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GENERALIZED COMMUTING MAPS ON THE SET OF SINGULAR MATRICES*

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Abstract. Let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over a field $\mathbb{K}$. In the present paper, additive mappings $G : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ such that $[[G(y), y], y] = 0$ for all singular matrix y will be characterized. Precisely, it will be proved that $G(x) = \lambda x + \mu(x)$ for all $x \in M_n(\mathbb{K})$, where $\lambda \in \mathbb{K}$ and μ is a central map. As an application, the description is given of all additive maps $g : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ such that $\sum_{k_1,k_2,k_3=1}^{m} [[g(y^{k_1}), y^{k_2}], y^{k_3}] = 0$ for all singular matrices $y \in M_n(\mathbb{K})$, where $m \in \mathbb{N}^*$.

Key words. Commuting maps, Generalized commuting maps, Functional identities, Singular matrices.

AMS subject classifications. 16R60, 15A99.

1. Introduction.

Let R be a ring with center Z. A mapping $T : R \to R$ is called *centralizing* (resp., *commuting*) on a subset H of R if $[T(h), h] = T(h)h - hT(h) \in Z$ (resp., $[T(h), h] = 0$) for all $h \in H$. In [2], Brešar proved, in the case that R is a prime ring, that every additive map f which is centralizing on R has the following *standard* form:

$$f(r) = \lambda r + \mu(r) \quad \text{for all } r \in R, \tag{1.1}$$

where λ lies in the extended centroid C of R and μ is an additive map from R into C.

In [1], Brešar showed that if G is an additive map of a prime ring R of characteristic not 2 such that $[[G(r), r], r] = 0$ for all $r \in R$, then G is commuting on R, that is, G has the form (1.1).

The first results on commuting mappings on subsets of matrices that are not closed under addition have appeared in the papers [3, 4]. Basically, it was proved that if the characteristic of a field is zero or strictly greater than 3, then the only possible additive maps which are either commuting on the set of invertible matrices (resp., singular matrices) or commuting on $R_k = \{\text{matrices that have rank } k\}$ (for $k > 1$) are the standard ones.

The case $k = 1$ is exceptional. In 2013, the first author [4] provided an example of an additive map which is commuting on R_1 that does not have the standard form (1.1). Later, in 2016, Franca [9] found the description of all additive maps which are commuting on R_1. Recently, in [8], we have shown that if we replace $\mathbb{K}$ with a noncommutative division ring $\mathbb{D}$, then an additive map $G : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ that is commuting on the set of rank-1 matrices has the standard form (1.1).

To see more results related to functional identities on some subsets which are not closed under addition, we recommend the papers [5, 6, 7, 10, 11].

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2. The main result. Throughout this work $M_n(\mathbb{K})$ will denote the ring of all $n\times n$ matrices over a field $\mathbb{K}$ whose characteristic is either zero or greater than 2. We set $\Omega = \{1,\dots,n\}$. For index sets $\Omega_1,\Omega_2 \subset \Omega$, we denote by $A[\Omega_1,\Omega_2]$ the (sub)matrix of entries that lie in the rows of A indexed by Ω_1 and the columns of A indexed by Ω_2. Furthermore, we represent by $\overline{\Omega}_1$ the complement of Ω_1 in Ω.

For each $r,s \in \Omega$, we write a_{rs} to represent the (r,s)-entry of a matrix $A \in M_n(\mathbb{K})$, where $A = \sum_{r,s=1}^{n} a_{rs}E_{rs}$. In this section, we fix an additive map $G : M_n(\mathbb{K}) \to M_n(\mathbb{K})$. Since $G(\alpha E_{pq}) \in M_n(\mathbb{K})$ for each $p,q \in \Omega$, and $\alpha \in \mathbb{K}$, we can write $G(\alpha E_{pq}) = \sum_{r,s=1}^{n} a_{rs}E_{rs}$, where $a_{rs} = G(\alpha E_{pq})_{rs}$.

Now, we will state our first result:

Proposition 2.1. *Let $n \in \mathbb{N}$ $(n \geq 2)$ and $G : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ be an additive map. Consider the following elements:*

(a) $N = \beta E_{ij}$;

(b) $N = \alpha E_{ii} + \theta E_{jj} + \beta E_{ij} + \gamma E_{\xi\xi}$;

(c) $N = \beta E_{ij} + \alpha E_{j\xi}$;

(d) $N = E_{ij} + E_{j\xi} + E_{\xi i}$.

