity if and only if x is a right eigenvector of P_k with eigenvalue 0. Moreover, we have

$$\operatorname{rev}_1 \mathcal{N}_P^\mu(0) = \left[\begin{array}{ccccc|ccc} P_k & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & & 0 & I_n & \cdots & 0 \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \cdots & I_n \\ \hline 0 & I_n & 0 & \cdots & 0 & & & \\ 0 & 0 & I_n & \cdots & 0 & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & 0 & \cdots & I_n & & & \end{array}\right].$$

Hence, any right eigenvector z of $\operatorname{rev}_1 \mathcal{N}_P^\mu(\lambda)$ with eigenvalue 0 is necessarily of the form $\begin{bmatrix} x^T & 0 & \cdots & 0 \end{bmatrix}^T$ for some eigenvector x of P_k with eigenvalue 0. Conclusively, the first block-entry of z, when seen as a block vector of length k, is an eigenvector of $P(\lambda)$ with eigenvalue infinity.

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Let us now prove the results for any Newton linearization $N(\lambda)$. By Remark 3.4, we have

$$N(\lambda) = \begin{bmatrix} I_{(\mu+1)n} & A \\ 0 & I_{(k-\mu-1)n} \end{bmatrix} \mathcal{N}_P^{\mu}(\lambda) \begin{bmatrix} I_{(k-\mu)n} & 0 \\ B & I_{\mu n} \end{bmatrix} \tag{3.33}$$

for some matrices A and B. The equivalence transformation (3.33) implies that z is a right eigenvector of $N(\lambda)$ with eigenvalue (finite or infinite) λ_0 if and only if $\widetilde{z} := \left[\begin{smallmatrix} I_{(k-\mu)n} & 0 \\ B & I_{\mu m} \end{smallmatrix}\right] z$ is an eigenvector of $\mathcal{N}_P^{\mu}(\lambda)$ with eigenvalue (finite or infinite) λ_0. To finish the proof, it suffices to notice that the first $k-\mu$ blocks of the eigenvectors z and $\widetilde{z}$ are the same. □

Remark 3.11. Until very recently, there was no consensus in the literature on how to properly define eigenvectors of singular matrix polynomials. In [10], such a definition is constructed using the concept of root polynomials, after extending their definition in [15] for regular matrix polynomials to the singular case. Root polynomials are also very closely related to Jordan chains of matrix polynomials. Thus, providing recovery formulas for maximal sets of root polynomials is useful in both the regular and the singular case. In the case of the Newton linearizations, it can be proven that if $\{r_1(\lambda), r_2(\lambda), \ldots, r_s(\lambda)\}$ is a maximal set of root polynomials at λ_0 (of orders $\ell_1 \geq \cdots \geq \ell_s$) for a Newton linearization $N(\lambda)$ associated with the parameter μ, and for all $j \in \{1, 2, \ldots, s\}$, $\widetilde{r}_i(\lambda)$ denotes the $(k-\mu)$th block of $r_i(\lambda)$, then $\{\widetilde{r}_1(\lambda), \ldots, \widetilde{r}_s(\lambda)\}$ is a maximal set of root polynomials at λ_0 for $P(\lambda)$ of orders $\ell_1 \geq \cdots \geq \ell_s$. The proof is identical to the proof of Theorem 8.5 in [10] taking into account that there exist unimodular matrices U and V such that

$$U(\lambda)N(\lambda)V(\lambda) = \begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

where the last block column of $V(\lambda)$ is of the form $[D_1^N(\lambda)^T X(\lambda)^T]^T$ for some matrix polynomial $X(\lambda)$ and D_1^N contains a block equal to I_n in position $k-\mu$. In fact, since multiplying a maximal set of root polynomials at λ_0 by a scalar polynomial $q(\lambda)$ such that $q(\lambda_0) \neq 0$ generates another maximal set of root polynomials at λ_0 of the same orders, taking into account that all the block entries of D_1^N are of the form $q(\lambda)I_n$ for some scalar polynomial $q(\lambda)$ such that $q(\lambda_0) \neq 0$ for all finite eigenvalues of P, the proof of Theorem 8.5 can be slightly modified to show that the block entries of $r_1, r_2, \ldots, r_s$ in positions $j \in \{1, 2, \ldots, k-\mu\}$ form a maximal set of root polynomials at λ_0 for $P(\lambda)$ of orders $\ell_1 \geq \cdots \geq \ell_s$, which is consistent with the recovery formulas for eigenvectors of regular Newton polynomials given in Theorem 3.10. A similar remark applies to Lagrange and Chebyshev polynomials. We do not include these results formally to keep the paper as concise as possible.

3.2. Recovery of minimal bases and minimal indices from Newton linearizations. Assume the $m \times n$ matrix polynomial $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ is singular. In this section, we show how to recover the minimal indices and minimal bases of $P(\lambda)$ from those of its Newton linearizations.

Theorem 3.12. (Recovery of minimal bases and minimal indices from Newton linearizations) *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, n_i(\lambda)$ be an $m \times n$ singular matrix polynomial expressed in the Newton basis associated with nodes $\{x_1, \ldots, x_k\}$. Let $N(\lambda)$ be a Newton linearization of $P(\lambda)$ with parameter μ as in* (3.31).

(a1) *Suppose that $\{z_1(\lambda), z_2(\lambda), \ldots, z_p(\lambda)\}$ is a minimal basis for the right nullspace of $N(\lambda)$, with vector polynomials z_i partitioned into blocks conformable with the blocks of $N(\lambda)$, and let $x_\ell(\lambda)$ be the $(k-\mu)$th block-entry of $z_\ell(\lambda)$ for $\ell = 1, 2, \ldots, p$. Then, $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ is a minimal basis for the right nullspace of $P(\lambda)$.*

(a2) *If $0 \leq \epsilon_1 \leq \cdots \leq \epsilon_p$ are the right minimal indices of $N(\lambda)$, then*

$$0 \leq \epsilon_1 - k + \mu + 1 \leq \epsilon_2 - k + \mu + 1 \leq \cdots \leq \epsilon_p - k + \mu + 1$$

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are the right minimal indices of $P(\lambda)$.

(b1) *Suppose that* $\{\omega_1(\lambda),\dots,\omega_q(\lambda)\}$ *is a minimal basis for the left nullspace of* $N(\lambda)$, *with vectors* ω_i *partitioned into blocks conformable with the blocks of* $N(\lambda)$, *and let* $y_\ell(\lambda)$ *be the* $(\mu+1)$*th block-entry of* $\omega_\ell(\lambda)$ *for* $\ell=1,2,\dots,q$. *Then,* $\{y_1(\lambda),\dots,y_q(\lambda)\}$ *is a minimal basis for the left nullspace of* $P(\lambda)$.

(b2) *If* $0\le\mu_1\le\cdots\le\mu_q$ *are the left minimal indices of* $N(\lambda)$, *then*

$$0\le\mu_1-\mu\le\mu_2-\mu\le\cdots\le\mu_p-\mu$$

are the left minimal indices of $P(\lambda)$.

Proof. The proof follows closely the proof of Theorem 3.10, so we just sketch it. First, using Theorem 2.15 together with the one-sided factorizations in Theorem 3.8 one proves the results for the Newton colleague pencil (3.30). Then, using the strict equivalence

$$N(\lambda)=\begin{bmatrix} I_{(\mu+1)m} & A\\ 0 & I_{(k-\mu-1)n}\end{bmatrix}\mathcal{N}_P^{\mu}(\lambda)\begin{bmatrix} I_{(k-\mu)n} & 0\\ B & I_{\mu m}\end{bmatrix},$$

that transforms the Newton colleague pencil into the Newton linearization $N(\lambda)$, one proves the result for $N(\lambda)$. ∎

4. Strong linearizations for matrix polynomials in the Lagrange basis. Let $\{x_1,\dots,x_{k+1}\}$ be a set of $k+1$ nodes, and let $P(\lambda)$ be a matrix polynomial expressed in the modified Lagrange form:

$$P(\lambda)=\ell(\lambda)\sum_{i=1}^{k+1}P_i\,\frac{w_i}{\gamma_i(\lambda)},\quad P_1,\dots,P_{k+1}\in\mathbb{C}^{m\times n}, \tag{4.34}$$

where $\gamma_i(\lambda)=\lambda-x_i$, and $\ell(\lambda)$ and w_i are as in (2.14). In this section, we present a family of strong linearizations of the polynomial $P(\lambda)$ that can be easily constructed from the coefficients P_i and the corresponding nodes.

Let $0\le\mu\le k-1$ be an integer. We define the following matrix pencils

$$K_1^L(\lambda):=\begin{bmatrix} \gamma_{k+1}(\lambda)I_n & -\gamma_{k-1}(\lambda)I_n & & & \\ & \gamma_k(\lambda)I_n & -\gamma_{k-2}(\lambda)I_n & & \\ & & \ddots & \ddots & \\ & & & \gamma_{\mu+3}(\lambda)I_n & -\gamma_{\mu+1}(\lambda)I_n \end{bmatrix} \tag{4.35}$$

and

$$K_2^L(\lambda):=\begin{bmatrix} \gamma_{\mu+2}(\lambda)I_m & -\gamma_{\mu}(\lambda)I_m & & & \\ & \gamma_{\mu+1}(\lambda)I_m & -\gamma_{\mu-1}(\lambda)I_m & & \\ & & \ddots & \ddots & \\ & & & \gamma_{3}(\lambda)I_m & -\gamma_{1}(\lambda)I_m \end{bmatrix}, \tag{4.36}$$

where the polynomials $\gamma_j(\lambda)$ are defined in (3.23). Notice that when $\mu=0$ (resp., $\mu=k-1$), the matrix pencil $K_2^L(\lambda)$ (resp., $K_1^L(\lambda)$) is an empty matrix.

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LEMMA 4.1. *Let $\{x_1,\dots,x_{k+1}\}$ be a set of nodes, and let $0\le\mu\le k-1$ be an integer. The matrix pencils $K_1^L(\lambda)$ and $K_2^L(\lambda)$ given in (4.35) and (4.36) are both minimal bases. Moreover, the matrix polynomials*

$$D_1^L(\lambda)^T=\begin{bmatrix}\dfrac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{k+1}(\lambda)\gamma_k(\lambda)}I_n\\ \dfrac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_k(\lambda)\gamma_{k-1}(\lambda)}I_n\\ \vdots\\ \dfrac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{\mu+2}(\lambda)\gamma_{\mu+1}(\lambda)}I_n\end{bmatrix}\quad\text{and}\quad D_2^L(\lambda)^T=\begin{bmatrix}\dfrac{n_1^{\mu+2}(\lambda)}{\gamma_{\mu+2}(\lambda)\gamma_{\mu+1}(\lambda)}I_m\\ \dfrac{n_1^{\mu+2}(\lambda)}{\gamma_{\mu+1}(\lambda)\gamma_\mu(\lambda)}I_m\\ \vdots\\ \dfrac{n_1^{\mu+2}(\lambda)}{\gamma_2(\lambda)\gamma_1(\lambda)}I_m\end{bmatrix}, \tag{4.37}$$

where the polynomials $n_i^j(\lambda)$ are defined in (3.24), are, respectively, dual bases of $K_1^L(\lambda)$ and $K_2^L(\lambda)$.

