Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 583-598, November 2019.

COMPUTING KEMENY'S CONSTANT FOR BARBELL-TYPE GRAPHS*

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Abstract. In a graph theory setting, Kemeny's constant is a graph parameter which measures a weighted average of the mean first passage times in a random walk on the vertices of the graph. In one sense, Kemeny's constant is a measure of how well the graph is 'connected'. An explicit computation for this parameter is given for graphs of order n consisting of two large cliques joined by an arbitrary number of parallel paths of equal length, as well as for two cliques joined by two paths of different length. In each case, Kemeny's constant is shown to be $O(n^3)$, which is the largest possible order of Kemeny's constant for a graph on n vertices. The approach used is based on interesting techniques in spectral graph theory and includes a generalization of using twin subgraphs to find the spectrum of a graph.

Key words. Kemeny's constant, Normalized Laplacian, Markov chain, Twin subgraphs.

AMS subject classifications. 05C50, 05C81.

1. Introduction.

A *graph* G consists of a set of vertices, $V(G)$, and a set of edges between those vertices, $E(G) \subseteq \{\{u,v\} \mid u,v \in V(G)\}$. An edge connecting u and v is written $\{u,v\}$, or more often as simply uv, and we say that u and v are *adjacent*. For a vertex $v \in V(G)$, the *neighbors* of v are the vertices u for which $uv \in E(G)$, and the *degree* of v, $\deg(v)$, is the number of neighbors of v. A graph is *connected* if for any pair of vertices $u,v \in V(G)$, there is a path in G from u to v. A graph is called *simple* if there are no loops (edges of the form uu) and no multiple edges between the same pair of vertices. The *connectivity* (or *vertex-connectivity* of a graph G is the minimum number of vertices that must be removed from G so that the remaining graph is disconnected. *Edge-connectivity* is defined similarly, being the minimum number of edges that must be removed to disconnect the graph.

A *subgraph* of G is any graph obtained by the removal of vertices and edges of G (recalling that when a vertex is deleted, all edges incident with that vertex are removed too). An *induced subgraph* on a subset of vertices $V_1 \subseteq V(G)$, denoted $G[V_1]$, is the subgraph obtained by removing all other vertices of G, leaving V_1 and all edges between vertices of V_1. A *clique* in a graph is an induced subgraph which is a complete graph; every vertex is adjacent to every other vertex in the subgraph, or all possible edges are present.

Given a connected simple graph G one can consider a random walk process on the vertices and edges of G, as a means of studying the properties of the graph. For example, if a graph exhibits clustering properties, or is 'poorly-connected' in some sense, this may be established readily using Markov chain techniques. Random walks occur in many models in mathematics and physics.

Formally, given a simple connected graph $G = (V,E)$, the process of a random walk is defined locally as follows: If our walker is currently occupying a vertex $v \in V$, then in the next step, the walker moves to one of the neighbors of v, choosing each with probability $\frac{1}{\deg(v)}$. That is, the walker chooses his/her next

*Received by the editors on August 14, 2019. Accepted for publication on October 5, 2019. Handling Editor: Dario Bini. Corresponding Author: Jane Breen.
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A publication of the International Linear Algebra Society
Volume 35, pp. 583-598, November 2019.
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vertex uniformly at random among the vertices adjacent to v. This stochastic process is a Markov chain (the interested reader may refer to [7] for background reading in this area, but deep knowledge of this topic is not necessary for what follows). Since the graph is simply connected, the diagonal matrix of vertex degrees D is invertible, and the probability transition matrix of this Markov chain is easily defined as $P = D^{-1}A$, where A is the adjacency matrix of G. Note that the (i,j) entry of this matrix, $p_{i,j}$, is the probability of moving from vertex i to vertex j in a single step; in particular, $p_{i,j} = \frac{1}{\deg(v_i)}$ if $\{v_i, v_j\} \in E$, and $p_{i.j} = 0$ if there is no edge between v_i and v_j in G.

In analyzing the behavior of a random walk on a graph – or a general Markov chain – there are two things we are interested in: Average long-term behavior and average short-term behavior. The long-term behavior is studied through repeated powers of the matrix P; the i^{th} row of the matrix P^k gives the probability distribution across all vertices after k steps, given that the random walker starts on v_i. For a general Markov chain which is irreducible and primitive, each row converges to the *stationary distribution* of the chain. It is well-known (see for example, [5, Section 1.5]) that if the graph G is connected and not bipartite, then this probability distribution converges to the vector consisting of the degrees of each vertex, scaled by the volume of the graph (twice the number of edges in G). This can be interpreted to mean that in the long run, independent of which vertex the walker starts on, the probability that the random walker occupies vertex v is proportional to the degree of v. Note that this does not hold in general for directed graphs, i.e., if the associated matrix is not symmetric. The short-term behavior of a Markov chain is examined via the *mean first passage times*. The mean first passage time from vertex i to vertex j, denoted $m_{i,j}$, is the *expected* time that it takes to reach v_j, given that the random walker begins in v_i.

A parameter for random walks which has garnered recent attention is Kemeny's constant, which we denote $\mathcal{K}(G)$, and is defined (among other ways) as

$$\mathcal{K}(G) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} x_i m_{i,j} x_j,$$

where $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top$ is the stationary vector of the Markov chain, and $m_{i,j}$ denote the mean first passage times. This is a weighted average of the mean first passage times of the random walk on G and can be thought of as the expected number of steps to get from a randomly-chosen starting vertex to a randomly-chosen terminal vertex (here "weighted" and "randomly-chosen" indicate that vertices are chosen proportional to the stationary distribution; i.e., proportional to their degrees). With this definition in mind, note that a small value of Kemeny's constant would indicate that the vertices of the graph are well-connected to one another, while a large value of Kemeny's constant would indicate that the graph is poorly-connected.

As with many parameters associated with random walks, Kemeny's constant can be computed through use of the spectrum of the probability transition matrix $P = D^{-1}A$. Note that since A is real and symmetric, its eigenvalues are real; thus, the matrix P also has real eigenvalues.

Proposition 1.1. (Levene and Loizou [6]) *Let A be the adjacency matrix of a connected simple graph G. Let $\rho_{n-1} \le \cdots \le \rho_1 < \rho_0 = 1$ be the eigenvalues of the probability transition $P = D^{-1}A$, where $D = diag(d)$, $d = Ae$, $e = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top$. Then,*

$$\mathcal{K}(G) = \sum_{i=1}^{n-1} \frac{1}{1-\rho_i}.$$

A closely related matrix to the probability transition matrix is the normalized Laplacian matrix, defined

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as $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$. It is easily seen that P is diagonally similar to $D^{-1/2}AD^{-1/2}$:

$$D^{-1/2}(D^{-1/2}AD^{-1/2})D^{1/2} = D^{-1}A = P.$$

Since P is similar to $I - \mathcal{L}$, we have ρ is an eigenvalue of P if and only if $1 - \rho$ is an eigenvalue of $\mathcal{L}$. Combining this with the fact that 0 is an eigenvalue of $\mathcal{L}$ and that the coefficients of the characteristic polynomial are formed from proper combinations of sums and products of eigenvalues, we get the following.

Proposition 1.2. *Let $p_{\mathcal{L}}(x) = \cdots + c_2x^2 + c_1x$ be the characteristic polynomial of $\mathcal{L}$ for a simple connected graph G. Then,*

$$\mathcal{K}(G) = -\frac{c_2}{c_1}.$$

This gives a way to compute Kemeny's constant without having to explicitly determine the entire spectrum, namely it suffices to find the coefficients c_2 and c_1 in the characteristic polynomial. We will use this to compute Kemeny's constant for the following families of graphs.

Definition 1.3. The graph $B(k, a, b, c)$ on $ka + b + c$ vertices is formed by taking k copies of P_a (path on a vertices) and putting in a clique at both ends to "glue" the paths together; we then connect all vertices of a K_b to one set of neighbors in the graph with degree k and a K_c to the other set of neighbors in the graph with degree k.

These graphs have vertex-connectivity at most 2 (since connectivity is always bounded by the minimum degree), but to separate the two large cliques you must break each of the k paths so that in some sense as you increase k the size of the bottleneck will also increase. The graph $B(3, 6, 4, 5)$ is shown in Figure 1. We expect the value of Kemeny's constant to be large for this graph, since there are many pairs of vertices v_i, v_j of relatively high degree for which the mean first passage time $m_{i,j}$ should be large. Intuitively, a random walk beginning at a vertex in one of the cliques will on average take a long time to reach a vertex in the other clique.