Assume that $[[G(N),N],N] = 0$ for all $i,j,\xi \in \Omega$, and for all $\alpha,\beta,\theta,\gamma \in \mathbb{K}$. Then G has the form (1.1).

The proof of the Proposition 2.1 will be divided in a series of technical lemmas in order to make it more transparent. From now on, let G be a mapping as in Proposition 2.1.

Before starting the proofs, we will make a short and relevant observation:

Remark 2.2. Notice that $[[G(N),N],N] = 0$ is equivalent to $[G(N),N]$ is in the centralizer of N.

Lemma 2.3. *$G(\beta E_{ii})$ is diagonal for each $i \in \Omega$ and $\beta \in \mathbb{K}$.*

Proof. This is clear for $\beta = 0$. Let us assume $\beta \neq 0$. Take $N = \beta E_{ii}$ (N has the form (a) for $i = j$). Remember that each matrix M in the centralizer of N has $M[\overline{\{i\}}, i] = 0$ and $M[i,\overline{\{i\}}] = 0$. Take $r,t \in \Omega$, and $s \in \overline{\{i\}}$. Recall that $[G(N),N]$ belongs to the centralizer of N, that is, $[G(N),N] = \xi_{ii}E_{ii} + \sum_{\substack{k,l=1\\ k,l\neq i}}^{n} \xi_{kl}E_{kl}$.

So,

$$\begin{aligned} E_{rs}G(N)E_{it} &= E_{rs}[G(N)E_{ii} - E_{ii}G(N)]E_{it} = E_{rs}[G(N),N]E_{it} \\ &= E_{rs}\left(\xi_{ii}E_{ii} + \sum_{\substack{k,l=1\\ k,l\neq i}}^{n} \xi_{kl}E_{kl}\right)E_{it} = \left(\sum_{\substack{l=1\\ l\neq i}}^{n} \xi_{sl}E_{rl}\right)E_{it} = 0. \end{aligned}$$

Hence,

$$E_{rs}G(N)E_{it} = 0.$$

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Therefore,

$$G(\beta E_{ii})[\overline{\{i\}},\{i\}]=0 \quad \text{for all } \ i\in\Omega.$$

By a similar argument, we see that

$$G(\beta E_{ii})[\{i\},\overline{\{i\}}]=0 \quad \text{for all } \ i\in\Omega.$$

Next, take $j\in\overline{\{i\}}$ and consider $N_1=\beta E_{ii}+\beta E_{jj}$ (N_1 has the form (b)). Remember that each matrix M in the centralizer of N_1 has $M[\overline{\{i,j\}},\{i,j\}]=0$ and $M[\{i,j\},\overline{\{i,j\}}]=0$. Since, $[G(N_1),N_1]$ lies in the centralizer of N_1, we have

$$G(\beta E_{ii}+\beta E_{jj})_{jk}=0 \quad \text{for all } \ k\in\overline{\{i,j\}}.$$

Using that the off diagonal entries of $G(\beta E_{jj})$ in the j-th row (resp., j-th column) are zero, and that G is additive, we conclude that the (j,k) entry of $G(\beta E_{ii})$ is equal to zero whenever $k\in\overline{\{i,j\}}$. Combined with the above, and allowing j vary over $\overline{\{i\}}$, we conclude that $G(\beta E_{ii})$ is a diagonal matrix. ∎

LEMMA 2.4. *Let* $i\in\Omega$ *and* $j\in\overline{\{i\}}$. *Then,* $G(\beta E_{ij})$ *is the sum of a diagonal matrix and a multiple of* E_{ij}.

Proof. This is clear if $\beta=0$. Let us take $\alpha\in\mathbb{K}\setminus\{0,\beta\}$. Set $N=\alpha E_{ii}+\alpha E_{jj}+\beta E_{ij}$ (using (b)). It can be derived directly from the equation $[[G(\beta E_{ij}),\beta E_{ij}],\beta E_{ij}]=0$ that $G(\beta E_{ij})_{ji}=0$. So, if $n=2$ we have established the claim. Otherwise, take $s,t\in\overline{\{i,j\}}$. Remember that each matrix M in the centralizer of N has $M[\overline{\{i,j\}},\{i,j\}]=0$, $M[\{i,j\},\overline{\{i,j\}}]=0$, $M_{ji}=0$ and $M_{ii}=M_{jj}$.