Proof. It is easy to check through straightforward computations that $K_1^L(\lambda)D_1^L(\lambda)^T=0$ and $K_2^L(\lambda)D_2^L(\lambda)^T=0$. The minimality of the four matrix polynomials follows from the characterization of minimal bases in Theorem 2.7. ∎

We now consider strong block minimal basis pencils of the form

$$L(\lambda)=\begin{bmatrix}M(\lambda) & K_2^L(\lambda)^T\\ K_1^L(\lambda) & 0\end{bmatrix}. \tag{4.38}$$

We will refer to (4.38) as a *Lagrange pencil*. In theorem 4.2, we show how to chose the body $M(\lambda)$ of a Lagrange pencil (4.38) so that the Lagrange pencil is a strong linearization of the matrix polynomial (4.34).

THEOREM 4.2. *Let $P(\lambda)$ be an $m\times n$ matrix polynomial as in (4.34). Let $0\le\mu\le k-1$ be an integer and let $M_\mu^L(\lambda):=$*

$$\begin{bmatrix}P_{k+1}w_{k+1}\gamma_k(\lambda)+P_kw_k\gamma_{k+1}(\lambda) & P_{k-1}w_{k-1}\gamma_k(\lambda) & \cdots & P_{\mu+1}w_{\mu+1}\gamma_{\mu+2}(\lambda)\\ & & & P_\mu w_\mu\gamma_{\mu+1}(\lambda)\\ & & & \vdots\\ & & & P_2w_2\gamma_3(\lambda)\\ & & & P_1w_1\gamma_2(\lambda)\end{bmatrix},$$

when $0\le\mu<k-1$; and

$$M_\mu^L(\lambda):=\begin{bmatrix}P_{k+1}w_{k+1}\gamma_k(\lambda)+P_kw_k\gamma_{k+1}(\lambda)\\ P_{k-1}w_{k-1}\gamma_k(\lambda)\\ \vdots\\ P_2w_2\gamma_3(\lambda)\\ P_1w_1\gamma_2(\lambda)\end{bmatrix},$$

when $\mu=k-1$. Then, the Lagrange pencil

$$L_P^\mu(\lambda)=\begin{bmatrix}M_\mu^L(\lambda) & K_2^L(\lambda)^T\\ K_1^L(\lambda) & 0\end{bmatrix}. \tag{4.39}$$

is a strong linearization of $P(\lambda)$. We will refer to (4.39) as the colleague Lagrange pencil *of $P(\lambda)$ associated with μ.*

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Proof. By direct matrix multiplication, we have $D_2^L(\lambda)M_\mu^L(\lambda)D_1^L(\lambda)^T = P(\lambda)$, where $D_1^L(\lambda)$ and $D_2^L(\lambda)$ are the dual minimal basis of $K_1^L(\lambda)$ and $K_2^L(\lambda)$, respectively. Thus, by Theorem 2.12, the colleague Lagrance pencil $L_P^\mu(\lambda)$ is a strong linearization of the matrix polynomial $P(\lambda)$. ∎

REMARK 4.3. Previously to this work, and as far as we know, the only strong linearization for matrix polynomials in the Lagrange basis as in (4.34) of size $nk \times nk$ explicitly constructed is

$$\begin{bmatrix} -\gamma_1 P_0 & -\gamma_2 P_1 & \cdots & -\gamma_{k-1}P_{k-2} & -\gamma_k P_{k-1} - \gamma_{k-1}\theta_k^{-1}P_k \\ \hline -\gamma_0 I & \gamma_2\theta_1 I & & & \\ & \ddots & \ddots & & \\ & & -\gamma_{k-3}I & \gamma_{k-1}\theta_{k-2}I & \\ & & & -\gamma_{k-2}I & \gamma_k\theta_{k-1}I \end{bmatrix}, \tag{4.40}$$

where $\theta_i = \frac{w_{i-1}}{w_i}$, for $i = 1, \ldots, k$, and where we omit the dependence on λ of the γ_i polynomials for lack of space. This strong linearization was introduced in [27], and it can be easily established to be strictly equivalent to the Lagrange colleague pencil (4.39) associated with $\mu = 0$.

By applying Theorem 2.13 to the colleague Lagrange pencil (4.39), we construct in Theorem 4.4 an infinite family of strong linearizations of a matrix polynomial $P(\lambda)$ expressed in the Lagrange basis.

THEOREM 4.4. *Let $P(\lambda)$ be a matrix polynomial expressed in the Lagrange basis as in (4.34). Let $0 \le \mu \le k-1$ be an integer and let M_μ^L be as in Theorem 4.2. Let A and B be two arbitrary matrices of size $(\mu+1)m \times (k-\mu-1)n$ and $\mu m \times (k-\mu)n$, respectively. Then, the pencil*

$$L(\lambda) := \left[\begin{array}{c|c} M_\mu^L(\lambda) + AK_1^L(\lambda) + K_2^L(\lambda)^T B & K_2^L(\lambda)^T \\ \hline K_1^L(\lambda) & 0 \end{array}\right] \tag{4.41}$$

is a strong linearization of $P(\lambda)$. We will refer to (4.41) as a Lagrange linearization *of the matrix polynomial $P(\lambda)$ with parameter μ.*

REMARK 4.5. Note that every Lagrange linearization (4.41) of a matrix polynomial $P(\lambda)$ is strictly equivalent to the colleague Lagrange pencil $L(\lambda)$ as in (4.39), since we have

$$L(\lambda) = \begin{bmatrix} I_{(\mu+1)m} & A \\ 0 & I_{(k-\mu-1)n} \end{bmatrix} \begin{bmatrix} M_\mu^L(\lambda) & K_2^L(\lambda)^T \\ K_1^L(\lambda) & 0 \end{bmatrix} \begin{bmatrix} I_{(k-\mu)n} & 0 \\ B & I_{\mu m} \end{bmatrix}.$$

Next we construct a few examples of Lagrange linearizations of a matrix polynomial of grade 5.

EXAMPLE 4.6. Let $P(\lambda)$ be a matrix polynomial expressed in the Lagrange basis as in (4.34) of grade 5. Then, the Lagrange colleague pencil of $P(\lambda)$ associated with $\mu = 2$ is given by $L_P^2(\lambda) =$

$$\left[\begin{array}{ccc|cc} P_6w_6\gamma_5 + P_5w_5\gamma_6 & P_4w_4\gamma_5 & P_3w_3\gamma_4 & \gamma_4 I_m & 0 \\ 0 & 0 & P_2w_2\gamma_3 & -\gamma_2 I_m & \gamma_3 I_m \\ 0 & 0 & P_1w_1\gamma_2 & 0 & -\gamma_1 I_m \\ \hline \gamma_6 I_n & -\gamma_4 I_n & 0 & 0 & 0 \\ 0 & \gamma_5 I_n & -\gamma_3 I_n & 0 & 0 \end{array}\right],$$

where, for lack of space, we omit the dependence in λ of the $\gamma_i(\lambda)$ polynomials. By Theorem 4.4, the following Lagrange pencils are also strong linearizations of $P(\lambda)$. They are obtained from the Lagrange colleague pencil by applying a finite number of elementary block-row or block-column operations, in the

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same spirit as in Example 3.5. Using the notation in Theorem 4.4, we specify the matrices A and B used to obtain the body of each particular linearization.

The following linearization has been obtained from $L_P^2(\lambda)$ by adding to the first block-row the fifth block-row multiplied by $-P_4w_4$:

$$L_1(\lambda) = \left[\begin{array}{ccc|cc} P_6w_6\gamma_5 + P_5w_5\gamma_6 & 0 & P_4w_4\gamma_3 + P_3w_3\gamma_4 & \gamma_4 I_m & 0 \\ 0 & 0 & P_2w_2\gamma_2 & -\gamma_2 I_m & \gamma_3 I_m \\ 0 & 0 & P_1w_1\gamma_2 & 0 & -\gamma_1 I_m \\ \hline \gamma_6 I_n & -\gamma_4 I_n & 0 & 0 & 0 \\ 0 & \gamma_5 I_n & -\gamma_3 I_n & 0 & 0 \end{array}\right],$$

In this case, we have $A = \begin{bmatrix} 0 & -P_4w_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = 0$.

The following linearization has been obtained from $L_1(\lambda)$ by adding to the first block-row the fourth block-row multiplied by $-P_5w_5$:

$$L_2(\lambda) = \left[\begin{array}{ccc|cc} P_6w_6\gamma_5 & P_5w_5\gamma_4 & P_4w_4\gamma_3 + P_3w_3\gamma_4 & \gamma_4 I_m & 0 \\ 0 & 0 & P_2w_2\gamma_3 & -\gamma_2 I_m & \gamma_3 I_m \\ 0 & 0 & P_1w_1\gamma_2 & 0 & -\gamma_1 I_m \\ \hline \gamma_6 I_n & -\gamma_4 I_n & 0 & 0 & 0 \\ 0 & \gamma_5 I_n & -\gamma_3 I_n & 0 & 0 \end{array}\right].$$

In this case, we have $A = \begin{bmatrix} -P_5w_5 & -P_4w_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = 0$.

Our next goal is to obtain recovery rules for eigenvectors, and minimal bases and minimal indices of a matrix polynomial $P(\lambda)$ from those of its Lagrange linearizations. We will need the following notation.