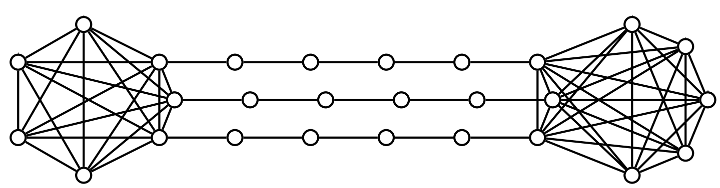

Figure 1. *The graph $B(3, 6, 4, 5)$.*

Definition 1.4. The graph $D(a, b, c, d)$ on $a + b + c + d + 2$ vertices is formed by taking a cycle on $a + b + 2$ vertices, we then identify one vertex on the cycle as a vertex in a K_{c+1}, then go a vertices in one direction and identify that as a vertex in a K_{d+1}.

The graph $D(a, b, c, d)$ is shown in Figure 2, where all the vertices in the indicated clique join at the indicated vertices (thus, forming a clique on one more vertex). These graphs are all 2-edge connected.

The graph $B(1, a, b, c)$ is sometimes referred to as the barbell graph, or the double kite, and is a well-known example of a graph which is poorly-connected. The following result shows that the barbell graph is the extremal graph for another measure of connectivity known as the relaxation time.

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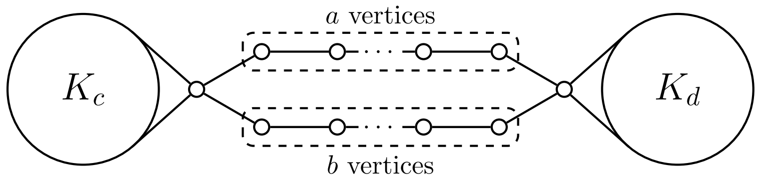

FIGURE 2. *The graph $D(a, b, c, d)$.*

THEOREM 1.5. (Aksoy et al. [1]) *If $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{n-1}$ are the eigenvalues of the normalized Laplacian matrix for $B(1, n/3, n/3, n/3)$, then $\lambda_1 = \big(1 + o(1)\big)\frac{54}{n^3}$. This is asymptotically the minimum value for λ_1 among all n-vertex graphs.*

Combining with Proposition 1.1, we have for $B(1, n/3, n/3, n/3)$ that

$$\mathcal{K}\big(B(1, n/3, n/3, n/3)\big) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \geq \frac{1}{\lambda_1} = \big(1 + o(1)\big)\frac{n^3}{54}.$$

We will show this is the correct size for $\mathcal{K}\big(B(1, n/3, n/3, n/3)\big)$, by generally establishing explicit formulas for $B(k, a, b, c)$ and $D(a, b, c, d)$. In other words, $\lambda_2, \ldots, \lambda_{n-1}$ do not make a significant contribution.

This shows that it is possible to get Kemeny's constant to be as large as order n^3. On the other hand it is known from Brightwell and Winkler [2] that the worst-case mean first passage time between a pair of vertices for any graph is at most order n^3, so the weighted average is also at most order n^3. Note that the extremal graph in that case consists of a large clique on approximately $\frac{2}{3}n$ vertices, with the remaining vertices in a path which has been attached to the clique.

The techniques we will use to approach the computation of Kemeny's constant are based in spectral graph theory. We will give a further refinement to the method of computing eigenvalues from twin structures in graphs (see Proposition 2.4), and give a complete statement of the procedure for computing the characteristic polynomial for the normalized Laplacian using cycle decompositions for weighted graphs (see Proposition 4.2). In both cases, we generalize previously-known tools [3]. The main contributions of this article are to prove and showcase these tools, and to produce classes of graphs with large (likely extremal) values of Kemeny's constant, in order to gain further understanding into the nature of Kemeny's constant and how it quantifies the 'connectedness' of a graph.

2. Tools.

For determining Kemeny's constant, we will need a few basic tools to manipulate graphs in order to find eigenvalues, or characteristic polynomial, of the normalized Laplacian. Most of these come from, or are extensions of, known results which can be found in a survey on the algebraic properties of the normalized Laplacian [3].

2.1. Weighted graphs, scaling, and harmonic eigenvectors.

While the graphs defined in Section 1 are simple graphs, we will find it convenient to switch to weighted graphs at a later point. A weighted graph is a graph where there is a weight function on the edges $w(u, v)$ which takes on any real values (including possibly negative values) satisfying $w(u, v) = w(v, u)$. There is also a weight function on the vertices $w(u)$ which takes on any real values. Simple graphs are ones in which $w(u, v) \in \{0, 1\}$ and $w(u) = 0$.

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Given a weighted graph, we define the corresponding adjacency matrix by $A_{u,v} = w(u,v)$. The diagonal degree matrix D has entries on the main diagonal which are the degrees of vertices, given by

$$D_{u,u} = d(u) = w(u) + \sum_{v} w(u,v).$$

Note that a loop is distinct from a vertex weight in that a loop will affect both A and D while a vertex weight will only affect D. With both A and D defined we have that the normalized Laplacian for a weighted graph is $\mathcal{L} = D^{-1/2}(D-A)D^{-1/2}$.

Proposition 2.1. (Butler [3]) *Let G be a weighted graph, and let αG be a new graph where* all *weights (edge and vertex) have been scaled by α. Then $\mathcal{L}_G = \mathcal{L}_{\alpha G}$. In particular, scaling all weights by a common factor does not affect the normalized Laplacian.*

The proof of the proposition follows immediately by noting that all scaling factors cancel.

Definition 2.2. Given a weighted graph G with no vertices of degree 0, and $\mathbf{x}$ an eigenvector of $\mathcal{L}$ we call $\mathbf{y} = D^{-1/2}\mathbf{x}$ the associated *harmonic eigenvector* of $\mathcal{L}$.

Starting from $\mathcal{L}\mathbf{x} = \lambda\mathbf{x}$, we can rewrite this eigenequation in terms of the harmonic eigenvector $\mathbf{y}$, giving $A\mathbf{y} = (1-\lambda)D\mathbf{y}$. This translates into the local condition that at each vertex v we have

$$\sum_{u} w(u,v)\mathbf{y}_u = (1-\lambda)d(v)\mathbf{y}_v. \tag{2.1}$$

Conversely, if we can find an appropriate vector satisfying this relationship at each vertex v of G, then we have found a harmonic eigenvector of $\mathcal{L}$ corresponding to the eigenvalue λ.

2.2. (Conjoined) Twin subgraphs. When determining the eigenvalues of a graph one tool is to look for portions of the graph with identical structure. Since harmonic eigenvectors may be determined by local conditions, these identical structures give a way to simplify the computation of the eigenvalues. These subgraphs with shared structure in a graph are called twin subgraphs.

Definition 2.3. We say G contains *twin subgraphs* $G[V_1]$ and $G[V_2]$ if the vertex set of G can be written as a disjoint union $V(G) = V_1 \dot\cup V_2 \dot\cup U$, and there is an automorphism $\pi : V(G) \to V(G)$ satisfying the following:

- π is involutory; i.e., $\pi^2 = \mathrm{id}$;
- vertices in U are fixed, $\pi(V_1) = V_2$, and $\pi(V_2) = V_1$;
- π is weight-preserving; i.e., $w\big(\pi(u)\big) = w(u)$ and $w\big(\pi(u), \pi(v)\big) = w(u,v)$.

The notion of twin subgraphs has appeared before in [3], but this is a generalization in that we allow for edges *between* the sets V_1 and V_2 (we call these *conjoined twins*). The following proposition extends Theorem 3.1 in [4] to the case where there exist edges between twin subgraphs.

Proposition 2.4. *Let G be a weighted graph with $V(G) = U \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$, where $G[V_i]$ and $G[V_j]$ are twin subgraphs for $1 \le i < j \le k$, and let π be the automorphism between the twin subgraphs $G[V_1]$ and $G[V_2]$. Then the spectrum of G can be found by the following:*

- *Let T be the graph formed by starting with $G[V_1]$ and:*
  - *for any edge $\{v_1, v_2\} \in E(G)$ with $v_1 \in V_1$ and $v_2 \in V_2$ decrease the weight of the edge in T between $\big\{v_1, \pi(v_2)\big\}$ by adding $-w(v_1, v_2)$;*

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- *change all vertex weights so that the degrees in T match the corresponding degrees of vertices in G.*

Then the spectrum of G includes the eigenvalues of T, each with multiplicity $(k-1)$.