So,

$$[G(N),N]=a_{ij}E_{ij}+\sum_{k=1}^{n}a_{kk}E_{kk}+\sum_{k,l\in\overline{\{i,j\}}}a_{kl}E_{kl}. \tag{2.2}$$

On the other hand, taking into account that $G(\alpha E_{ii})$ and $G(\alpha E_{jj})$ are diagonal, we have

$$\begin{aligned}[G(N),N]&=[G(\alpha E_{ii}),\beta E_{ij}]+[G(\alpha E_{jj}),\beta E_{ij}]+[G(\beta E_{ij}),\alpha E_{ii}]+[G(\beta E_{ij}),\alpha E_{jj}]\\&\quad+[G(\beta E_{ij}),\beta E_{ij}]\\&=\epsilon E_{ij}+[G(\beta E_{ij}),\alpha E_{ii}]+[G(\beta E_{ij}),\alpha E_{jj}]+[G(\beta E_{ij}),\beta E_{ij}],\end{aligned}$$

for some $\epsilon\in\mathbb{K}$. After multiplying $[G(N),N]$ by E_{it} on the left, we arrive at

$$\begin{aligned}E_{it}[G(N),N]&=\alpha\left(E_{it}G(\beta E_{ij})E_{ii}\right)+\alpha\left(E_{it}G(\beta E_{ij})E_{jj}\right)+\beta\left(E_{it}G(\beta E_{ij})E_{ij}\right)\\&=\alpha\left(G(\beta E_{ij})_{ti}E_{ii}+G(\beta E_{ij})_{tj}E_{ij}\right)+\beta G(\beta E_{ij})_{ti}E_{ij}\\&=E_{it}[G(N),N]\overset{(2.2)}{=}a_{tt}E_{it}+\sum_{l\in\overline{\{i,j\}}}a_{tl}E_{il}.\end{aligned}$$

Then, after comparing the two last lines in the above equality, we conclude that

$$G(\beta E_{ij})_{ti}=G(\beta E_{ij})_{tj}=0 \quad \text{for all } \ t\in\overline{\{i,j\}}.$$

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On the other hand, after multiplying $[G(N), N]$ by E_{si} on the right and proceeeding similarly as we did before, we see that

$$\begin{aligned}[G(N), N]E_{si} &= -\alpha(E_{ii}G(\beta E_{ij})E_{si} + E_{jj}G(\beta E_{ij})E_{si}) - \beta(E_{ij}G(\beta E_{ij})E_{si}) \\ &= -\alpha(G(\beta E_{ij})_{is}E_{ii} + G(\beta E_{ij})_{js}E_{ji}) - \beta G(\beta E_{ij})_{js}E_{ii} \\ &\overset{(2.2)}{=} a_{ss}E_{si} + \sum_{k\in\overline{\{i,j\}}} a_{ks}E_{ki}.\end{aligned}$$

So,

$$G(\beta E_{ij})_{is} = G(\beta E_{ij})_{js} = 0 \quad \text{for all } \ s \in \overline{\{i,j\}}.$$

Therefore,

$$G(\beta E_{ij})[\{i,j\}, \overline{\{i,j\}}] = 0, \quad G(\beta E_{ij})[\overline{\{i,j\}}, \{i,j\}] = 0, \quad \text{and } \ G(\beta E_{ij})_{ji} = 0. \tag{2.3}$$

Now, take $\xi \in \overline{\{i,j\}}$, $\gamma \in \mathbb{K} \setminus \{\alpha, 0\}$ and consider $N_1 = \alpha E_{ii} + \alpha E_{jj} + \beta E_{ij} + +\gamma E_{\xi\xi}$. The additivity of G combined with (2.3) and the previous lemma allow us to conclude that $G(N_1)[\{i,j\}, \overline{\{i,j\}}] = 0$, $G(N_1)[\overline{\{i,j\}}, \{i,j\}] = 0$, and $G(N_1)_{ji} = 0$. In particular, $G(N_1)_{i,\xi} = G(N_1)_{j,\xi} = G(N_1)_{\xi,i} = G(N_1)_{\xi,j} = 0$. Now, we will show that $G(N_1)_{k\xi} = G(N_1)_{\xi l} = 0$ for all $k, l \in \overline{\{i,j,\xi\}}$. Indeed, note that $G(N_1)$ can be written as the following:

$$G(N_1) = \sum_{k=1}^{n} a_{kk}E_{kk} + a_{ij}E_{ij} + \sum_{k,l\in\overline{\{i,j\}}} a_{kl}E_{kl}.$$