Associated with the matrix polynomial $P(\lambda)$ in (4.34), we define the matrix polynomials

$$T_j^P(\lambda) := \ell(\lambda)\sum_{i=1}^{j} P_i \frac{w_i}{\gamma_i(\lambda)} \quad \text{and} \quad S_j^P(\lambda) := \ell(\lambda)\sum_{i=j}^{k+1} P_i \frac{w_i}{\gamma_i(\lambda)} \quad (j = 1, \dots, k+1),$$

where, we recall, $\ell(\lambda) = n_1^{k+1}(\lambda) = \prod_{i=1}^{k+1}(\lambda - x_i)$. Observe that $T_{k+1}^P(\lambda) = S_1^P(\lambda) = P(\lambda)$. Moreover, we have

$$S_{j+1}^P(\lambda) + T_j^P(\lambda) = P(\lambda), \quad \text{for } j = 1, 2, \dots, k.$$

Let $0 \le \mu \le k-1$ be an integer and let $a_{\mu+1}, \dots, a_2, a_1$ be the coordinates of the (scalar) polynomial $p(x) = 1$ "in the basis $D_2^L(\lambda)$", that is,

$$a_{\mu+1}\frac{n_1^{\mu+2}(\lambda)}{\gamma_{\mu+2}(\lambda)\gamma_{\mu+1}(\lambda)} + a_\mu \frac{n_1^{\mu+2}(\lambda)}{\gamma_{\mu+1}(\lambda)\gamma_\mu(\lambda)} + \cdots + a_2\frac{n_1^{\mu+2}(\lambda)}{\gamma_3(\lambda)\gamma_2(\lambda)} + a_1\frac{n_1^{\mu+2}(\lambda)}{\gamma_2(\lambda)\gamma_1(\lambda)} = 1. \tag{4.42}$$

We call $[a_{\mu+1}, a_\mu, \dots, a_2, a_1]$ the *μ-2-coordinates of* 1. We notice that, by evaluating the expression (4.42) at the nodes x_1 and $x_{\mu+2}$, respectively, we get the values of a_1 and $a_{\mu+2}$, namely,

$$a_1 = \frac{1}{\prod_{i=3}^{\mu+2}(x_1 - x_i)} \quad \text{and} \quad a_{\mu+1} = \frac{1}{\prod_{i=1}^{\mu}(x_{\mu+2} - x_i)}.$$

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The rest of the coordinates can be obtained from the recurrence relation

$$1 = a_i \left.\frac{n_1^{\mu+2}(\lambda)}{\gamma_{i+1}(\lambda)\gamma_i(\lambda)}\right|_{\lambda=x_i} + a_{i-1} \left.\frac{n_1^{\mu+2}(\lambda)}{\gamma_i(\lambda)\gamma_{i-1}(\lambda)}\right|_{\lambda=x_i},$$

which is the result of evaluating (4.42) at the node x_i $(i = 2, \ldots, \mu+1)$.

Similarly, let $b_{\mu+1}, b_{\mu+2}, \ldots, b_k$ be the coordinates of the polynomial $p(x) = 1$ "in the basis $D_1^L(\lambda)$", that is,

$$b_k \frac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{k+1}(\lambda)\gamma_k(\lambda)} + \cdots + b_{\mu+2} \frac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{\mu+3}(\lambda)\gamma_{\mu+2}(\lambda)} + b_{\mu+1} \frac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{\mu+2}(\lambda)\gamma_{\mu+1}(\lambda)} = 1. \tag{4.43}$$

We call $[b_k, \ldots, b_{\mu+2}, b_{\mu+1}]$ the μ-1-coordinates of 1. The numbers b_i can be obtained using the same approach used to compute the μ-2-coordinates of 1.

Finally, we denote

$$\mathcal{P}_j^{\mu}(\lambda) := -\sum_{i=1}^{j} a_i \frac{\gamma_{j+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_{j+1}^P(\lambda) + \sum_{i=j+1}^{\mu+1} a_i \frac{\gamma_{j+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_j^P(\lambda),$$

for $j = 1, 2, \ldots, \mu$, and

$$\mathcal{Q}_j^{\mu}(\lambda) := -\sum_{i=\mu+1}^{j} b_i \frac{\gamma_{j+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_{j+1}^P(\lambda) + \sum_{i=j+1}^{k} b_i \frac{\gamma_{j+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_j^P(\lambda)$$

for $j = \mu+1, \ldots, k-1$. We observe that both $\mathcal{P}_j^{\mu}(\lambda)$ $(j = 1, \ldots, \mu)$ and $\mathcal{Q}_j^{\mu}(\lambda)$ $(j = \mu+1, \ldots, k-1)$ are matrix polynomials.

Theorem 4.7 gives right- and left-sided factorizations of the Lagrange colleague pencil (4.39).

THEOREM 4.7. *Let $P(\lambda)$ be a matrix polynomial of degree k as in (4.34), let $L_P^{\mu}(\lambda)$ be the Lagrange colleague pencil associated wih μ given in (4.39), and let $D_1^L(\lambda)$ and $D_2^L(\lambda)$ be the minimal bases in (4.37).*

For $0 < \mu \leq k-1$, let

$$H_L^{\mu}(\lambda)^T := \begin{bmatrix} D_1^L(\lambda) & \mathcal{P}_{\mu}^{\mu}(\lambda)^T & \mathcal{P}_{\mu-1}^{\mu}(\lambda)^T & \cdots & \mathcal{P}_1^{\mu}(\lambda)^T \end{bmatrix}^T$$

and for $\mu = 0$, let $H_L^{\mu}(\lambda)^T := D_1^L(\lambda)$.

For $0 \leq \mu < k-1$, let

$$G_L^{\mu}(\lambda) := \begin{bmatrix} D_2^L(\lambda) & \mathcal{Q}_{k-1}^{\mu}(\lambda) & \cdots & \mathcal{Q}_{\mu+1}^{\mu}(\lambda) \end{bmatrix},$$

and for $\mu = k-1$, let $G_L^{\mu}(\lambda) := D_2^L(\lambda)$. Then, the following right- and left-sided factorizations hold

$$L_P^{\mu}(\lambda) H_L^{\mu}(\lambda) = \left(\sum_{i=1}^{\mu+1} a_{\mu+2-i} e_i\right) \otimes P(\lambda) \quad \text{and} \quad G_L^{\mu}(\lambda) L_P^{\mu}(\lambda) = \left(\sum_{i=\mu+1}^{k} b_i e_i^T\right) \otimes P(\lambda),$$

where e_i denotes the ith column of the $k \times k$ identity matrix, and where $\begin{bmatrix} a_{\mu+1} & \cdots & a_2 & a_1 \end{bmatrix}$ and $\begin{bmatrix} b_k & \cdots & b_{\mu+2} & b_{\mu+1} \end{bmatrix}$ are, respectively, the μ-2-coordinates and μ-1-coordinates of 1.

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Proof. We prove the right-sided factorization. The left-sided factorization can be proven similarly.

By the duality of the minimal bases $K_1^L(\lambda)$ and $D_1^L(\lambda)$, it is clear that the ith block entry, with $i \in \{\mu+1, \mu+2, \ldots, k\}$, of $L_P^\mu(\lambda) H_L^\mu(\lambda)$ is zero.

Let $i \in \{1, 2, \ldots, \mu+1\}$. We need to compute the product of the ith block row of $H_L^\mu(\lambda)$ and $L_P^\mu(\lambda)$. To do this, we have to distinguish three cases:

Case I: Let $i = 1$. By direct matrix multiplication, the product of the first block row of $H_L^\mu(\lambda)$ and $L_P^\mu(\lambda)$ is given by

$$\begin{aligned}
n_{\mu+1}^k(\lambda) \sum_{i=\mu+1}^{k+1} \frac{P_i w_i}{\gamma_i(\lambda)} + \gamma_{\mu+2}(\lambda)\mathcal{P}_\mu^\mu(\lambda) &= \frac{S_{\mu+1}^P(\lambda)}{n_1^\mu(\lambda)} + \gamma_{\mu+2}(\lambda)\mathcal{P}_\mu^\mu(\lambda) \\
&= \frac{S_{\mu+1}^P(\lambda)}{n_1^\mu(\lambda)} - \sum_{i=1}^{\mu} \frac{a_i \gamma_{\mu+1}(\lambda)\gamma_{\mu+2}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_{\mu+1}^P(\lambda) + a_{\mu+1} T_\mu^P(\lambda) \\
&= \frac{S_{\mu+1}^P(\lambda)}{n_1^\mu(\lambda)} - \sum_{i=1}^{\mu} \frac{a_i n_1^{\mu+2}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} \frac{S_{\mu+1}^P(\lambda)}{n_1^\mu(\lambda)} + a_{\mu+1} T_\mu^P(\lambda) \\
&= \frac{S_{\mu+1}^P(\lambda)}{n_1^\mu(\lambda)} - \left(1 - \frac{a_{\mu+1}}{\gamma_{\mu+1}(\lambda)\gamma_{\mu+2}(\lambda)} n_1^{\mu+2}(\lambda)\right) \frac{S_{\mu+1}^P(\lambda)}{n_1^\mu(\lambda)} + a_{\mu+1} T_\mu^P(\lambda) \\
&= a_{\mu+1}\left(S_{\mu+1}^P(\lambda) + T_\mu^P(\lambda)\right) = a_{\mu+1} P(\lambda),
\end{aligned}$$

which is the desired result.

Case II: Let $i \in \{2, 3, \ldots, \mu\}$, and let $r = \mu + 2 - i$. The product of the ith block row of $H_L^\mu(\lambda)$ and $L_P^\mu(\lambda)$ is given by

$$\begin{aligned}
&P_r w_r \gamma_{r+1}(\lambda) \frac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{\mu+1}(\lambda)\gamma_{\mu+2}(\lambda)} - \gamma_r(\lambda)\mathcal{P}_r^\mu(\lambda) + \gamma_{r+1}(\lambda)\mathcal{P}_{r-1}^\mu(\lambda) \\
&= P_r w_r \gamma_{r+1} n_{\mu+3}^{k+1}(\lambda) \\
&\quad - \gamma_r(\lambda)\left(-\sum_{i=1}^{r} a_i \frac{\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_{r+1}^P(\lambda) + \sum_{i=r+1}^{\mu+1} a_i \frac{\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_r^P(\lambda)\right) \\
&\quad + \gamma_{r+1}(\lambda)\left(-\sum_{i=1}^{r-1} a_i \frac{\gamma_r(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_r^P(\lambda) + \sum_{i=r}^{\mu+1} a_i \frac{\gamma_r(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_{r-1}^P(\lambda)\right).
\end{aligned} \tag{4.44}$$

Taking into account that $S_r^P(\lambda) = S_{r+1}^P(\lambda) + n_1^{k+1}(\lambda) P_r w_r / \gamma_r(\lambda)$, we get

$$\begin{aligned}
&\sum_{i=1}^{r} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_{r+1}^P(\lambda) - \sum_{i=1}^{r-1} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_r^P(\lambda) \\
&= \sum_{i=1}^{r} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} S_{r+1}^P(\lambda) - \sum_{i=1}^{r-1} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} \left(S_{r+1}^P(\lambda) + n_1^{k+1}(\lambda) P_r \frac{w_r}{\gamma_r(\lambda)}\right) \\
&= a_r S_{r+1}^P(\lambda) - \sum_{i=1}^{r-1} a_i P_r w_r n_1^{k+1}(\lambda) \frac{\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)}.
\end{aligned} \tag{4.45}$$