- *Let H be the graph formed by starting with $G[U\dot{\cup}V_1]$ and:*
  - *for any edge $\{v_1, u\} \in E(G)$ with $v_1 \in V_1$ and $u \in U\dot{\cup}V_1$ increase the weight of the edge in H between $\{v_1, u\}$ by $(k-1)w(v_1, u)$;*
  - *for any edge $\{v_1, v_2\} \in E(G)$ with $v_1 \in V_1$ and $v_2 \in V_2$ increase the weight of the edge in H between $\big\{v_1, \pi(v_2)\big\}$ by $k(k-1)w(v_1, v_2)$.*

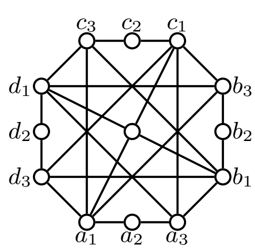
Then the spectrum of G includes the eigenvalues of H, each with multiplicity 1.

Example 2.5. Let G be the graph shown in Figure 3(a); this has a set of four conjoined twin subgraphs, $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, $\{c_1, c_2, c_3\}$, and $\{d_1, d_2, d_3\}$. The graphs T and H are shown in Figure 3(b) and 3(c) (respectively). The characteristic polynomials of the normalized Laplacian of these graphs are as follows:

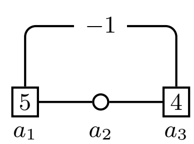
$$\begin{aligned} p_G(x) &= \left(x^3 - 3x^2 + \tfrac{109}{40}x - \tfrac{31}{40}\right)^3\left(x^4 - 4x^3 + \tfrac{41}{8}x^2 - \tfrac{21}{10}x\right), \\ p_T(x) &= x^3 - 3x^2 + \tfrac{109}{40}x - \tfrac{31}{40}, \\ p_H(x) &= x^4 - 4x^3 + \tfrac{41}{8}x^2 - \tfrac{21}{10}x. \end{aligned}$$

For H, we could scale all edge weights by $\frac{1}{4}$ before computing the characteristic polynomial.

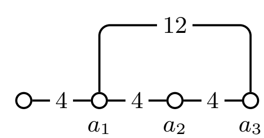
If G also had edges $\{a_1, a_3\}$, $\{b_1, b_3\}$, and $\{c_1, c_3\}$, then in T there would be no edge between vertices a_1 and a_3 (i.e., $1 + (-1) = 0$) while in H the weight of the edge between a_1 and a_3 would be 16.



(a) G



(b) T



(c) H

Figure 3. *An example of using conjoined twins to collapse a large graph into smaller pieces. The numbers inside a vertex indicate a vertex weight; any unlabeled edge corresponds to an edge with weight 1.*

Proof of Proposition 2.4. Let $\mathbf{x}$ be a harmonic eigenvector of the normalized Laplacian matrix of T associated with eigenvalue λ. Then we can construct $k-1$ harmonic eigenvectors of G, say $\widehat{\mathbf{x}}_2, \ldots, \widehat{\mathbf{x}}_k$, all associated with λ, by

$$\widehat{\mathbf{x}}_i(v) = \begin{cases} \mathbf{x}(v) & \text{if } v \in V_1, \\ -\mathbf{x}(v) & \text{if } v \in V_i, \\ 0 & \text{else.} \end{cases}$$

The verification that these are eigenvectors follows from using (2.1). (Note that the reason we need to put an edge with negative weight is because of the difference in signs in the two parts; the degree weights also guarantee that the degree term in (2.1) will match.)

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Let $\mathbf{y}$ be a harmonic eigenvector of H associated with an eigenvalue μ. Then we can construct a harmonic eigenvector of G associated with μ by

$$\widehat{\mathbf{y}}(v) = \begin{cases} \mathbf{y}\big(\pi_i(v)\big) & \text{if } v \in V_1 \dot{\cup} \cdots \dot{\cup} V_k, \\ \mathbf{y}(v) & \text{else.} \end{cases}$$

Here, π_i is an involutory map that connects the twin subgraphs $G[V_1]$ and $G[V_i]$ (with π_1 taken as the identity). Again this comes down to verification using (2.1). If a vertex in U connects to a vertex in any V_i, then the vertex connects to the corresponding vertex in all, hence has an edge weight k times as large. Locally, inside of the V_i, all terms from H have been scaled by a factor of $1/k$ which can be cancelled out in (2.1).

Finally, we note that this produces $(k-1)|V_1| + |U \dot{\cup} V_1| = |V(G)|$ harmonic eigenvectors. The set of harmonic eigenvectors from T is linearly independent and the set of harmonic eigenvectors from H is linearly independent; furthermore, the harmonic harmonic eigenvectors coming from T are orthogonal to those from H. Hence, we have a linearly independent set of $|V(G)|$ harmonic eigenvectors, and have found all eigenvalues of G. ∎

2.3. Terminal coefficients for a special tridiagonal matrix. To find Kemeny's constant we will need to recover the last two coefficients in the characteristic polynomial for the graph. We will first need the following result for a specific family of tridiagonal matrices. These matrices represent the submatrix of the normalized Laplacian corresponding to a long path in a graph, which appear in the families of graphs we wish to examine (or in the graphs T and H obtained using Proposition 2.4). While ultimately we will only need the last two coefficients to compute Kemeny's constant, in this case, we must give ourselves some cushion as we know that there will be cancellation occurring later; hence, why the result looks at the last three coefficients.

LEMMA 2.6. *Let M_n be the $n \times n$ tridiagonal matrix with*

$$M_n = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

and let $p_n(x)$ be the characteristic polynomial of M_n. Then,

$$p_n(x) = \cdots + (-\tfrac{1}{2})^{n-2}\tbinom{n+3}{5}x^2 + (-\tfrac{1}{2})^{n-1}\tbinom{n+2}{3}x + (-\tfrac{1}{2})^{n}(n+1).$$

Proof. The expression is easily verified for $n = 0, 1$, with $p_0(x) = 1$ and $p_1(x) = x - 1$. By using cofactor expansion for $\det(xI - M_n)$ along the first row, we get the recursion $p_n(x) = (x-1)p_{n-1}(x) - \frac{1}{4}p_{n-2}(x)$. We now apply this recursion to verify inductively the form of the coefficients (the base cases having already been established).

For the constant term, we have

$$-\underbrace{(-\tfrac{1}{2})^{n-1}n}_{\text{from } p_{n-1}(x)} - \tfrac{1}{4}\underbrace{(-\tfrac{1}{2})^{n-2}(n-1)}_{\text{from } p_{n-2}(x)} = (-\tfrac{1}{2})^{n}\big(2n - (n-1)\big) = (-\tfrac{1}{2})^{n}(n+1).$$

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For the first order term, we have

$$\underbrace{(-\tfrac{1}{2})^{n-1}n}_{\text{from } p_{n-1}(x)} - \underbrace{(-\tfrac{1}{2})^{n-2}\tbinom{n+1}{3}}_{\text{from } p_{n-1}(x)} - \tfrac{1}{4}\underbrace{(-\tfrac{1}{2})^{n-3}\tbinom{n}{3}}_{\text{from } p_{n-2}(x)} = (-\tfrac{1}{2})^{n-1}\big(n + 2\tbinom{n+1}{3} - \tbinom{n}{3}\big) = (-\tfrac{1}{2})^{n-1}\tbinom{n+2}{3}.$$

For the second order term, we have

$$\underbrace{(-\tfrac{1}{2})^{n-2}\tbinom{n+1}{3}}_{\text{from } p_{n-1}(x)} - \underbrace{(-\tfrac{1}{2})^{n-3}\tbinom{n+2}{5}}_{\text{from } p_{n-1}(x)} - \tfrac{1}{4}\underbrace{(-\tfrac{1}{2})^{n-4}\tbinom{n+1}{5}}_{\text{from } p_{n-2}(x)}$$
$$= (-\tfrac{1}{2})^{n-2}\big(\tbinom{n+1}{3} + 2\tbinom{n+2}{5} - \tbinom{n+1}{5}\big) = (-\tfrac{1}{2})^{n-2}\tbinom{n+3}{5}. \qquad \square$$

3. Kemeny's constant for $B(k,a,b,c)$. We are now ready for the computation of Kemeny's constant for $B(k,a,b,c)$. In what follows we will assume that $a \geq 2$.

We can apply Proposition 2.4 to three sets of twin subgraphs in $B(k,a,b,c)$; namely the individual vertices of the K_b, the individual vertices of the K_c, and the k copies of the path P_a. The spectrum (and hence, characteristic polynomial) comes from the four graphs shown in Figure 4. For convenience in our calculations we have scaled the bottom left graph by a factor of $1/k$; by Proposition 2.1 this will not affect the spectrum.