Then,

$$\begin{aligned}[G(N_1), N_1] &= [G(N_1), \alpha E_{ii}] + [G(N_1), \alpha E_{jj}] + [G(N_1), \beta E_{ij}] + [G(N_1), \gamma E_{\xi\xi}] \\ &= [a_{ij}E_{ij}, \alpha E_{ii}] + [a_{ij}E_{ij}, \alpha E_{jj}] + \sum_{k=1}^{n}[a_{kk}E_{kk}, \beta E_{ij}] \\ &\quad + \sum_{k,l\in\overline{\{i,j\}}}[a_{kl}E_{kl}, \gamma E_{\xi\xi}] \\ &= -\alpha a_{ij}E_{ij} + \alpha a_{ij}E_{ij} + \beta(a_{ii} - a_{jj})E_{ij} + \sum_{k,l\in\overline{\{i,j\}}}[a_{kl}E_{kl}, \gamma E_{\xi\xi}] \\ &= \beta(a_{ii} - a_{jj})E_{ij} + \gamma\left(\sum_{k\in\overline{\{i,j\}}} a_{k\xi}E_{k\xi} - \sum_{l\in\overline{\{i,j\}}} a_{\xi l}E_{\xi l}\right).\end{aligned}$$

Remember that $[G(N_1), N_1]\left[\{\xi\}, \overline{\{\xi\}}\right] = 0$ and $[G(N_1), N_1]\left[\overline{\{\xi\}}, \{\xi\}\right] = 0$, since $[G(N_1), N_1]$ belongs to the centralizer of N_1. Then, $G(N_1)_{k\xi} = G(N_1)_{\xi l} = 0$ for all $k, l \in \overline{\{i,j,\xi\}}$.

Hence, all off-diagonal entries in row or column ξ of $G(N_1)$ are equal to zero. As $G(\alpha E_{ii})$, $G(\alpha E_{jj})$ and $G(\gamma E_{\xi\xi})$ are diagonal and G is additive, we conclude that all off-diagonal entries in row or column ξ of $G(\beta E_{ij})$ are zero. Now, letting ξ vary over $\overline{\{i,j\}}$ the claim is established. ∎

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LEMMA 2.5. *Let $i \in \Omega$ and $j \in \overline{\{i\}}$. Then, there is a field element λ_{ij} not depending on β such that $G(\beta E_{ij})$ is the sum of a scalar matrix and $\lambda_{ij}\beta E_{ij}$. Besides $\lambda_{ij} = \lambda_{ji}$.*

Proof. Let $\alpha, \beta \in \mathbb{K}^*$ and consider $N = \beta E_{ij} + \alpha E_{ji}$ (using (c) with $\xi = i$). By the additivity of G and the previous lemma, we conclude that

$$G(N) = G(\beta E_{ij}) + G(\alpha E_{ji}) = \sum_{k=1}^{n} a_{kk}E_{kk} + a_{ij}E_{ij} + a_{ji}E_{ji}.$$

So,

$$\begin{aligned}[G(N), N] &= [G(N), \beta E_{ij}] + [G(N), \alpha E_{ji}] \\ &= \sum_{k=1}^{n}[a_{kk}E_{kk}, \beta E_{ij}] + [a_{ij}E_{ij}, \beta E_{ij}] + [a_{ji}E_{ji}, \beta E_{ij}] + \sum_{k=1}^{n}[a_{kk}E_{kk}, \alpha E_{ji}] \\ &\quad + [a_{ij}E_{ij}, \alpha E_{ji}] + [a_{ji}E_{ji}, \alpha E_{ji}] \\ &= \beta(a_{ii} - a_{jj})E_{ij} + a_{ji}\beta(E_{jj} - E_{ii}) + \alpha(a_{jj} - a_{ii})E_{ji} + \alpha a_{ij}(E_{ii} - E_{jj}).\end{aligned}$$