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Taking into account that $T_r^P(\lambda) = T_{r-1}^P(\lambda) + n_1^{k+1}(\lambda)P_r w_r/\gamma_r(\lambda)$, we obtain

$$
\begin{aligned}
& -\sum_{i=r+1}^{\mu+1} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_r^P(\lambda) + \sum_{i=r}^{\mu+1} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_{r-1}^P(\lambda) \\
&= -\sum_{i=r+1}^{\mu+1} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} \left(T_{r-1}^P(\lambda) + n_1^{k+1}(\lambda) P_r \frac{w_r}{\gamma_r(\lambda)} \right) + \sum_{i=r}^{\mu+1} a_i \frac{\gamma_r(\lambda)\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_{r-1}^P(\lambda) \\
&= a_r T_{r-1}^P(\lambda) - \sum_{i=r+1}^{\mu+1} a_i P_r w_r n_1^{k+1}(\lambda) \frac{\gamma_{r+1}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)}.
\end{aligned} \tag{4.46}
$$

Substituting (4.45) and (4.46) into (4.46) yields

$$
\begin{aligned}
& P_r w_r \gamma_{r+1}(\lambda) \frac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{\mu+1}(\lambda)\gamma_{\mu+2}(\lambda)} - \gamma_r(\lambda)\mathcal{P}_r^{\mu}(\lambda) + \gamma_{r+1}(\lambda)\mathcal{P}_{r-1}^{\mu}(\lambda) \\
&= P_r w_r \gamma_{r+1}(\lambda) n_{\mu+3}^{k+1}(\lambda) + a_r S_{r+1}^P(\lambda) + a_r T_{r-1}^P(\lambda) \\
&\quad + \left(\sum_{i=1}^{r-1} \frac{a_i n_1^{\mu+2}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} + \sum_{i=r+1}^{\mu+1} \frac{a_i n_1^{\mu+2}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} \right) n_{\mu+3}^{k+1}(\lambda)\gamma_{r+1}(\lambda) P_r w_r \\
&= P_r w_r \gamma_{r+1}(\lambda) n_{\mu+3}^{k+1}(\lambda) + a_r S_{r+1}^P(\lambda) + a_r T_{r-1}^P(\lambda) \\
&\quad + \left(1 - \frac{a_r n_1^{\mu+2}(\lambda)}{\gamma_r(\lambda)\gamma_{r+1}(\lambda)} \right) n_{\mu+3}^{k+1}(\lambda)\gamma_{r+1}(\lambda) P_r w_r \\
&= a_r S_{r+1}^P(\lambda) + a_r T_{r-1}^P(\lambda) + a_r n_1^{k+1}(\lambda) P_r \frac{w_r}{\gamma_r(\lambda)} = a_r \left(S_{r+1}^P(\lambda) + T_r^P(\lambda) \right) = a_r P(\lambda),
\end{aligned}
$$

as we wanted to show.

Case III: Let $i = \mu + 1$. The product of the $(\mu+1)$th block row of $H_L^{\mu}(\lambda)$ and $L_P^{\mu}(\lambda)$ is given by

$$
\begin{aligned}
P_1 w_1 \gamma_2(\lambda) \frac{n_{\mu+1}^{k+1}(\lambda)}{\gamma_{\mu+2}(\lambda)\gamma_{\mu+1}(\lambda)} - \gamma_1(\lambda)\mathcal{P}_1^{\mu}(\lambda) &= \frac{T_1^P(\lambda)}{n_3^{\mu+2}(\lambda)} - \gamma_1(\lambda)\mathcal{P}_1^{\mu}(\lambda) \\
&= \frac{T_1^P(\lambda)}{n_3^{\mu+2}(\lambda)} + a_1 S_2^P(\lambda) - \sum_{i=2}^{\mu+1} \frac{a_i \gamma_1(\lambda)\gamma_2(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} T_1^P(\lambda) \\
&= \frac{T_1^P(\lambda)}{n_3^{\mu+2}(\lambda)} + a_1 S_2^P(\lambda) - \sum_{i=2}^{\mu+1} \frac{a_i n_1^{\mu+2}(\lambda)}{\gamma_i(\lambda)\gamma_{i+1}(\lambda)} \frac{T_1^P(\lambda)}{n_3^{\mu+2}(\lambda)} \\
&= \frac{T_1^P(\lambda)}{n_3^{\mu+2}(\lambda)} + a_1 S_2^P(\lambda) - \left(1 - \frac{a_1 n_1^{\mu+2}(\lambda)}{\gamma_1(\lambda)\gamma_2(\lambda)} \right) \frac{T_1^P(\lambda)}{n_3^{\mu+2}(\lambda)} \\
&= a_1 \left(S_2^P(\lambda) + T_1^P(\lambda) \right) = a_1 P(\lambda),
\end{aligned}
$$

as we wanted to prove. □

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4.1. Recovery of eigenvectors from Lagrange linearizations. Assume the matrix polynomial (2.12) is regular. In this section, we provide recovery formulas for the (left and right) eigenvectors of $P(\lambda)$ from those of its Lagrange linearizations.

Theorem 4.8 provides explicit formulas for the eigenvectors of the Lagrange colleague pencil.

THEOREM 4.8. *Let $P(\lambda)$ be a regular matrix polynomial expressed in the modified Lagrange basis associated with nodes $\{x_1, \ldots, x_{k+1}\}$. Let λ_0 be a finite eigenvalue of $P(\lambda)$. Let $L_P^{\mu}(\lambda)$ be the Lagrange colleague pencil associated with μ given in (4.39). Then, z (resp., w) is a right (resp., left) eigenvector of $L_P^{\mu}(\lambda)$ associated with λ_0 if and only if $z = H_L^{\mu}(\lambda_0)x$ (resp., $G_L^{\mu}(\lambda_0)^T y$), where x (resp., y) is a right (resp., left) eigenvector of $P(\lambda)$ associated with λ_0.*

Proof. The eigenvector formulas follow from Theorems 2.15 and 4.7. ◻

Theorem 4.9 provides recovery formulas of eigenvectors (associated with finite and infinite eigenvalues) of the matrix polynomial $P(\lambda)$ from those of its Lagrange linearizations. We note that, in this theorem, we only consider finite eigenvalues λ that are not an interpolation node, which is the most likely case in applications, since when λ is a node, many sub-cases need to be considered and make the theorem difficult to read. In any case, in Remark 4.10, all those sub-cases are presented for completion.

THEOREM 4.9. (Recovery of eigenvectors from Lagrange linearizations) *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial expressed in the modified Lagrange basis as in (4.34), and let λ_0 be an eigenvalue (finite or infinite) of $P(\lambda)$. Let $L(\lambda)$ be a Lagrange linearization of $P(\lambda)$ as in (4.41). Let z and ω be, respectively, a right and a left eigenvector of $L(\lambda)$ associated with λ_0.*

1. *Assume λ_0 is finite and $\lambda_0 \notin \{x_1, x_2, \ldots, x_{k+1}\}$. Then,*
   - *the block-entries $z(1), z(2), \ldots, z(k-\mu)$ are right eigenvectors of $P(\lambda)$ associated with λ_0, and*
   - *the block-entries $\omega(1), \omega(2), \ldots, \omega(\mu+1)$ are left eigenvectors of $P(\lambda)$ associated with λ_0.*
2. *Assume λ_0 is infinite. Then,*
   - *the block entries $z(1), z(2), \ldots, z(k-\mu)$ are right eigenvectors of $P(\lambda)$ associated with λ_0, and*
   - *the block-entries $\omega(1), \omega(2), \ldots, \omega(\mu+1)$ are left eigenvectors of $P(\lambda)$ associated with λ_0.*

Proof. We prove the result for the right eigenvectors. The proof is similar for the left eigenvectors.

We show first that the theorem holds for the Lagrange colleague pencil $L_P^{\mu}(\lambda)$.

Case I: Assume that λ_0 is a finite eigenvalue such that $\lambda_0 \notin \{x_1, x_2, \ldots, x_{k+1}\}$, and let z be a right eigenvector of the Lagrange colleague pencil $L_P^{\mu}(\lambda)$ associated with λ_0. By Theorem 4.8, we have $z = H_L^{\mu}(\lambda_0)x$ for some right eigenvector x of $P(\lambda)$ with eigenvalue λ_0. Then, it is clear that the top $k-\mu$ block entries of z are all nonzero multiples of the eigenvector x.

Case II: Assume that λ_0 is an infinite eigenvalue of $P(\lambda)$. This means that zero is an eigenvalue of $\mathrm{rev}_k P(\lambda)$ and $\mathrm{rev}_1 L_P^{\mu}(\lambda)$. By Lemma 2.3, we have

$$\mathrm{rev}_k P(\lambda) = \sum_{i=1}^{k+1} P_i \, \mathrm{rev}_k \ell_i(\lambda) = \sum_{i=1}^{k+1} P_i \, \widetilde{\ell}_i(\lambda),$$

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where $\widetilde{\ell}_i(\lambda) = w_i \prod_{j=1,\, j\neq i}^{k}(1 - x_j\lambda)$. Thus, $\mathrm{rev}_k P(0) = \sum_{i=1}^{k+1} w_i P_i$. Moreover, we also have $\mathrm{rev}_1 L_P^{\mu}(0) =$

$$\left[\begin{array}{ccccc|cccc}
P_{k+1}w_{k+1} + P_k w_k & P_{k-1}w_{k-1} & P_{k-2}w_{k-3} & \cdots & P_{\mu+1}w_{\mu+1} & I_n & 0 & \cdots & 0 \\
 & & & & P_{\mu}w_{\mu} & -I_n & I_n & \ddots & \vdots \\
 & & & & \vdots & 0 & \ddots & \ddots & 0 \\
 & & & & P_2 w_2 & \vdots & \ddots & -I_n & I_n \\
 & & & & P_1 w_1 & 0 & \cdots & 0 & -I_n \\
\hline
I_n & -I_n & 0 & \cdots & 0 & & & & \\
0 & I_n & -I_n & \ddots & \vdots & & & & \\
\vdots & \ddots & \ddots & \ddots & 0 & & & & \\
0 & \cdots & 0 & I_n & -I_n & & & &
\end{array}\right].$$

From the structure of the matrix $\mathrm{rev}_1 L_P^{\mu}(0)$, it follows that any right eigenvector of $\mathrm{rev}_1 L_P^{\mu}(\lambda)$ with eigenvalue zero must be of the form

$$z = \begin{bmatrix} x & \cdots & x & -\sum_{i=\mu+1}^{k+1} P_i w_i x & -\sum_{i=\mu}^{k+1} P_i w_i x & \cdots & -\sum_{i=2}^{k+1} P_i w_i x \end{bmatrix}^{\mathcal{B}}$$

for some eigenvector x of $\mathrm{rev}_k P(\lambda)$ with eigenvalue zero. Hence, we can recover x from any of the top $k-\mu$ block-entries of z.