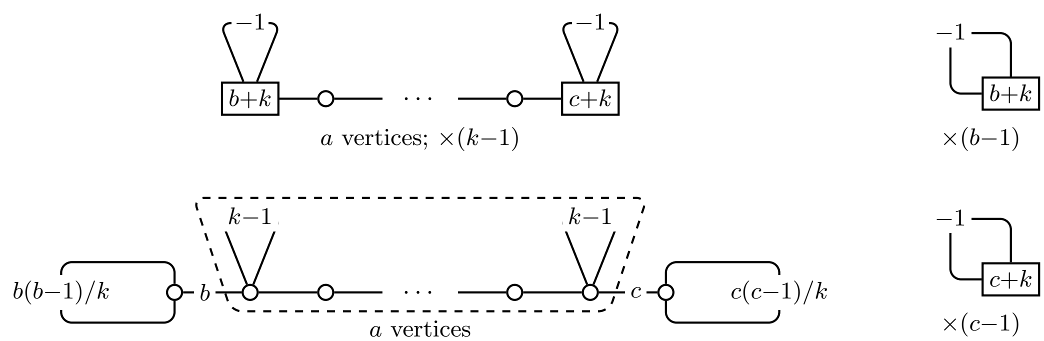

FIGURE 4. *The four graphs needed to find the characteristic polynomial for $B(k,a,b,c)$.*

The characteristic polynomials for the loops are $(x - \frac{b+k}{b+k-1})$ and $(x - \frac{c+k}{c+k-1})$, respectively. So their contribution to the characteristic polynomial (remembering we only need the last two coefficients and can pull out any common scaling factors) will be:

$$\begin{aligned}
(x - \tfrac{b+k}{b+k-1})^{b-1} &= \cdots + (b-1)(-1)^{b-2}(\tfrac{b+k}{b+k-1})^{b-2}x + (-1)^{b-1}(\tfrac{b+k}{b+k-1})^{b-1} \\
&= (-1)^{b-1}\tfrac{(b+k)^{b-2}}{(b+k-1)^{b-1}}\big(\cdots - (b-1)(b+k-1)x + (b+k)\big), \\
(x - \tfrac{c+k}{c+k-1})^{c-1} &= \cdots + (c-1)(-1)^{c-2}(\tfrac{c+k}{c+k-1})^{c-2}x + (-1)^{c-1}(\tfrac{c+k}{c+k-1})^{c-1} \\
&= (-1)^{c-1}\tfrac{(c+k)^{c-2}}{(c+k-1)^{c-1}}\big(\cdots - (c-1)(c+k-1)x + (c+k)\big).
\end{aligned}$$

Next we turn to the graph in the upper left of Figure 4 (the template for the twin paths). The characteristic polynomial for this comes from the following tridiagonal matrix (where M_{a-2} is as defined in Lemma

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2.6):

$$\left(\begin{array}{c|ccc|c} \frac{b+k+1}{b+k} & -\frac{1}{\sqrt{2(b+k)}} & 0 & \cdots & 0 \\ \hline -\frac{1}{\sqrt{2(b+k)}} & & & & 0 \\ 0 & & M_{a-2} & & \vdots \\ \vdots & & & & -\frac{1}{\sqrt{2(c+k)}} \\ \hline 0 & \cdots & 0 & -\frac{1}{\sqrt{2(c+k)}} & \frac{c+k+1}{c+k} \end{array}\right).$$

The key to computing this part of the characteristic polynomial is to observe that this is nearly the matrix given in Lemma 2.6. So we can "peel off" the corners at the ends by doing cofactor expansion and then apply the result; this can be done in four different ways. If we let $p_n(x)$ be as in the statement of Lemma 2.6, we get the following for a single copy:

$$p_{a-2}(x)(x-\tfrac{b+k+1}{b+k})(x-\tfrac{c+k+1}{c+k})+p_{a-4}(x)\tfrac{1}{2(b+k)}\tfrac{1}{2(c+k)} \\ -p_{a-3}(x)(x-\tfrac{b+k+1}{b+k})\tfrac{1}{2(c+k)}-p_{a-3}(x)(x-\tfrac{c+k+1}{c+k})\tfrac{1}{2(b+k)}.$$

Expanding this out, we get $D(\cdots+d_1x+d_0)$, where D is a common term which has been factored out and

$$\begin{aligned} d_0 &= -3abc-3abk-3ack-3ak^2+3bc+3bk+3ck+3k^2-3b-3c-6k, \\ d_1 &= a^3bc+a^3bk+a^3ck+a^3k^2-3a^2bc-3a^2bk-3a^2ck-3a^2k^2+3a^2b+3a^2c \\ &\quad +8abc+6a^2k+8abk+8ack+8ak^2-9ab-9ac-6bc-18ak-6bk-6ck \\ &\quad -6k^2+6a+9b+9c+18k-12. \end{aligned}$$

Since we want $k-1$ copies of this we note that the total contribution will be

$$\big(D(\cdots+d_1x+d_0)\big)^{k-1}=D^{k-1}d_0^{k-2}(\cdots+(k-1)d_1x+d_0).$$

Finally, we turn to the graph in the lower left of Figure 4. The characteristic polynomial for this comes from the following tridiagonal matrix (again M_{a-2} is as defined in Lemma 2.6):

$$\left(\begin{array}{cc|ccc|cc} \frac{bk}{b^2-b+bk} & -\beta & 0 & & & & \\ -\beta & \frac{b+1}{b+k} & -\frac{1}{\sqrt{2(b+k)}} & & & & \\ \hline 0 & -\frac{1}{\sqrt{2(b+k)}} & & & & & \\ & & & M_{a-2} & & & \\ & & & & & -\frac{1}{\sqrt{2(c+k)}} & \\ \hline & & & & -\frac{1}{\sqrt{2(c+k)}} & \frac{c+1}{c+k} & -\gamma \\ & & & & 0 & -\gamma & \frac{ck}{c^2-c+ck} \end{array}\right),$$

where

$$\beta=\frac{b\sqrt{k}}{\sqrt{(b+k)(b^2-b+bk)}} \quad\text{and}\quad \gamma=\frac{c\sqrt{k}}{\sqrt{(c+k)(c^2-c+ck)}}.$$

We proceed as in the previous case, only now there are more ways to peel off the corners (nine possibilities). So, we have the following:

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$$
\begin{aligned}
&p_{a-2}(x)(x-\tfrac{bk}{b^2-b+bk})(x-\tfrac{b+1}{b+k})(x-\tfrac{c+1}{c+k})(x-\tfrac{ck}{c^2-c+ck})-p_{a-2}(x)\tfrac{b^2k}{(b+k)(b^2-b+bk)}(x-\tfrac{c+1}{c+k})(x-\tfrac{ck}{c^2-c+ck})\\
&-p_{a-2}(x)(x-\tfrac{bk}{b^2-b+bk})(x-\tfrac{b+1}{b+k})\tfrac{c^2k}{(c+k)(c^2-c+k)}+p_{a-2}(x)\tfrac{b^2k}{(b+k)(b^2-b+bk)}\tfrac{c^2k}{(c+k)(c^2-c+k)}\\
&-p_{a-3}(x)(x-\tfrac{bk}{b^2-b+bk})\tfrac{1}{2(b+k)}(x-\tfrac{c+1}{c+k})(x-\tfrac{ck}{c^2-c+ck})+p_{a-3}(x)(x-\tfrac{bk}{b^2-b+bk})\tfrac{1}{2(b+k)}\tfrac{c^2k}{(c+k)(c^2-c+k)}\\
&-p_{a-3}(x)(x-\tfrac{bk}{b^2-b+bk})(x-\tfrac{b+1}{b+k})\tfrac{1}{2(c+k)}(x-\tfrac{ck}{c^2-c+ck})+p_{a-3}(x)\tfrac{b^2k}{(b+k)(b^2-b+bk)}\tfrac{1}{2(c+k)}(x-\tfrac{ck}{c^2-c+ck})\\
&+p_{a-4}(x)(x-\tfrac{bk}{b^2-b+bk})\tfrac{1}{2(b+k)}\tfrac{1}{2(c+k)}(x-\tfrac{ck}{c^2-c+ck}).
\end{aligned}
$$