Therefore,

$$\begin{aligned}0 &= [[G(N), N], \beta E_{ij} + \alpha E_{ji}] = [[G(N), N], \beta E_{ij}] + [[G(N), N], \alpha E_{ji}] \\ &= \beta\{a_{ji}\beta(-2E_{ij}) + \alpha(a_{jj} - a_{ii})(E_{jj} - E_{ii}) + \alpha a_{ij}(2E_{ij})\} \\ &\quad + \alpha\{\beta(a_{ii} - a_{jj})(E_{ii} - E_{jj}) + a_{ji}\beta(2E_{ji}) + \alpha a_{ij}(-2E_{ji})\}.\end{aligned}$$

So,

$$2\beta(-a_{ji}\beta + \alpha a_{ij})E_{ij} = 0,$$

and

$$2\beta\alpha(a_{jj} - a_{ii})E_{jj} = 0.$$

Thus, $\alpha G(N)_{ij} = \beta G(N)_{ji}$ and $G(N)_{ii} = G(N)_{jj}$ for all $\alpha, \beta \in \mathbb{K}^*$, where $N = = \beta E_{ij} + \alpha E_{ji}$. Once again, using the previous lemma and the additivity of G, we see

$$\alpha G(\beta E_{ij})_{ij} = \beta G(\alpha E_{ji})_{ji} \quad \text{for all } \alpha, \beta \in \mathbb{K}^*, \tag{2.4}$$

and

$$G(\beta E_{ij})_{ii} + G(\alpha E_{ji})_{ii} = G(\beta E_{ij})_{jj} + G(\alpha E_{ji})_{jj} \quad \text{for all } \alpha, \beta \in \mathbb{K}^*. \tag{2.5}$$

Fix $\alpha \in \mathbb{K}^*$ and let β vary over $\mathbb{K}^*$ in (2.4). Then,

$$\frac{G(\beta E_{ij})_{ij}}{\beta} = \frac{G(\alpha E_{ji})_{ji}}{\alpha} = \lambda_{ij}, \quad \text{for all } \beta \in \mathbb{K}^*, \text{ where } \lambda_{ij} \in \mathbb{K}.$$

Hence, $G(\beta E_{ij})_{ij} = \beta\lambda_{ij}\ \forall\ \beta \in \mathbb{K}^*$. Similarly, we conclude that $G(\alpha E_{jj})_{ji} = \alpha\lambda_{ji}$ for all $\alpha \in \mathbb{K}^*$. Besides, $\lambda_{ij} = \lambda_{ji}$ (note that these equalities hold for $\alpha = 0$ and $\beta = 0$).

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From (2.5), we obtain that $G(\beta E_{ij})_{ii} - G(\beta E_{ij})_{jj} = G(\alpha E_{ji})_{jj} - G(\alpha E_{ji})_{ii}$ for all $\alpha, \beta \in \mathbb{K}^*$. Fixing $\alpha \in \mathbb{K}^*$, and letting β vary over $\mathbb{K}^*$, we see that

$$G(\beta E_{ij})_{ii} - G(\beta E_{ij})_{jj} = v, \text{ for all } \beta \in \mathbb{K}^*, \text{ where } v \in \mathbb{K}.$$

Take $\beta_1, \beta_2 \in \mathbb{K}^*$ such that $\beta_1 + \beta_2 \in \mathbb{K}^*$. Note

$$\begin{aligned} v &= G((\beta_1 + \beta_2)E_{ij})_{ii} - G(\beta_1 + \beta_2)E_{ij})_{jj} \\ &= G(\beta_1 E_{ij})_{ii} + G(\beta_2 E_{ij})_{ii} - G(\beta_1 E_{ij})_{jj} - G(\beta_2 E_{ij})_{jj} \\ &= v + v = 2v. \end{aligned}$$

Thus,

$$G(\beta E_{ij})_{ii} = G(\beta E_{ij})_{jj} \quad \text{for all } \beta \in \mathbb{K}^*. \tag{2.6}$$