The results for the Lagrange linearization $L(\lambda)$ in (4.41) follow from the results for the Lagrange colleague pencil $L_P^{\mu}(\lambda)$ and Remark (4.5). ☐

REMARK 4.10. In the unlikely case that $\lambda_0 \in \{x_1, x_2, \ldots, x_{k+1}\}$, right and left eigenvectors of $P(\lambda)$ can still be recovered from the eigenvectors of a Lagrange linearization. With the notation used in Theorem 4.9, we have:

- If $\lambda_0 = x_1$ (resp., $\lambda_0 = x_{\mu+2}$), then $z(1), \ldots, z(k-\mu)$ (resp., $z(k-\mu-1)$ and $z(k-\mu)$) are right eigenvectors of $P(\lambda)$ associated with λ_0, and $\omega(\mu+1)$ (resp., $\omega(1)$) is a left eigenvector of $P(\lambda)$ associated with λ_0.
- If $\lambda_0 = x_j \in \{x_2, \ldots, x_\mu\}$, then $z(1), \ldots, z(k-\mu)$ are right eigenvectors of $P(\lambda)$ associated with λ_0, and $\omega(\mu-j+2)$ and $\omega(\mu-j+3)$ are left eigenvectors of $P(\lambda)$ associated with λ_0.
- If $\lambda_0 = x_{\mu+1}$ (resp., $\lambda_0 = x_{k+1}$), then $z(k-\mu)$ (resp., $z(1)$) is a right eigenvector of $P(\lambda)$ associated with λ_0, and $\omega(1)$ and $w(2)$ (resp., $w(1), \ldots w(\mu+1)$) are left eigenvectors of $P(\lambda)$ associated with λ_0.
- If $\lambda_0 = x_j \in \{x_{\mu+3}, \ldots, x_k\}$, then $z(k-j+1)$ and $z(k-j+2)$ are right eigenvectors of $P(\lambda)$ associated with λ_0, and $\omega(1), \ldots, \omega(\mu+1)$ are left eigenvectors of $P(\lambda)$ associated with λ_0.

4.2. Recovery of minimal bases and minimal indices from Lagrange linearizations. Assume the matrix polynomial $P(\lambda)$ in (4.34) is singular. In this section, we show how to recover the minimal indices and minimal bases of $P(\lambda)$ from those of its Lagrange linearizations.

THEOREM 4.11. (Recovery of minimal bases and minimal indices from Lagrange linearizations) *Let $P(\lambda)$ be a singular matrix polynomial expressed in the modified Lagrange basis as in (4.34). Let $L(\lambda)$ be a Lagrange linearization of $P(\lambda)$ with parameter μ as in (4.41). Let $a_{\mu+1}, \ldots, a_1$ and $b_k, \ldots, b_{\mu+1}$ be, respectively, the μ-1- and μ-2-coordinates of 1.*

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(a1) *Suppose that* $\{z_1(\lambda), z_2(\lambda), \ldots, z_p(\lambda)\}$ *is a minimal basis for the right nullspace of* $L(\lambda)$*, with vector polynomials* z_i *partitioned into blocks conformable with the blocks of* $L(\lambda)$*. Let*

$$x_i(\lambda) = \begin{bmatrix} b_k I_n & \cdots & b_{\mu+1} I_n & 0 & \cdots & 0 \end{bmatrix} z_i(\lambda) \quad (i = 1, \ldots, p).$$

Then, $\{x_1(\lambda), x_2(\lambda), \ldots, x_p(\lambda)\}$ *is a minimal basis for the right nullspace of* $P(\lambda)$*.*

(a2) *If* $0 \leq \epsilon_1 \leq \cdots \leq \epsilon_p$ *are the right minimal indices of* $L(\lambda)$*, then*

$$0 \leq \epsilon_1 - k + \mu + 1 \leq \cdots \leq \epsilon_p - k + \mu + 1$$

are the right minimal indices of $P(\lambda)$*.*

(b1) *Suppose that* $\{w_1(\lambda), w_2(\lambda), \ldots, w_q(\lambda)\}$ *is a minimal basis for the left nullspace of* $L(\lambda)$*, with vector polynomials* w_i *partitioned into blocks conformable with the blocks of* $L(\lambda)$*. Let*

$$y_i(\lambda) = \begin{bmatrix} a_{\mu+1} I_m & \cdots & a_1 I_m & 0 & \cdots & 0 \end{bmatrix} w_i(\lambda) \quad (i = 1, \ldots, q).$$

Then $\{y_1(\lambda), y_2(\lambda), \ldots, y_q(\lambda)\}$ *is a minimal basis for the left nullspace of* $P(\lambda)$*.*

(b2) *If* $0 \leq \mu_1 \leq \cdots \leq \mu_q$ *are the left minimal indices of* $L(\lambda)$*, then*

$$0 \leq \mu_1 - \mu \leq \cdots \leq \mu_q - \mu$$

are the left minimal indices of $P(\lambda)$*.*

Proof. We prove the result for the right minimal indices and bases. The results for the left minimal indices and bases can be proven similarly.

Let $B(\lambda)$ be a matrix whose columns form a basis for the right nullspace of $P(\lambda)$. From Theorems 2.15 and 4.7, we have that the columns of $H_L^{\mu}(\lambda)B(\lambda)$ form a basis for the right nullspace of the Lagrange colleague pencil $L_P^{\mu}(\lambda)$ in (4.39). From the definition of the μ-1-coordinates of 1, we have

$$\begin{bmatrix} b_k I_n & \cdots & b_{\mu+1} I_n & 0 & \cdots & 0 \end{bmatrix} H_L^{\mu}(\lambda) B(\lambda) = B(\lambda).$$

Hence, part (a1) holds for the Lagrange colleague pencil. Part (a2) follows also from Theorems 2.15 and 4.7, together with the fact $\deg(D_1^L(\lambda)) = k - \mu - 1$, in the case that $L(\lambda)$ is the Lagrange colleague pencil. When $L(\lambda)$ is a Lagrange linearization other than the Lagrange colleague pencil, parts (a1) and (a2) follow from Remark (4.5), together with parts (a1) and (a2) applied to the Lagrange colleague pencil. ∎

5. Strong linearizations for matrix polynomials in the Chebyshev basis. We finish the paper with the Chebyshev bases. Some of the information that we include here can be found in [20], where an infinite family of block minimal basis linearizations of a matrix polynomial expressed in either the Chebyshev basis of the first kind or the second kind is presented.

In order to write the results in a more compact way, we use a nonstandard notation to represent the Chebyshev polynomials. We denote by $\phi_n^{(1)}(\lambda)$ (resp., $\phi_n^{(2)}(\lambda)$) the nth Chebyshev polynomial of the first kind (resp., of the second kind). Our goal is, then, to construct strong linearizations for matrix polynomials of the form

$$P(\lambda) = \sum_{i=0}^{k} P_i \, \phi_i^{(r)}(\lambda), \quad P_0, \ldots, P_k \in \mathbb{C}^{n \times n}, \quad r \in \{1, 2\}. \tag{5.47}$$

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Let $0 \le \epsilon \le k-1$ be an integer, and let n and m be positive integers. We define the matrix pencils

$$K_1^{(C,i)}(\lambda) = \begin{bmatrix} I_n & -2\lambda I_n & I_n & & & \\ & I_n & -2\lambda I_n & I_n & & \\ & & \ddots & \ddots & \ddots & \\ & & & I_n & -2\lambda I_n & I_n \\ & & & & I_n & -\phi_1^{(i)}(\lambda) I_n \end{bmatrix}_{\epsilon n \times (\epsilon+1)n}, \tag{5.48}$$

$$K_2^{(C,j)}(\lambda) = \begin{bmatrix} I_n & -2\lambda I_n & I_n & & & \\ & I_n & -2\lambda I_n & I_n & & \\ & & \ddots & \ddots & \ddots & \\ & & & I_n & -2\lambda I_n & I_n \\ & & & & I_n & -\phi_1^{(j)}(\lambda) I_n \end{bmatrix}_{(k-1-\epsilon)m \times (k-\epsilon)m}, \tag{5.49}$$

where $i, j \in \{1, 2\}$.

Lemma 5.1. *Let $0 \le \epsilon \le k-1$ be an integer, and let $i, j \in \{1,2\}$. The matrix pencils $K_1^{(C,i)}(\lambda)$ and $K_2^{(C,j)}(\lambda)$ given in (5.48) and (5.49) are both minimal bases. Moreover, the matrix polynomials*

$$D_1^{(C,i)}(\lambda)^T = \begin{bmatrix} \phi_\epsilon^{(i)}(\lambda) I_n \\ \vdots \\ \phi_1^{(i)}(\lambda) I_n \\ \phi_0^{(i)}(\lambda) I_n \end{bmatrix} \quad \text{and} \quad D_2^{(C,j)}(\lambda)^T = \begin{bmatrix} \phi_{k-1-\epsilon}^{(j)}(\lambda) I_m \\ \vdots \\ \phi_1^{(j)}(\lambda) I_m \\ \phi_0^{(j)}(\lambda) I_m \end{bmatrix} \tag{5.50}$$

are, respectively, dual minimal bases of $K_1^{(C,i)}(\lambda)$ and $K_2^{(C,j)}(\lambda)$.

Proof. The minimality of the four matrix polynomials follows readily from the characterization of minimal bases in Theorem 2.7. Moreover, by using the recurrence relationship of Chebyshev polynomials (2.16), one can establish the duality by direct matrix multiplication. ∎

We now consider strong block minimal basis pencils of the form

$$C(\lambda) = \begin{bmatrix} M(\lambda) & K_2^{(C,j)}(\lambda) \\ K_1^{(C,i)}(\lambda) & 0 \end{bmatrix} \tag{5.51}$$

We will refer to (5.51) as a *Chebyshev pencil*. The following theorem shows how to choose the body $M(\lambda)$ so that the Chebyshev pencil (5.51) is a strong linearization of the matrix polynomial (5.47).