Expanding this part out, we get $E(\cdots+e_2x^2+e_1x)$, where E is a common term which has been factored out and

$$
\begin{aligned}
e_1 &= -3b^2k-3c^2k-6ak^2-6bk^2-6ck^2-6k^3+3bk+3ck+12k^2,\\
e_2 &= 3ab^2c^2+3a^2b^2k+6ab^2ck+3a^2c^2k+6abc^2k+2a^3k^2+6a^2bk^2+3ab^2k^2+6a^2ck^2\\
&\quad+12abck^2+3ac^2k^2+6a^2k^3+6abk^3+6ack^3+3ak^4-3ab^2c-3abc^2-3b^2c^2\\
&\quad-3a^2bk-9ab^2k-3a^2ck-12abck-6b^2ck-9ac^2k-6bc^2k-12a^2k^2-21abk^2\\
&\quad-3b^2k^2-21ack^2-12bck^2-3c^2k^2-18ak^3-6bk^3-6ck^3-3k^4+3abc\\
&\quad+6b^2c+6bc^2+15abk+12b^2k+15ack+24bck+12c^2k+34ak^2+30bk^2+30ck^2\\
&\quad+24k^3-3b^2-9bc-3c^2-12ak-30bk-30ck-48k^2+3b+3c+24k.
\end{aligned}
$$

Combining all four of these parts together and factoring out common terms, we get the characteristic polynomial for $B(k,a,b,c)$ is $F(\cdots+f_2x^2+f_1x)$. The expression for f_1 contains 106 terms and the one for f_2 contains 347; the full expressions are given in the appendix.

We now can conclude that $\mathcal{K}\big(B(k,a,b,c)\big)=-\frac{f_2}{f_1}$. If we assume that a,b,c are all of order n (i.e., are each a positive fraction of the vertices) then most of the terms are lower order (so will not drive asymptotic growth). Taking the highest order terms in f_1 and f_2, we have

$$\mathcal{K}\big(B(k,a,b,c)\big)\approx\frac{ab^2c^2}{b^2+c^2}.$$

If we let $b=pn$ and $c=qn$ (the number of vertices in the two cliques), where $0<p,q<1$, then $a=n(1-p-q)/k$ (the number of vertices in each path), and we have established the following result.

THEOREM 3.1. *Let $0<p,q$ and $p+q<1$. Then,*

$$\mathcal{K}\big(B(k,n(1-p-q)/k,pn,qn)\big)=\frac{(1-p-q)p^2q^2}{k(p^2+q^2)}n^3+O(n^2).$$

A simple optimization of p and q shows that this is optimized when $p=q=\frac{1}{3}$. So we have

$$\mathcal{K}\big(B(k,n/3k,n/3,n/3)\big)=\frac{1}{54k}n^3+O(n^2).$$

For $k=1$, this shows the inequality established by Aksoy et al. [1] is actually an equality.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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4. Kemeny's constant for $D(a,b,c,d)$. One of the key tools in establishing the previous result is the presence of duplicate structures (twins) in the graph. This allowed us to 'collapse' portions of the graph until the remaining analysis was carried out on a set of graphs which were essentially paths. If the parallel paths connecting the two cliques had been of differing length, then we would not have been able to carry the computation out in this manner, as the paths would no longer have the twin subgraph structure.

We can still compute the last two terms of the characteristic polynomial in this case (and so also Kemeny's constant). This is done by the use of decompositions to find the characteristic polynomials.

DEFINITION 4.1. A *decomposition*, $\mathcal{D}$, of a graph G is a spanning subgraph in which the components of the subgraph consist of isolated vertices, edges between two vertices (no loops), and cycles of length at least three.

The key idea is that when expanding $\det(xI - \mathcal{L})$, interpreted as a sum over all permutations, the only nonzero terms correspond to these decompositions. This allows us to write the determinant as follows (see [3] for a similar result for simple graphs).

PROPOSITION 4.2. *For a weighted graph G without isolated vertices, the characteristic polynomial for $\mathcal{L}$ is*

$$\frac{1}{\prod_{u\in V} d(u)} \sum_{\mathcal{D}} (-1)^{Even(\mathcal{D})} 2^{Long(\mathcal{D})} \prod_{u\in U(\mathcal{D})} \big(d(u)(x-1)+w(u,u)\big) \prod_{e\in F(\mathcal{D})} \big(w(e)\big)^2 \prod_{e\in C(\mathcal{D})} w(e),$$

where the sum is taken over all decompositions of G, and where $Even(\mathcal{D})$ is the number of cycles of even length including edges, $Long(\mathcal{D})$ is the number of cycles of length at least three, $U(\mathcal{D})$ is the set of isolated vertices, $F(\mathcal{D})$ is the set of isolated edges, $C(\mathcal{D})$ is the set of edges in cycles of length at least three, and for an edge $e = \{u, v\}$ we have $w(e) = w(u, v)$.

This idea of cycle decompositions has already been implicitly used earlier when we did our cofactor expansion and then made an appeal to Lemma 2.6. In that case, we were grouping decompositions based on what was happening near the ends of the path; part of our decomposition came from the choice of what happened on the ends and the rest of the decompositions were built into the lemma.

The first step in computing Kemeny's constant for $D(a,b,c,d)$ is the same as before, namely take all the twin vertices in each of the cliques and collapse to get the three graphs shown in Figure 5 to be used to determine the characteristic polynomial.

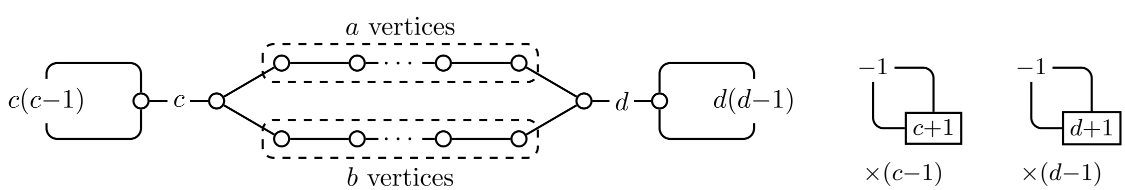

FIGURE 5. *The graph $D(a,b,c,d)$ with the twins in the cliques collapsed.*

The two smaller graphs will contribute to the characteristic polynomials as they did before:

$$(x - \tfrac{b+1}{b})^{b-1} = (-1)^{b-1} \tfrac{(b+1)^{b-2}}{b^{b-1}} \big(\cdots - (b-1)bx + (b+1)\big),$$

$$(x - \tfrac{c+1}{c})^{c-1} = (-1)^{c-1} \tfrac{(c+1)^{c-2}}{c^{c-1}} \big(\cdots - (c-1)cx + (c+1)\big).$$

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A publication of the International Linear Algebra Society
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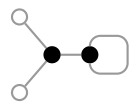
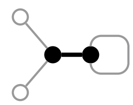
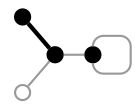
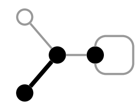
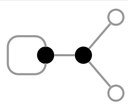
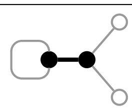
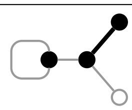
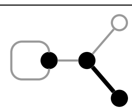
TABLE 1
Possible decompositions of the two ends involving no long cycle for $D(a,b,c,d)$. The terms in the table are the remaining graphs for which a decomposition must be found.

	 $(x-\frac{1}{d})(x-1)$	 $\frac{-1}{d+2}$	 $(x-\frac{1}{d})\frac{-1}{2(d+2)}$	 $(x-\frac{1}{d})\frac{-1}{2(d+2)}$
 $(x-\frac{1}{c})(x-1)$	$P_a \cup P_b$	$P_a \cup P_b$	$P_a \cup P_{b-1}$	$P_{a-1} \cup P_b$
 $\frac{-1}{c+2}$	$P_a \cup P_b$	$P_a \cup P_b$	$P_a \cup P_{b-1}$	$P_{a-1} \cup P_b$
 $(x-\frac{1}{c})\frac{-1}{2(c+2)}$	$P_a \cup P_{b-1}$	$P_a \cup P_{b-1}$	$P_a \cup P_{b-2}$	$P_{a-1} \cup P_{b-1}$
 $(x-\frac{1}{c})\frac{-1}{2(c+2)}$	$P_{a-1} \cup P_b$	$P_{a-1} \cup P_b$	$P_{a-1} \cup P_{b-1}$	$P_{a-2} \cup P_b$

For the larger graph we now turn to cycle decompositions. Because of the nature of the graph, there is exactly one decomposition that has a long cycle of length ≥ 3, and in that case, the remaining part of the decomposition is two isolated vertices. All of the edges involved in this decomposition have weight 1, so this decomposition will contribute

$$\begin{aligned}(4.2)\quad \frac{1}{2^{a+b}c^2d^2(c+2)(d+2)}&(-1)^{a+b+1}2\big(c^2(x-1)+(c^2-c)\big)\big(d^2(x-1)+(d^2-d)\big)\\ &= \frac{(-1)^{a+b+1}(x-\frac{1}{c})(x-\frac{1}{d})}{2^{a+b-1}(c+2)(d+2)}.\end{aligned}$$

The remaining decompositions consist of disjoint unions of edges and vertices. For these we group based on what is happening around the two vertices that correspond to the cliques. In particular there are four possibilities at each end, giving sixteen possible combinations. For each combination, the remaining part for which we need to find decompositions will be two paths as given in Table 1. In this table we have marked the contribution from the decomposition involving the vertices that come from the two cliques. The remaining portion of the decomposition coming from the paths can be found by directly applying Lemma 2.6 (that is, in each case, the remaining part of the decompositions combine together to give the characteristic polynomial of a matrix with 1 on the diagonal and $-\frac{1}{2}$ on the super and sub-diagonal).