Now, we will show that all diagonal entries of $G(\beta E_{ij})$ are equal. Indeed, consider $N_1 = \beta E_{ij} + \alpha E_{j\xi}$, where $\xi \in \overline{\{i, j\}}$. By the previous lemma and the first part of this proof, we can infer that $G(N_1) = \sum_{k=1}^{n} a_{kk} E_{kk} + \lambda_{ij} \beta E_{ij} + \lambda_{j\xi} \alpha E_{j\xi}$, where λ_{ij} (resp., $\lambda_{j\xi}$) does not depend on β (resp., α). Observe that

$$\begin{aligned} [G(N_1), N_1] &= [G(N_1), \beta E_{ij}] + [G(N_1), \alpha E_{j\xi}] \\ &= \sum_{k=1}^{n} [a_{kk} E_{kk}, \beta E_{ij}] + [\lambda_{ij} \beta E_{ij}, \beta E_{ij}] + [\lambda_{j\xi} \alpha E_{j\xi}, \beta E_{ij}] \\ &\quad + \sum_{k=1}^{n} [a_{kk} E_{kk}, \alpha E_{j\xi}] + [\lambda_{ij} \beta E_{ij}, \alpha E_{j\xi}] + [\lambda_{j\xi} \alpha E_{j\xi}, \alpha E_{j\xi}] \\ &= \beta(a_{ii} - a_{jj}) E_{ij} + \alpha(a_{jj} - a_{\xi\xi}) E_{j\xi} + \alpha\beta(\lambda_{ij} - \lambda_{j\xi}) E_{i\xi}. \end{aligned}$$

So,

$$\begin{aligned} 0 = [[G(N_1), N_1], N_1] &= [\beta(a_{ii} - a_{jj}) E_{ij} + \alpha(a_{jj} - a_{\xi\xi}) E_{j\xi} + \alpha\beta(\lambda_{ij} - \lambda_{j\xi}) E_{i\xi}, N_1] \\ &= \alpha(a_{jj} - a_{\xi\xi})[E_{j\xi}, \beta E_{ij}] + \lambda\beta(\lambda_{ij} - \lambda_{j\xi})[E_{i\xi}, \beta E_{ij}] + \beta(a_{ii} - a_{jj})[E_{ij}, \alpha E_{j\xi}] \\ &\quad + \alpha\beta(\lambda_{ij} - \lambda_{j\xi})[E_{i\xi}, \alpha E_{j\xi}] \\ &= \alpha(a_{\xi\xi} - a_{jj})\beta E_{i\xi} + \beta(a_{ii} - a_{jj})\alpha E_{i\xi}. \end{aligned}$$

Thus, $(a_{ii} - a_{jj}) = (a_{jj} - a_{\xi\xi})$. Hence, $G(N_1)_{ii} - G(N_1)_{jj} = G(N_1)_{jj} - G(N_1)_{\xi\xi}$. This last equality yields

$$\begin{aligned} &(G(\beta E_{ij})_{ii} + G(\alpha E_{j\xi})_{ii}) - (G(\beta E_{ij})_{jj} + G(\alpha E_{j\xi})_{jj}) \\ &= (G(\beta E_{ij})_{jj} + G(\alpha E_{j\xi})_{jj}) - (G(\beta E_{ij})_{\xi\xi} + G(\alpha E_{j\xi})_{\xi\xi}). \end{aligned}$$

Employing equation (2.6) (twice), we can deduce that $G(\alpha E_{j\xi})_{ii} - G(\alpha E_{j\xi})_{jj} = G(\beta E_{ij})_{jj} - G(\beta E_{ij})_{\xi\xi}$ for all $\alpha, \beta \in \mathbb{K}^*$. Repeating an earlier argument, we see that $G(\beta E_{ij})_{jj} - G(\beta E_{ij})_{\xi\xi} = 0$ for all $\xi \in \overline{\{i, j\}}$. Then,

$$G(\beta E_{ij})_{ii} \overset{(2.6)}{=} G(\beta E_{ij})_{jj} = G(\beta E_{ij})_{\xi\xi} \quad \text{for all } \xi \in \overline{\{i, j\}}.$$

Hence, all diagonal entries of $G(\beta E_{ij})$ are equal. □

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LEMMA 2.6. *Let $i \in \Omega$ and $j \in \overline{\{i\}}$. Then, $\lambda_{ij} = \lambda$, that is, λ_{ij} does not depend on i and j. In particular, $G(\beta E_{ij}) - \lambda\beta E_{ij}$ is a scalar matrix, where $\lambda \in \mathbb{K}$.*