Theorem 5.2. *Let $P(\lambda) = \sum_{i=0}^{k} P_i \, \phi_i^{(r)}(\lambda)$, where $r \in \{1, 2\}$, be an $m \times n$ matrix polynomial expressed in a Chebyshev basis. Let $0 \le \epsilon \le k-1$ be an integer, and let*

$$M_\epsilon^C(\lambda) := \begin{bmatrix} 2\lambda P_k + P_{k-1} & -P_k & 0 & \cdots & \cdots & 0 \\ P_{k-2} - P_k & -P_{k-1} & \vdots & \ddots & & \vdots \\ P_{k-3} & \vdots & \vdots & & \ddots & \vdots \\ \vdots & -P_{\varepsilon+2} & 0 & \cdots & \cdots & 0 \\ P_\varepsilon & P_{\varepsilon-1} - P_{\varepsilon+1} & P_{\varepsilon-2} & P_{\varepsilon-3} & \cdots & P_0 \end{bmatrix},$$

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when $1 \leq \epsilon \leq k-2$*;*

$$M_\epsilon^C(\lambda) := \begin{bmatrix} 2\lambda P_k + P_{k-1} & P_{k-2} - P_k & P_{k-3} & \cdots & P_1 & P_0 \end{bmatrix},$$

when $\epsilon = k-1$*;*

$$M_\epsilon^C(\lambda) := \frac{1}{2}\begin{bmatrix} 2\lambda P_k^T + P_{k-1}^T & P_{k-2}^T - 2P_k^T & P_{k-3}^T - P_{k-1}^T & \cdots & P_1^T - P_3^T & 2P_0^T - P_2^T \end{bmatrix}^T,$$

when $\epsilon = 0$ *and* $r = 1$*; and*

$$M_\epsilon^C(\lambda) := \begin{bmatrix} \lambda P_k^T + P_{k-1}^T & P_{k-2}^T - P_k^T & P_{k-3}^T & \cdots & P_1^T & P_0^T \end{bmatrix}^T,$$

when $\epsilon = 0$ *and* $r = 2$.

(a) *If* $P(\lambda)$ *is expressed in the Chebyshev basis of the first kind, then the Chebyshev pencil*

$$C_P^\epsilon(\lambda) = \begin{bmatrix} M_\epsilon^C(\lambda) & K_2^{(C,2)}(\lambda)^T \\ K_1^{(C,1)}(\lambda) & 0 \end{bmatrix} \tag{5.52}$$

is a strong linearization of $P(\lambda)$.

(b) *If* $P(\lambda)$ *is expressed in the Chebyshev basis of the second kind, then the Chebyshev pencil*

$$C_P^\epsilon(\lambda) = \begin{bmatrix} M_\epsilon^C(\lambda) & K_2^{(C,2)}(\lambda)^T \\ K_1^{(C,2)}(\lambda) & 0 \end{bmatrix} \tag{5.53}$$

is a strong linearization of $P(\lambda)$.

We will refer to (5.52)-(5.53) *as the* colleague Chebyshev pencil *of* $P(\lambda)$ *associated with the parameter* ϵ*.*

Proof. The proof follows by using Theorem 2.12, together with Lemmas 2.2 and 5.1. ☐

Remark 5.3. In the case where the matrix polynomial $P(\lambda)$ is expressed in the Chebyshev polynomial basis of the first kind, one could consider a colleague pencil of the form

$$C_P(\lambda) = \begin{bmatrix} M(\lambda) & K_2^{(C,1)}(\lambda)^T \\ K_1^{(C,2)}(\lambda) & 0 \end{bmatrix}.$$

The construction of linearizations of this form is very similar to the case (5.52), so we do not pursue this further. One could also consider a colleague pencil of the form

$$C_P(\lambda) = \begin{bmatrix} M(\lambda) & K_2^{(C,1)}(\lambda)^T \\ K_1^{(C,1)}(\lambda) & 0 \end{bmatrix}.$$

However, when constructing linearizations of this form, some of the block entries of $M(\lambda)$ become linear combinations of a large number of matrix coefficients of $P(\lambda)$ and thus, may cause numerical problems due to cancellation errors; see, for example, [20, Remark 3.8].

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EXAMPLE 5.4. Let $P(\lambda) = \sum_{i=0}^{5} P_i\, \phi_i^{(1)}(\lambda)$ be an $m \times n$ matrix polynomial of degree 5 expressed in the Chebyshev basis of the first kind. Let $\epsilon = 3$. Then,

$$\mathcal{C}_P^{\epsilon}(\lambda) = \left[\begin{array}{cccc|c} 2\lambda P_5 + P_4 & -P_5 & 0 & 0 & I_m \\ P_3 - P_5 & P_2 - P_4 & P_1 & P_0 & -2\lambda I_m \\ \hline I_n & -2\lambda I_n & I_n & 0 & 0 \\ 0 & I_n & -2\lambda I_n & I_n & 0 \\ 0 & 0 & I_n & -\lambda I_n & 0 \end{array}\right]$$

is the colleague Chebyshev pencil of $P(\lambda)$ associated with $\epsilon = 3$.

Let $P(\lambda) = \sum_{i=0}^{5} P_i\, \phi_1^{(2)}(\lambda)$ be an $m \times n$ matrix polynomial of degree 5 expressed in the Chebyshev basis of the second kind. Let $\epsilon = 1$. Then

$$\mathcal{C}_P^{\epsilon}(\lambda) = \left[\begin{array}{cc|ccc} 2\lambda P_5 + P_4 & -P_5 & I_m & 0 & 0 \\ P_3 - P_5 & -P_4 & -2\lambda I_m & I_m & 0 \\ P_2 & -P_3 & I_m & -2\lambda I_m & I_m \\ P_1 & P_0 - P_2 & 0 & I_m & -2\lambda I_m \\ \hline I_n & -2\lambda I_n & 0 & 0 & 0 \end{array}\right]$$

is the colleague Chebyshev pencil of $P(\lambda)$ associated with $\epsilon = 1$.

REMARK 5.5. A drawback of the Chebyshev colleague linearizations of a matrix polynomial $P(\lambda)$ is that they are not companion forms since the matrix coefficient corresponding to the zero-degree term of these linearizations contains blocks which are sums of matrix coefficients of $P(\lambda)$. The Newton and Lagrange colleague linearizations are companion forms though.

An infinite family of linearizations for matrix polynomials in the Chebyshev basis (of the first kind or the second kind) can be constructed combining the colleague Chebyshev pencil and Theorem 2.13.

THEOREM 5.6. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i^{(r)}(\lambda)$, where $r \in \{1, 2\}$, be an $m \times n$ matrix polynomial expressed in a Chebyshev basis. Let $0 \le \epsilon \le k-1$ be an integer and let $M_{\epsilon}^{C}(\lambda)$ be as in Theorem 5.2. Let A and B be two arbitrary matrices of sizes $(k-\epsilon)m \times \epsilon n$ and $(k-1-\epsilon)m \times (\epsilon+1)n$, respectively.*

(a) *If $r = 1$, then the Chebyshev pencil*

$$C(\lambda) = \begin{bmatrix} M_{\epsilon}^{C}(\lambda) + AK_1^{(C,1)}(\lambda) + K_2^{(C,2)}(\lambda)^T B & K_2^{(C,2)}(\lambda)^T \\ K_1^{(C,1)}(\lambda) & 0 \end{bmatrix} \tag{5.54}$$

is a strong linearization of $P(\lambda)$.

(b) *If $r = 2$, then the Chebyshev pencil*

$$C(\lambda) = \begin{bmatrix} M_{\epsilon}^{C}(\lambda) + AK_1^{(C,2)}(\lambda) + K_2^{(C,2)}(\lambda)^T B & K_2^{(C,2)}(\lambda)^T \\ K_1^{(C,2)}(\lambda) & 0 \end{bmatrix} \tag{5.55}$$

is a strong linearization of $P(\lambda)$.

We will refer to a Chebyshev pencil of the form (5.54)-(5.55) as a Chebyshev linearization *of $P(\lambda)$ with parameter ϵ.*

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REMARK 5.7. Observe that every Chebyshev linearization (5.54)-(5.55) is strictly equivalent to the colleague pencil (5.52)-(5.53):

$$C(\lambda) = \begin{bmatrix} I_{(k-\epsilon)m} & A \\ 0 & I_{\epsilon n} \end{bmatrix} C_P^{\epsilon}(\lambda) \begin{bmatrix} I_{(\epsilon+1)n} & 0 \\ B & I_{(k-1-\epsilon)m} \end{bmatrix}. \tag{5.56}$$

In the following two sections, we obtain recovery rules for eigenvectors, and minimal bases and minimal indices of a matrix polynomial $P(\lambda)$ from those of its Chebyshev linearizations. We will need the following definitions and results.

DEFINITION 5.8. (Chebyshev-Horner shifts) Let k and $0 \leq \epsilon \leq k-1$ be integers. Given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i^{(r)}(\lambda)$ expressed in the Chebyshev basis of the rth kind, where $r \in \{1,2\}$, the *ith Chebyshev-Horner shift* of $P(\lambda)$ associated with ϵ is given by

$$P^i_{\epsilon,r}(\lambda) := P_k \phi^{(r)}_{\epsilon+i}(\lambda) + P_{k-1}\phi^{(r)}_{\epsilon+i-1}(\lambda) + \cdots + P_{k-i+1}\phi^{(r)}_{\epsilon+1}(\lambda) + P_{k-i}\phi^{(r)}_{\epsilon}(\lambda)$$

for $i = 0, 1, \ldots, k-\epsilon$. Note that $P^0_{0,r}(\lambda) = P_k$ and $P^k_{0,r}(\lambda) = P(\lambda)$ for $r = 1, 2$.

Lemma 5.9 provides a property of the Chebyshev-Horner shifts of a matrix polynomial that will be useful to prove Theorem 5.10.

LEMMA 5.9. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i^{(r)}(\lambda)$, with $r \in \{1,2\}$, be a matrix polynomial of degree k expressed in the Chebyshev basis of rth kind. Then, the ith Chebyshev Horner shift polynomial $P^i_{\epsilon,r}(\lambda)$ is a polynomial of degree $\epsilon + i$ and*

$$P^{i+1}_{\epsilon,r}(\lambda) = 2\lambda P^i_{\epsilon,r}(\lambda) - P^i_{\epsilon-1,r}(\lambda) + P_{k-i-1}\phi^{(r)}_{\epsilon}(\lambda) \quad (i = 1, \ldots, k-1).$$

Proof. From $\phi^{(r)}_j(\lambda) = 2\lambda\phi^{(r)}_{j-1}(\lambda) - \phi^{(r)}_{j-2}(\lambda)$, $r \in \{1,2\}$, we obtain $2\lambda P^i_{\epsilon,r}(\lambda) - P^i_{\epsilon-1,r}(\lambda) = P_k\phi^{(r)}_{\epsilon+i+1}(\lambda) + P_{k-1}\phi^{(r)}_{\epsilon+i}(\lambda) + \cdots + P_{k-i}\phi^{(r)}_{\epsilon+1}(\lambda)$. The result now follows from the definition of Chebyshev Horner shift of $P(\lambda)$ and the fact that the Chebyshev bases are degree-graded bases. ∎

Theorem 5.10 gives right- and left-sided factorizations of the colleague Chebyshev pencil (5.52)-(5.53).