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A publication of the International Linear Algebra Society
Volume 35, pp. 583-598, November 2019.
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So, summing the single term from the long cycle (4.2) together with all sixteen terms coming from Table 1 (each entry multiplicatively combining the term corresponding to the row, to the column, and to the cell), we get the tail of the characteristic polynomial for the large component in Figure 5 will be $R(\cdots + r_2x^2 + r_1x)$, where

$$\begin{aligned} r_1 &= -3ac^2 - 3bc^2 - 3ad^2 - 3bd^2 - 6a^2 - 12ab - 6b^2 - 3ac - 3bc - 6c^2 \\ &\quad - 3ad - 3bd - 6d^2 - 24a - 24b - 6c - 6d - 24, \\ r_2 &= 3abc^2d^2 + a^3c^2 + 3a^2bc^2 + 3ab^2c^2 + b^3c^2 + 3abc^2d + a^3d^2 + 3a^2bd^2 + 3ab^2d^2 \\ &\quad + b^3d^2 + 3abcd^2 + 3ac^2d^2 + 3bc^2d^2 + a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 + a^3c \\ &\quad + 3a^2bc + 3ab^2c + b^3c + 6a^2c^2 + 12abc^2 + 6b^2c^2 + a^3d + 3a^2bd + 3ab^2d \\ &\quad + b^3d + 3abcd + 6ac^2d + 6bc^2d + 6a^2d^2 + 12abd^2 + 6b^2d^2 + 6acd^2 + 6bcd^2 \\ &\quad + 3c^2d^2 + 8a^3 + 24a^2b + 24ab^2 + 8b^3 + 12a^2c + 24abc + 12b^2c + 14ac^2 \\ &\quad + 14bc^2 + 12a^2d + 24abd + 12b^2d + 9acd + 9bcd + 9c^2d + 14ad^2 + 14bd^2 \\ &\quad + 9cd^2 + 23a^2 + 46ab + 23b^2 + 35ac + 35bc + 12c^2 + 35ad + 35bd + 15cd \\ &\quad + 12d^2 + 28a + 28b + 30c + 30d + 12. \end{aligned}$$

With the tail parts of all three components determined, we can put them together to get the characteristic polynomial for $D(a, b, c, d)$ of the form $S(\cdots + s_2x^2 + s_1x)$. The expression for s_1 contains 48 terms and the one for s_2 contains 168; the full expressions are given in the appendix.

We can now conclude that $\mathcal{K}\big(D(a, b, c, d)\big) = -\frac{s_2}{s_1}$. If we assume that a, b, c, d are all of order n (have a positive fraction of the vertices) then most of the terms are lower order (so will not drive asymptotic growth). Taking the highest order terms in s_1 and s_2 we have established the following result.

Theorem 4.3. *Let a, b, c, d be $\theta(n)$. Then,*

$$\mathcal{K}\big(D(a, b, c, d)\big) = \frac{ab}{a+b} \cdot \frac{c^2d^2}{c^2+d^2} + O(n^2).$$

If we optimize this expression, then the maximum value is attained when $2a = 2b = c = d = \frac{1}{3}n$, giving a Kemeny's constant of $\frac{1}{108}n^3 + O(n^2)$.

5. Concluding remarks. The graph $B(1, a, b, c)$ is 1-connected while $B(k, a, b, c)$ is 2-connected when $k \geq 2$. One variant that can be explored with the techniques outlined here is where we put cliques in along the length of the path, such as is shown in Figure 6. These graphs will be k-connected for each choice of k as every vertex is adjacent to some vertex in any of the paths, so to disconnect the graph all paths would have to be broken.

There are again three sets of twins and so the computation of the characteristic polynomial reduces down to the four graphs shown in Figure 7.

While much of the work and bookkeeping is the same, there is a key difference. Namely, the results of Lemma 2.6 do not apply so cleanly, making it more difficult to work with the resulting expressions. To illustrate more precisely: If we worked with an $n \times n$ tridiagonal matrix which is t on the main diagonal and

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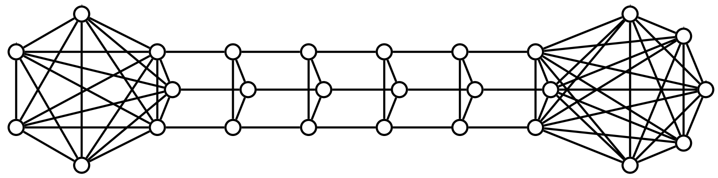

FIGURE 6. *A modification where cliques are added along the length of the path.*

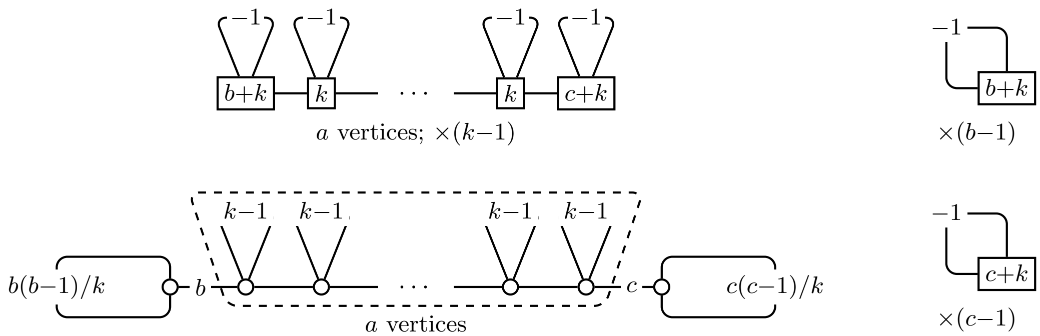

FIGURE 7. *The four graphs needed to find the characteristic polynomial for a modification of $B(k,a,b,c)$ which is k-connected.*

-1 on the super- and sub-diagonal, then the constant term of the characteristic polynomial for the matrix is

$$\begin{cases} (-1)^n(n+1) & \text{if } t=2,\\[2ex] (-\frac{1}{2})^n\dfrac{(t+\sqrt{t^2-4})^{n+1}-(t-\sqrt{t^2-4})^{n+1}}{2\sqrt{t^2-4}} & \text{if } t>2.\end{cases}$$

The next two terms (c_1 and c_2) also have complicated expressions. What we worked with is equivalent to the $t=2$ case which allowed us to find the answer with relative ease.

Acknowledgments. The research was started as part of an Iowa State Mathematics Research Team (ISMaRT) project in 2019. The authors also wish to thank two anonymous referees whose thoughtful comments improved this article.

REFERENCES

[1] S. Aksoy, F. Chung, M. Tait, and J. Tobin. The maximum relaxation time of a random walk. *Advances in Applied Mathematics*, 101:1–14, 2018.
[2] G. Brightwell and P. Winkler. Maximum hitting time for random walks on graphs. *Random Structures & Algorithms*, 1(3):263–276, 1990.
[3] S. Butler. Algebraic aspects of the normalized Laplacian. In: A. Beveridge, J. Griggs, L. Hogben, G. Musiker, and P. Tetali (editors), *Recent Trends in Combinatorics*, The IMA Volumes in Mathematics and its Applications, Springer, 295–315, 2016.
[4] S. Butler. Using twins and scaling to construct cospectral graphs for the normalized Laplacian. *Electronic Journal of Linear Algebra*, 28:54–68, 2015.
[5] F.R.K. Chung. *Spectral Graph Theory*, second edition. American Mathematical Society, 1997.
[6] M. Levene and G. Loizou. Kemeny's constant and the random surfer. *American Mathematical Monthly*, 109:741–745, 2002.
[7] E. Seneta *Non-negative Matrices and Markov chains.* Springer, New York, 2006.