Proof. Consider $N = E_{ij} + E_{j\xi} + E_{\xi i}$, where $\xi \in \overline{\{i,j\}}$ (using (d)). Then, $G(N) = = cI + \lambda_{ij}E_{ij} + \lambda_{j\xi}E_{j\xi} + \lambda_{\xi i}E_{\xi i}$ for some $c \in \mathbb{K}$. Observe

$$\begin{aligned}[G(N), N] &= [G(N), E_{ij} + E_{j\xi} + E_{\xi i}] = \lambda_{ij}([E_{ij}, E_{j\xi}] + [E_{ij}, E_{\xi i}]) + \lambda_{j\xi}([E_{j\xi}, E_{ij}] \\ &\quad + [E_{j\xi}, E_{\xi i}]) + \lambda_{\xi i}([E_{\xi i}, E_{ij}] + [E_{\xi i}, E_{j\xi}]) \\ &= \lambda_{ij}(E_{i\xi} - E_{\xi j}) + \lambda_{j\xi}(-E_{i\xi} + E_{ji}) + \lambda_{\xi i}(E_{\xi j} - E_{ji}) \\ &= (\lambda_{ij} - \lambda_{j\xi})E_{i\xi} + (\lambda_{\xi i} - \lambda_{ij})E_{\xi j} + (\lambda_{j\xi} - \lambda_{\xi i})E_{ji}.\end{aligned}$$

So,

$$\begin{aligned}0 &= [[G(N), N], N] = [[G(N), N], E_{ij} + E_{j\xi} + E_{\xi i}] \\ &= (\lambda_{ij} - \lambda_{j\xi})\left([E_{i\xi}, E_{ij}] + [E_{i\xi}, E_{j\xi}] + [E_{i\xi}, E_{\xi i}]\right) + (\lambda_{\xi i} - \lambda_{ij})([E_{\xi j}, E_{ij}] \\ &\quad + [E_{\xi j}, E_{j\xi}] + [E_{\xi j}, E_{\xi i}]) + (\lambda_{j\xi} - \lambda_{\xi i})\left([E_{ji}, E_{ij}] + [E_{ji}, E_{j\xi}] + [E_{ji}, E_{\xi i}]\right).\end{aligned}$$

Therefore,

$$\begin{aligned}0 &= \left((\lambda_{ij} - \lambda_{j\xi}) - (\lambda_{j\xi} - \lambda_{\xi i})\right)E_{ii} + \left((\lambda_{j\xi} - \lambda_{\xi i}) - (\lambda_{\xi i} - \lambda_{ij})\right)E_{jj} \\ &\quad + \left((\lambda_{\xi i} - \lambda_{ij}) - (\lambda_{ij} - \lambda_{j\xi})\right)E_{\xi\xi}.\end{aligned}$$

Thus, we arrive in the following system

$$\left\{\begin{array}{ccccccc} \lambda_{ij} & - & 2\lambda_{j\xi} & + & \lambda_{\xi i} & = & 0 \\ \lambda_{ij} & + & \lambda_{j\xi} & - & 2\lambda_{\xi i} & = & 0 \\ -2\lambda_{ij} & + & \lambda_{j\xi} & + & \lambda_{\xi i} & = & 0 \end{array}\right. .$$

Let A be the matrix formed by the coefficients of the above matrix, that is,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix}.$$

Note that after some elementary row operations, the matrix A is equivalent to the following matrix:

$$B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the solutions are $\lambda_{ij} = \lambda_{j\xi} = \lambda_{\xi i}$ for all distinct i, j, and ξ. This implies that λ_{ij} is independent of i and j, since $\lambda_{ij} = \lambda_{ji}$ (using $\lambda_{ji} = \lambda_{\xi j} = \lambda_{i\xi}$). In particular, $G(\beta E_{ij}) - \lambda\beta E_{ij}$ is a scalar matrix. ◻

LEMMA 2.7. *Let $i \in \Omega$. Then, $G(\alpha E_{ii}) - (\lambda\alpha)E_{ii}$ is a scalar matrix, where $\lambda \in \mathbb{K}$ is in accordance with Lemma 2.6.*