THEOREM 5.10. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i^{(r)}(\lambda)$, where $r \in \{1,2\}$, be a matrix polynomial expressed in the Chebyshev basis of the rth kind. Let $C_P^{r,\epsilon}(\lambda)$ be the colleague Chebyshev pencil (5.52)-(5.53) of $P(\lambda)$ associated with ϵ, and let $D_1^{(C,i)}(\lambda)$ and $D_2^{(C,j)}(\lambda)$ be the minimal bases defined in (5.50).*

For $0 < \epsilon < k-1$ and $r \in \{1,2\}$, define

$$H_C^{\epsilon}(\lambda)^T = \begin{bmatrix} D_1^{(C,r)}(\lambda) & -P^1_{\epsilon,r}(\lambda)^T & -P^2_{\epsilon,r}(\lambda)^T & \cdots & -P^{k-\epsilon-1}_{\epsilon,r}(\lambda)^T \end{bmatrix}^T$$

and

$$G_C^{\epsilon}(\lambda) = \begin{bmatrix} D_2^{(C,2)}(\lambda) & -P^{k-\epsilon}_{0,2}(\lambda) & -P^{k-\epsilon+1}_{0,2}(\lambda) & \cdots & -P^{k-1}_{0,2}(\lambda) \end{bmatrix}.$$

For $\epsilon = 0$ and $r = 1$, define

$$H_C^{\epsilon}(\lambda)^T := \begin{bmatrix} I_n & \left(-P^1_{0,1}(\lambda) + \frac{P_{k-1}}{2}\right)^T & \left(-P^2_{0,1}(\lambda) + \frac{P_{k-2}}{2}\right)^T & \cdots & \left(-P^{k-1}_{0,1}(\lambda) + \frac{P_1}{2}\right)^T \end{bmatrix}^T$$

and $G_C^{\epsilon}(\lambda) := D_2^{(C,2)}(\lambda)$.

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For $\epsilon = 0$ and $r = 2$, define

$$H_C^{\epsilon}(\lambda)^T := \begin{bmatrix} I_n & -P_{0,2}^{1}(\lambda)^T & -P_{0,2}^{2}(\lambda)^T & \cdots & -P_{0,2}^{k-1}(\lambda)^T \end{bmatrix}^T$$

and $G_C^{\epsilon}(\lambda) := D_2^{(C,2)}(\lambda)$.

For $\epsilon = k-1$ and $r \in \{1,2\}$, define $H_C^{r,\epsilon}(\lambda)^{\mathcal{B}} := D_1^{(C,r)}(\lambda)$ *and*

$$G_C^{\epsilon}(\lambda) := \begin{bmatrix} I_n & -P_{0,2}^{1}(\lambda) & -P_{0,2}^{2}(\lambda) & \cdots & -P_{0,2}^{k-1}(\lambda) \end{bmatrix}.$$

Then, the following right- and left-sided factorizations hold:

$$\mathcal{C}_P^{\epsilon}(\lambda) H_C^{\epsilon}(\lambda) = e_{k-\epsilon} \otimes P(\lambda) \quad \text{and} \quad G_C^{\epsilon}(\lambda)\mathcal{C}_P^{\epsilon}(\lambda) = e_{\epsilon+1}^T \otimes P(\lambda),$$

where e_i denotes the ith column of the $k \times k$ identity matrix.

Proof. By using Lemma 5.9, the results can be easily shown using straightforward but tedious calculations. ∎

5.1. Recovery of eigenvectors from Chebyshev linearizations.

Assume that the matrix polynomial $P(\lambda)$ is regular. In this section, we show how to recover (left and right) eigenvectors of $P(\lambda)$ from those of its Chebyshev linearizations.

First, Theorem 5.11 gives a close formula for the right and left eigenvectors of the Chebyshev pencil (5.52)-(5.53) associated with its finite eigenvalues.

THEOREM 5.11. *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i^{(r)}(\lambda)$ be an $n \times n$ regular matrix polynomial expressed in the Chebyshev basis of rth kind, where $r \in \{1,2\}$. Let λ_0 be a finite eigenvalue of $P(\lambda)$. Let $\mathcal{C}_P^{\epsilon}(\lambda)$ be the Chebyshev colleague pencil of $P(\lambda)$ associated with ϵ (defined in (5.52)-(5.53)). Then, z (resp., w) is a right (resp., left) eigenvector of $\mathcal{C}_P^{\epsilon}(\lambda)$ associated with λ_0 if and only if $z = H_C^{\epsilon}(\lambda_0)x$ (resp., $w = G_C^{\epsilon}(\lambda_0)^T y$), where x (resp., y) is a right (resp., left) eigenvector of $P(\lambda)$ with eigenvalue λ_0.*

Proof. This result is an immediate consequence of Theorems 2.15 and 5.10. ∎

Theorem 5.12 shows how to recover the eigenvectors of the matrix polynomial $P(\lambda)$ from those of its Chebyshev linearizations.

THEOREM 5.12. (Recovery of eigenvectors from Chebyshev linearizations) *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i^{(r)}(\lambda)$ be an $n \times n$ regular matrix polynomial expressed in the Chebyshev basis of rth kind, where $r \in \{1,2\}$, and let λ_0 be an eigenvalue (finite or infinite) of $P(\lambda)$. Let $C(\lambda)$ be a Chebyshev linearization of $P(\lambda)$ as in* (5.54)-(5.55). *Let z and ω be, respectively, a right and a left eigenvector of $C(\lambda)$ associated with λ_0.*

1. *Assume λ_0 is finite. Then,*
   - *the block entry $z(\epsilon+1)$ is a right eigenvector of $P(\lambda)$ with eigenvalue λ_0, and*
   - *the block entry $w(k-\epsilon)$ is a left eigenvector of $P(\lambda)$ with eigenvalue λ_0.*
2. *Assume λ_0 is infinite. Then,*
   - *the block entry $z(1)$ is a right eigenvector of $P(\lambda)$ with eigenvalue at infinity, and*
   - *the block entry $w(1)$ is a left eigenvector of $P(\lambda)$ with eigenvalue at infinity.*

Proof. We prove the result for the right eigenvectors. The proof for the left eigenvectors is analogous.

We first show that the theorem holds for the Chebyshev colleague pencil $C_P^{\epsilon}(\lambda)$ defined in (5.52)-(5.53).

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Case I: Assume that λ_0 is a finite eigenvalue, and let z be a right eigenvector of the Chebyshev colleague pencil associated with λ_0. From Theorem 5.11, we obtain that $z = H_C^{\epsilon}(\lambda_0)x$ for some right eigenvector x of $P(\lambda)$ with eigenvalue λ_0. Then, the recovery rule follows from the fact that the $\epsilon+1$ block-entry of $H_C^{\epsilon}(\lambda_0)x$ is the vector x.

Case II: Assume that λ_0 is an infinite eigenvalue. Since the Chebyshev bases are degree-graded, we have that $\operatorname{rev}_k P(0) = P_k$. Hence, x is an eigenvector of $P(\lambda)$ with eigenvalue at infinity if and only if $x \neq 0$ and $P_k x = 0$. Moreover, if $0 < \epsilon < k-1$, by evaluating the reversal of the Chebyshev colleague pencil at $\lambda = 0$, we obtain

$$\operatorname{rev}_1 \mathcal{C}_P^{\epsilon}(0) = \begin{bmatrix} 2P_k & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -2I_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -2I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -2I_n \\ 0 & -2I_n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -2I_n & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -rI_n & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, every right eigenvector z of $\mathcal{C}_P^{\epsilon}(\lambda)$ with eigenvalue at infinity must be of the form $\begin{bmatrix} x & 0 & \cdots & 0 \end{bmatrix}$ for some right eigenvector x of $P(\lambda)$ with eigenvalue at infinity. A similar argument shows that this is also the case when $\epsilon = 0$ or $\epsilon = k-1$.

The recovery rules when $C(\lambda)$ is a Chebyshev linearization other than the Chebyshev colleague pencil follow from the Chebyshev colleague's recovery rules and the equivalence transformation in (5.56). ◻

5.2. Recovery of minimal bases and minimal indices from Chebyshev linearizations. Assume that the matrix polynomial $P(\lambda)$ is singular. In this section, we show how to recover the minimal indices and minimal bases of $P(\lambda)$ from those of its Chebyshev linearizations.

Theorem 5.13. (Recovery of minimal bases and minimal indices from Chebyshev linearizations, [20]) *Let $P(\lambda) = \sum_{i=0}^{k} P_i\, \phi_i^{(r)}(\lambda)$ be an $m \times n$ singular matrix polynomial expressed in the Chebyshev basis of rth kind, where $r \in \{1, 2\}$. Let $C(\lambda)$ be a Chebyshev linearization of $P(\lambda)$ with parameter ϵ as in (5.54)-(5.55).*

- (a1) *Suppose that $\{z_1(\lambda), \ldots, z_p(\lambda)\}$ is any right minimal basis of $C(\lambda)$, with vectors partitioned into blocks conformable to the blocks of $C(\lambda)$, and let $x_\ell(\lambda)$ be the $(\epsilon+1)$th block of $z_\ell(\lambda)$ for $\ell = 1, 2, \ldots, p$. Then, $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ is a right minimal basis of $P(\lambda)$.*
- (a2) *If $0 \le \epsilon_1 \le \cdots \le \epsilon_p$ are the right minimal indices of $C(\lambda)$, then*
$$0 \le \epsilon_1 - \epsilon \le \epsilon_2 - \epsilon \le \cdots \le \epsilon_p - \epsilon$$
are the right minimal indices of $P(\lambda)$.
- (b1) *Suppose that $\{w_1(\lambda), \ldots, w_q(\lambda)\}$ is any left minimal basis of $C(\lambda)$, with vectors partitioned into blocks conformable to the blocks of $C(\lambda)$, and let $y_\ell(\lambda)$ be the $(k-\epsilon)$th block of $w_\ell(\lambda)$ for $\ell = 1, 2, \ldots, q$. Then, $\{y_1(\lambda), \ldots, y_q(\lambda)\}$ is a left minimal basis of $P(\lambda)$.*
- (b2) *If $0 \le \mu_1 \le \cdots \le \mu_q$ are the left minimal indices of $C(\lambda)$, then*
$$0 \le \mu_1 - k + 1 + \epsilon \le \epsilon_2 - k + 1 + \epsilon \le \cdots \le \epsilon_p - k + 1 + \epsilon$$
are the left minimal indices of $P(\lambda)$.