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A publication of the International Linear Algebra Society
Volume 35, pp. 583-598, November 2019.
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Appendix: Full expressions for the last two coefficients (up to common scaling factor). Kemeny's constant for $B(k,a,b,c)$ is $-\dfrac{f_2}{f_1}$, where f_1 and f_2 are as follows:

$f_1 = 3ab^4c^2k + 3ab^2c^4k + 6ab^4ck^2 + 6a^2b^2c^2k^2 + 12ab^3c^2k^2 + 12ab^2c^3k^2 + 6abc^4k^2 + 3ab^4k^3 + 12a^2b^2ck^3 +$
$24ab^3ck^3 + 12a^2bc^2k^3 + 36ab^2c^2k^3 + 24abc^3k^3 + 3ac^4k^3 + 6a^2b^2k^4 + 12ab^3k^4 + 24a^2bck^4 + 48ab^2ck^4 + 6a^2c^2k^4 +$
$48abc^2k^4 + 12ac^3k^4 + 12a^2bk^5 + 21ab^2k^5 + 12a^2ck^5 + 48abck^5 + 21ac^2k^5 + 6a^2k^6 + 18abk^6 + 18ack^6 + 6ak^7 -$
$3ab^3c^2k - 3b^4c^2k - 3ab^2c^3k - 3b^2c^4k - 6ab^3ck^2 - 6b^4ck^2 - 30ab^2c^2k^2 - 12b^3c^2k^2 - 6abc^3k^2 - 12b^2c^3k^2 - 6bc^4k^2 -$
$3ab^3k^3 - 3b^4k^3 - 51ab^2ck^3 - 24b^3ck^3 - 51abc^2k^3 - 36b^2c^2k^3 - 3ac^3k^3 - 24bc^3k^3 - 3c^4k^3 - 24ab^2k^4 - 12b^3k^4 -$
$84abck^4 - 48b^2ck^4 - 24ac^2k^4 - 48bc^2k^4 - 12c^3k^4 - 39abk^5 - 21b^2k^5 - 39ack^5 - 48bck^5 - 21c^2k^5 - 18ak^6 -$
$18bk^6 - 18ck^6 - 6k^7 + 3b^4ck + 6b^3c^2k + 6b^2c^3k + 3bc^4k + 3b^4k^2 + 6ab^2ck^2 + 24b^3ck^2 + 6abc^2k^2 + 42b^2c^2k^2 +$
$24bc^3k^2 + 3c^4k^2 + 6ab^2k^3 + 18b^3k^3 + 24abck^3 + 84b^2ck^3 + 6ac^2k^3 + 84bc^2k^3 + 18c^3k^3 + 18abk^4 + 48b^2k^4 +$
$18ack^4 + 120bck^4 + 48c^2k^4 + 12ak^5 + 57bk^5 + 57ck^5 + 24k^6 - 3b^3ck - 6b^2c^2k - 3bc^3k - 3b^3k^2 - 27b^2ck^2 -$
$27bc^2k^2 - 3c^3k^2 - 21b^2k^3 - 66bck^3 - 21c^2k^3 - 42bk^4 - 42ck^4 - 24k^5,$

$f_2 = -a^3b^4c^2k^2 - a^3b^2c^4k^2 - 2a^3b^4ck^3 - 2a^4b^2c^2k^3 - 4a^3b^3c^2k^3 - 4a^3b^2c^3k^3 - 2a^3bc^4k^3 - a^3b^4k^4 - 4a^4b^2ck^4 -$
$8a^3b^3ck^4 - 4a^4bc^2k^4 - 12a^3b^2c^2k^4 - 8a^3bc^3k^4 - a^3c^4k^4 - 2a^4b^2k^5 - 4a^3b^3k^5 - 8a^4bck^5 - 16a^3b^2ck^5 - 2a^4c^2k^5 -$
$16a^3bc^2k^5 - 4a^3c^3k^5 - 4a^4bk^6 - 7a^3b^2k^6 - 4a^4ck^6 - 16a^3bck^6 - 7a^3c^2k^6 - 2a^4k^7 - 6a^3bk^7 - 6a^3ck^7 - 2a^3k^8 -$
$3a^2b^4c^4 - 2a^3b^4c^2k - 12a^2b^4c^3k - 2a^3b^2c^4k - 12a^2b^3c^4k - 4a^3b^4ck^2 - 7a^3b^3c^2k^2 - 15a^2b^4c^2k^2 - 7a^3b^2c^3k^2 -$
$48a^2b^3c^3k^2 - 4a^3bc^4k^2 - 15a^2b^2c^4k^2 - 2a^3b^4k^3 - 14a^3b^3ck^3 - 6a^2b^4ck^3 - 10a^3b^2c^2k^3 - 60a^2b^3c^2k^3 - 14a^3bc^3k^3 -$
$60a^2b^2c^3k^3 - 2a^3c^4k^3 - 6a^2bc^4k^3 - 7a^3b^3k^4 - 7a^3b^2ck^4 - 24a^2b^3ck^4 - 7a^3bc^2k^4 - 72a^2b^2c^2k^4 - 7a^3c^3k^4 -$
$24a^2bc^3k^4 - 2a^3b^2k^5 + 12a^3bck^5 - 24a^2b^2ck^5 - 2a^3c^2k^5 - 24a^2bc^2k^5 + 9a^3bk^6 + 3a^2b^2k^6 + 9a^3ck^6 + 3a^2c^2k^6 +$
$6a^3k^7 + 6a^2bk^7 + 6a^2ck^7 + 3a^2k^8 + 3a^2b^4c^3 + 3a^2b^3c^4 + 6ab^4c^4 + 2a^3b^3c^2k + 15a^2b^4c^2k - 3ab^5c^2k + 2a^3b^2c^3k +$
$24a^2b^3c^3k + 21ab^4c^3k + 15a^2b^2c^4k + 21ab^3c^4k - 3ab^2c^5k + 4a^3b^3ck^2 + 18a^2b^4ck^2 - 6ab^5ck^2 + 12a^3b^2c^2k^2 +$
$66a^2b^3c^2k^2 + 10ab^4c^2k^2 + 4a^3bc^3k^2 + 66a^2b^2c^3k^2 + 72ab^3c^3k^2 + 18a^2bc^4k^2 + 10ab^2c^4k^2 - 6abc^5k^2 + 2a^3b^3k^3 +$
$6a^2b^4k^3 - 3ab^5k^3 + 12a^3b^2ck^3 + 60a^2b^3ck^3 - 19ab^4ck^3 + 12a^3bc^2k^3 + 98a^2b^2c^2k^3 + 52ab^3c^2k^3 + 2a^3c^3k^3 +$
$60a^2bc^3k^3 + 52ab^2c^3k^3 + 6a^2c^4k^3 - 19abc^4k^3 - 3ac^5k^3 + 2a^3b^2k^4 + 15a^2b^3k^4 - 14ab^4k^4 + 40a^2b^2ck^4 - 28ab^3ck^4 +$
$2a^3c^2k^4 + 40a^2bc^2k^4 + 24ab^2c^2k^4 + 15a^2c^3k^4 - 28abc^3k^4 - 14ac^4k^4 - 8a^3bk^5 - 7a^2b^2k^5 - 29ab^3k^5 - 8a^3ck^5 -$
$40a^2bck^5 - 53ab^2ck^5 - 7a^2c^2k^5 - 53abc^2k^5 - 29ac^3k^5 - 8a^3k^6 - 38a^2bk^6 - 38ab^2k^6 - 38a^2ck^6 - 68abck^6 -$
$38ac^2k^6 - 22a^2k^7 - 30abk^7 - 30ack^7 - 10ak^8 - 3a^2b^3c^3 - 12ab^4c^3 - 12ab^3c^4 - 3b^4c^4 - 21a^2b^3c^2k - 25ab^4c^2k +$
$3b^5c^2k - 21a^2b^2c^3k - 90ab^3c^3k - 9b^4c^3k - 25ab^2c^4k - 9b^3c^4k + 3b^2c^5k + 4a^3b^2ck^2 - 30a^2b^3ck^2 - 8ab^4ck^2 +$
$6b^5ck^2 + 4a^3bc^2k^2 - 60a^2b^2c^2k^2 - 134ab^3c^2k^2 + 6b^4c^2k^2 - 30a^2bc^3k^2 - 134ab^2c^3k^2 - 24b^3c^3k^2 - 8abc^4k^2 +$
$6b^2c^4k^2 + 6bc^5k^2 + 4a^3b^2k^3 - 12a^2b^3k^3 + 5ab^4k^3 + 3b^5k^3 + 16a^3bck^3 - 30a^2b^2ck^3 - 22ab^3ck^3 + 27b^4ck^3 + 4a^3c^2k^3 -$
$30a^2bc^2k^3 - 122ab^2c^2k^3 + 12b^3c^2k^3 - 12a^2c^3k^3 - 22abc^3k^3 + 12b^2c^3k^3 + 5ac^4k^3 + 27bc^4k^3 + 3c^5k^3 + 12a^3bk^4 +$
$9a^2b^2k^4 + 34ab^3k^4 + 15b^4k^4 + 12a^3ck^4 + 75a^2bck^4 + 94ab^2ck^4 + 60b^3ck^4 + 9a^2c^2k^4 + 94abc^2k^4 + 60b^2c^2k^4 +$
$34ac^3k^4 + 60bc^3k^4 + 15c^4k^4 + 8a^3k^5 + 75a^2bk^5 + 107ab^2k^5 + 33b^3k^5 + 75a^2ck^5 + 246abck^5 + 93b^2ck^5 + 107ac^2k^5 +$
$93bc^2k^5 + 33c^3k^5 + 54a^2k^6 + 138abk^6 + 42b^2k^6 + 138ack^6 + 84bck^6 + 42c^2k^6 + 60ak^7 + 30bk^7 + 30ck^7 + 9k^8 +$
$6ab^4c^2 + 18ab^3c^3 + 9b^4c^3 + 6ab^2c^4 + 9b^3c^4 + 6ab^4ck - 3b^5ck + 12a^2b^2c^2k + 79ab^3c^2k + 6b^4c^2k + 79ab^2c^3k + 60b^3c^3k +$
$6abc^4k + 6b^2c^4k - 3bc^5k - 3b^5k^2 + 62ab^3ck^2 - 30b^4ck^2 + 150ab^2c^2k^2 + 36b^3c^2k^2 + 62abc^3k^2 + 36b^2c^3k^2 - 30bc^4k^2 -$
$3c^5k^2 - 12a^2b^2k^3 + ab^3k^3 - 27b^4k^3 - 60a^2bck^3 - 15ab^2ck^3 - 102b^3ck^3 - 12a^2c^2k^3 - 15abc^2k^3 - 72b^2c^2k^3 + ac^3k^3 -$
$102bc^3k^3 - 27c^4k^3 - 60a^2bk^4 - 92ab^2k^4 - 87b^3k^4 - 60a^2ck^4 - 294abck^4 - 258b^2ck^4 - 92ac^2k^4 - 258bc^2k^4 - 87c^3k^4 -$
$48a^2k^5 - 223abk^5 - 156b^2k^5 - 223ack^5 - 330bck^5 - 156c^2k^5 - 130ak^6 - 147bk^6 - 147ck^6 - 54k^7 - 6ab^3c^2 - 9b^4c^2 -$
$6ab^2c^3 - 21b^3c^3 - 9b^2c^4 - 6ab^3ck + 3b^4ck - 66ab^2c^2k - 69b^3c^2k - 6abc^3k - 69b^2c^3k + 3bc^4k + 9b^4k^2 + 12a^2bck^2 -$
$25ab^2ck^2 + 21b^3ck^2 - 25abc^2k^2 - 78b^2c^2k^2 + 21bc^3k^2 + 9c^4k^2 + 12a^2bk^3 + 29ab^2k^3 + 57b^3k^3 + 12a^2ck^3 + 152abck^3 +$
$177b^2ck^3 + 29ac^2k^3 + 177bc^2k^3 + 57c^3k^3 + 12a^2k^4 + 159abk^4 + 177b^2k^4 + 159ack^4 + 423bck^4 + 177c^2k^4 +$
$124ak^5 + 252bk^5 + 252ck^5 + 120k^6 + 3b^4c + 15b^3c^2 + 15b^2c^3 + 3bc^4 + 6ab^2ck + 15b^3ck + 6abc^2k + 84b^2c^2k +$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 583-598, November 2019.
http://repository.uwyo.edu/ela

$15bc^3k-9b^3k^2-24abck^2+12b^2ck^2+12bc^2k^2-9c^3k^2-42abk^3-60b^2k^3-42ack^3-186bck^3-60c^2k^3-48ak^4-177bk^4-177ck^4-120k^5-3b^3c-6b^2c^2-3bc^3-18b^2ck-18bc^2k+3b^2k^2+6bck^2+3c^2k^2+42bk^3+42ck^3+48k^4.$

Kemeny's constant for $D(a,b,c,d)$ is $-\dfrac{s_2}{s_1}$, where s_1 and s_2 are as follows:

$s_1 = -3ac^3d-3bc^3d-3acd^3-3bcd^3-3ac^3-3bc^3-6a^2cd-12abcd-6b^2cd-6ac^2d-6bc^2d-6c^3d-6acd^2-6bcd^2-3ad^3-3bd^3-6cd^3-6a^2c-12abc-6b^2c-6ac^2-6bc^2-6c^3-6a^2d-12abd-6b^2d-30acd-30bcd-12c^2d-6ad^2-6bd^2-12cd^2-6d^3-6a^2-12ab-6b^2-27ac-27bc-12c^2-27ad-27bd-36cd-12d^2-24a-24b-30c-30d-24,$

$s_2 = 3abc^3d^3+a^3c^3d+3a^2bc^3d+3ab^2c^3d+b^3c^3d+6abc^3d^2+a^3cd^3+3a^2bcd^3+3ab^2cd^3+b^3cd^3+6abc^2d^3+3ac^3d^3+3bc^3d^3+a^3c^3+3a^2bc^3+3ab^2c^3+b^3c^3+a^4cd+4a^3bcd+6a^2b^2cd+4ab^3cd+b^4cd+2a^3c^2d+6a^2bc^2d+6ab^2c^2d+2b^3c^2d+6a^2c^3d+15abc^3d+6b^2c^3d+3ac^4d+3bc^4d+2a^3cd^2+6a^2bcd^2+6ab^2cd^2+2b^3cd^2+12abc^2d^2+12ac^3d^2+12bc^3d^2+a^3d^3+3a^2bd^3+3ab^2d^3+b^3d^3+6a^2cd^3+15abcd^3+6b^2cd^3+12ac^2d^3+12bc^2d^3+3c^3d^3+3acd^4+3bcd^4+a^4c+4a^3bc+6a^2b^2c+4ab^3c+b^4c+2a^3c^2+6a^2bc^2+6ab^2c^2+2b^3c^2+6a^2c^3+12abc^3+6b^2c^3+3ac^4+3bc^4+a^4d+4a^3bd+6a^2b^2d+4ab^3d+b^4d+10a^3cd+30a^2bcd+30ab^2cd+10b^3cd+24a^2c^2d+54abc^2d+24b^2c^2d+17ac^3d+17bc^3d+6c^4d+2a^3d^2+6a^2bd^2+6ab^2d^2+2b^3d^2+24a^2cd^2+54abcd^2+24b^2cd^2+36ac^2d^2+36bc^2d^2+18c^3d^2+6a^2d^3+12abd^3+6b^2d^3+17acd^3+17bcd^3+18c^2d^3+3ad^4+3bd^4+6cd^4+a^4+4a^3b+6a^2b^2+4ab^3+b^4+9a^3c+27a^2bc+27ab^2c+9b^3c+24a^2c^2+48abc^2+24b^2c^2+14ac^3+14bc^3+6c^4+9a^3d+27a^2bd+27ab^2d+9b^3d+35a^2cd+73abcd+35b^2cd+82ac^2d+82bc^2d+15c^3d+24a^2d^2+48abd^2+24b^2d^2+82acd^2+82bcd^2+60c^2d^2+14ad^3+14bd^3+15cd^3+6d^4+8a^3+24a^2b+24ab^2+8b^3+29a^2c+58abc+29b^2c+70ac^2+70bc^2+12c^3+29a^2d+58abd+29b^2d+53acd+53bcd+78c^2d+70ad^2+70bd^2+78cd^2+12d^3+23a^2+46ab+23b^2+39ac+39bc+60c^2+39ad+39bd+27cd+60d^2+28a+28b+18c+18d+12.$