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6. Conclusions. When solving a polynomial eigenvalue problem (PEP) $P(\lambda)x = 0$, the polynomial $P(\lambda)$ is sometimes expressed in a basis other than the monomial basis, for example, when it is the approximation of a nonlinear eigenvalue problem. In particular, the Chebyshev, Newton and Lagrange bases are the most commonly used. The solution of a PEP usually involves a linearization. In the literature, most of the available linearizations are constructed from the coefficients of the polynomial expressed in the monomial basis. From the numerical point of view, it is not wise to do the computations necessary to express $P(\lambda)$ in the monomial basis, when it is originally expressed in a non-monomial basis, in order to use one of the linearizations in the literature. A much better approach is to construct linearizations that can directly be constructed from the matrix coefficients of $P(\lambda)$ regardless of the basis it is expressed in. In this paper, we have constructed three families of block minimal basis pencils that are strong linearizations of $P(\lambda)$ when it is expressed in one of the three non-monomial bases mentioned above. These linearizations are easy to construct from the coefficients of $P(\lambda)$ and they include the so-called "colleague linearizations" for each type of basis used in the literature. Additionally, we have shown that it is easy to recover the eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from those of the linearizations. We notice though that not all of the families are equally convenient when solving a nonlinear eigenvalue problem $T(\lambda)x = 0$. While the Newton and Lagrange bases can be used when the domain of T is a subset of the complex numbers, the Chebyshev basis can only be used when the domain of T is a subset of the real numbers or a parametrizable curve. Moreover, the linearizations that we construct as well as the few available in the literature are companion forms in the Newton and Lagrange case while those in the Chebyshev family are not. However, the Chebyshev basis is the most commonly used basis in these applications. Our goal, in a subsequent paper, is to compare the linearizations in these three families from the numerical point of view, that is, in terms of conditioning of eigenvalues and backward errors with the objective of providing a guidance on what bases to use in each situation and, once chosen a basis, provide information about what linearization, within the family, has a better performance.

Appendix A. Proof of Theorem 2.15. Parts (e) and (f) have been proven in [8, Theorem 3.6]. Moreover, parts (b) and (d) follow from applying parts (a) and (c) to $L(\lambda)^T$ and $P(\lambda)^T$ and then taking transposes. Hence, we only need to prove parts (a) and (c).

Proof of part (a). Let λ_0 be a finite eigenvalue of $P(\lambda)$ and let $g := \dim \mathcal{N}_r(P(\lambda_0))$. Since $L(\lambda)$ is a strong linearization of $P(\lambda)$, we have that λ_0 is an eigenvalue of $L(\lambda)$ and $\dim \mathcal{N}_r(L(\lambda_0)) = g$.

Let $\{x_1, \ldots, x_g\}$ be a basis for $\mathcal{N}_r(P(\lambda_0))$, and consider the vectors

$$v_i = \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} x_i \quad (i = 1, \ldots, g).$$

We are going to prove that $\{v_1, \ldots, v_g\}$ is a basis for $\mathcal{N}_r(L(\lambda_0))$. First, we note that the vectors v_i are nonzero because $D_1(\lambda)^T$ has full column rank for any $\lambda \in \mathbb{C}$ since it is a minimal basis. Second, from the right-sided factorization, we get

$$L(\lambda_0)v_i = L(\lambda_0) \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} x_i = (v \otimes I_n)P(\lambda)x_i = 0.$$

Hence, $v_i \in \mathcal{N}_r(L(\lambda_0))$. To finish the proof, it suffices to show that the vectors v_i are linearly independent. Assume they are not independent, that is, assume there are constants c_i, not all zero, such that $c_1v_1 + \cdots +$

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$c_p v_p = 0$. Then,

$$0 = c_1 v_1 + \cdots + c_p v_p = \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} (c_1 x_1 + \cdots + c_p x_p),$$

which implies $c_1 x_1 + \cdots + c_p x_p = 0$. But this contradicts the fact that the x_i vectors are linearly independent. Thus, the vectors v_i must be independent and form a basis for $\mathcal{N}_r(L(\lambda_0))$.

Let $\{v_1, \ldots, v_g\}$ be a basis for $\mathcal{N}_r(L(\lambda_0))$. We are going to show that

$$v_i = \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} x_i \quad (i = 1, \ldots, p)$$

for some basis $\{x_1, \ldots, x_p\}$ of $\mathcal{N}_r(P(\lambda_0))$. Let $\{\widetilde{x}_1, \ldots, \widetilde{x}_p\}$ be some basis for $\mathcal{N}_r(P(\lambda_0))$. Then, we have that

$$\left\{ \widetilde{v}_1 := \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} \widetilde{x}_1, \ldots, \widetilde{v}_p := \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} \widetilde{x}_p \right\}$$

is a basis for $\mathcal{N}_r(L(\lambda_0))$, as proven above. Hence,

$$v_i = \sum_{j=1}^{p} c_j^{(i)} \widetilde{v}_i = \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} \sum_{j=1}^{p} c_j^{(i)} \widetilde{x}_i =: \begin{bmatrix} D_1(\lambda_0)^T \\ X(\lambda_0) \end{bmatrix} x_i \quad (i = 1, \ldots, p)$$

for some constants $c_j^{(i)}$. To finish the proof, it suffices to show that the vectors $x_i \in \mathcal{N}_r(P(\lambda_0))$ are linearly independent. But their independence follows easily from the fact that the v_i vectors are independent.

Proof of part (c). Since $L(\lambda)$ is a strong linearization of $P(\lambda)$, $p := \dim \mathcal{N}_r(P(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$.

Let $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ be a minimal basis of $\mathcal{N}_r(P(\lambda))$ and let $\epsilon_i := \deg x_i(\lambda)$ for $i = 1, \ldots, p$. Without loss of generality, assume $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_p$. Consider the polynomial vectors

$$v_i(\lambda) = \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda) \end{bmatrix} x_i(\lambda) \quad (i = 1, \ldots, p).$$

From the right-sided factorization, we obtain

$$L(\lambda) v_i(\lambda) = L(\lambda) \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda) \end{bmatrix} x_i(\lambda) = (v \otimes I_m) P(\lambda) x_i(\lambda) = 0.$$

Thus, $v_i(\lambda) \in \mathcal{N}_r(L(\lambda))$ for $i = 1, \ldots, p$. Furthermore, the polynomial vectors $v_i(\lambda)$ are linearly independent because the polynomial vectors $x_i(\lambda)$ are independent and $D_1(\lambda)^T$ has full column rank. Hence, according to part (e), to show that $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ is a basis for $\mathcal{N}_r(L(\lambda))$, it suffices to show that $\deg v_i(\lambda) = \epsilon_i + \deg D_1(\lambda)$ for $i = 1, \ldots, p$. This degree shifting property follows from the following argument. From $L(\lambda) v_i(\lambda) = 0$, we get

$$K_2(\lambda)^T X(\lambda) x_i(\lambda) = -M(\lambda) D_1^T(\lambda) x_i(\lambda). \tag{A.57}$$

We note that

$$\deg K_2(\lambda)^T X(\lambda) x_i(\lambda) = \deg K_2(\lambda)^T + \deg X(\lambda) x_i(\lambda) = 1 + \deg X(\lambda) x_i(\lambda),$$

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where the first equality follows from the fact that $K_2(\lambda)$ is a minimal basis. Moreover, $\deg M(\lambda)D_1^T(\lambda)x_i(\lambda) \leq 1 + \deg D_1^T(\lambda)x_i(\lambda)$. Then, by (A.57), we get $\deg X(\lambda)x_i(\lambda) \leq \deg D_1(\lambda)x_i(\lambda)$ for $i = 1, \ldots, p$. Therefore,

$$\begin{aligned}\deg v_i(\lambda) =&\deg \begin{bmatrix} D_1(\lambda)^T x_i(\lambda) \\ X(\lambda)x_i(\lambda) \end{bmatrix} = \max\{\deg D_1(\lambda)^T x_i(\lambda), \deg X(\lambda)x_i(\lambda)\} = \\ &\deg D_1(\lambda)^T x_i(\lambda) = \deg x_i(\lambda) + \deg D_1(\lambda) = \epsilon_i + \deg D_1(\lambda),\end{aligned} \tag{A.58}$$

where the fourth equality follows from the fact that $D_1(\lambda)$ is a minimal basis. This proves the claim.

Now we prove the converse. Let $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ be a minimal basis for $\mathcal{N}_r(L(\lambda))$ ordered so that $\deg v_1(\lambda) \geq \cdots \geq \deg v_p(\lambda)$. We are going to show that

$$v_i(\lambda) = \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda) \end{bmatrix} x_i(\lambda) \quad (i = 1, \ldots, p)$$

for some minimal basis $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ of $\mathcal{N}_r(P(\lambda))$. Let $\{\widetilde{x}_1(\lambda), \ldots, \widetilde{x}_p(\lambda)\}$ be some minimal basis for $\mathcal{N}_r(P(\lambda))$. Then, by the previous proof of part (c), we have that

$$\left\{\widetilde{v}_1(\lambda) := \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda) \end{bmatrix} \widetilde{x}_1(\lambda), \ldots, \widetilde{v}_p(\lambda) := \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda) \end{bmatrix} \widetilde{x}_p(\lambda)\right\}$$

is a minimal basis for $\mathcal{N}_r(L(\lambda))$. Hence,

$$v_i(\lambda) = \sum_{j=1}^{p} c_j^{(i)}(\lambda)\widetilde{v}_i(\lambda) = \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda) \end{bmatrix} \sum_{j=1}^{p} c_j^{(i)}(\lambda)\widetilde{x}_i(\lambda) =: \begin{bmatrix} D_1(\lambda)^T \\ X(\lambda) \end{bmatrix} x_i(\lambda) \quad (i = 1, \ldots, p)$$

for some (scalar) polynomials $c_j^{(i)}(\lambda)$ (see [14], Part 4 in Main Theorem). We observe that the polynomial vectors $x_i(\lambda) \in \mathcal{N}_r(P(\lambda))$ form a basis for $\mathcal{N}_r(P(\lambda))$, since they are linearly independent. Moreover, the degree-shifting property (A.58) implies $\deg v_i(\lambda) = \deg D_1(\lambda) + \deg x_i(\lambda)$, and part (e) implies $\deg v_i(\lambda) = \epsilon_i + \deg D_1(\lambda)$, where $\epsilon_1, \ldots, \epsilon_p$ are the right minimal indices of $P(\lambda)$. Hence, $\deg x_i(\lambda) = \epsilon_i$ for $i = 1, \ldots, p$. Therefore, $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ is a minimal basis for $\mathcal{N}_r(P(\lambda))$.

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