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Proof. Let $j \in \overline{\{i\}}$. Consider $N = \alpha E_{ii} + \gamma E_{jj} + \beta E_{ij}$ (using (b)), where α, β and $\gamma \in \mathbb{K}^*$ with $\alpha \neq \gamma$. Using the previous lemmas, we know that $G(N) = \sum_{k=1}^{n} a_{kk} E_{kk} + +\lambda\beta E_{ij}$. So,

$$\begin{aligned}[G(N), N] &= \alpha[G(N), E_{ii}] + \gamma[G(N), E_{jj}] + \beta[G(N), E_{ij}] \\ &= \alpha\lambda\beta[E_{ij}, E_{ii}] + \gamma\lambda\beta[E_{ij}, E_{jj}] + \beta \cdot \sum_{k=1}^{n} a_{kk}[E_{kk}, E_{ij}] \\ &= -\alpha\lambda\beta E_{ij} + \gamma\lambda\beta E_{ij} + \beta(a_{ii} - a_{jj})E_{ij} = \rho E_{ij}.\end{aligned}$$

Then,

$$\begin{aligned}[[G(N), N], N] &= \rho[E_{ij}, N] = \rho(\alpha[E_{ij}, E_{ii}] + \gamma[E_{ij}, E_{jj}]) \\ &= \rho(-\alpha E_{ij} + \gamma E_{ij}) = \rho(\gamma - \alpha)E_{ij} = 0.\end{aligned}$$

Thus, $\rho(\gamma - \alpha) = 0$. So, $\rho = \beta(-\alpha\lambda + \gamma\lambda + (a_{ii} - a_{jj})) = 0$, because $\alpha \neq \gamma$. Furthermore, since $\beta \in \mathbb{K}^*$, we see that $a_{ii} + \gamma\lambda = a_{jj} + \alpha\lambda$.

Therefore,

$$G(N)_{ii} + \gamma\lambda = G(N)_{jj} + \alpha\lambda.$$

Let us recall that G is additive, $N = \alpha E_{ii} + \gamma E_{jj} + \beta E_{ij}$, and $G(\beta E_{ij})_{ii} = G(\beta E_{ii})_{jj}$ (by Lemma 2.6). Hence,

$$G(\alpha E_{ii})_{ii} + G(\gamma E_{jj})_{ii} + \gamma\lambda = G(\alpha E_{ii})_{jj} + G(\gamma E_{jj})_{jj} + \alpha\lambda.$$

Then,

$$G(\alpha E_{ii})_{ii} - G(\alpha E_{ii})_{jj} - \lambda\alpha = G(\gamma E_{jj})_{jj} - G(\gamma E_{jj})_{ii} - \gamma\lambda.$$

Observe that the left (resp., right) hand side of the above equation only depends on α (resp., γ). Letting γ vary on $\mathbb{K}^*$, we see $H(\alpha) = G(\alpha E_{ii})_{ii} - G(\alpha E_{ii})_{jj} - \alpha\lambda = v$, where $v \in \mathbb{K}$. Therefore, $G(\alpha E_{ii})_{ii} - G(\alpha E_{ii})_{jj} - \lambda\alpha = 0$, since $H(\alpha)$ is additive. So, $G(\alpha E_{ii})_{ii} - \lambda\alpha = G(\alpha E_{ii})_{jj}$ for all $\alpha \in \mathbb{K}^*$ and $j \in \overline{\{i\}}$. And this allows us to conclude that $G(\alpha E_{ii})_{jj} = G(\alpha E_{ii})_{\xi\xi}$ for all $j, \xi \in \overline{\{i\}}$ and $\alpha \in \mathbb{K}^*$. Therefore, $G(\alpha E_{ii}) - (\lambda\alpha)E_{ii}$ is a scalar matrix. And this completes Proposition's 1.1 proof. ∎

Now, we are in a position to prove our main result:

Theorem 2.8. *Let $\mathbb{K}$ be a field whose characteristic is either zero or greater than 2, and $n \geq 4$. Let $G : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ be an additive map such that*

$$[[G(y), y], y] = 0 \quad \textit{for all singular } y \in M_n(\mathbb{K}).$$

Then, there exist an element $\lambda \in \mathbb{K}$ and a central map μ such that

$$G(x) = \lambda x + \mu(x) \quad \textit{for each } x \in M_n(\mathbb{K}).$$

Proof. The result follows immediately from Proposition 1.1, because if N has one of the froms $(a), (b), (c)$ or (d) then N is singular for all $n \geq 4$. ∎

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As an application, we have:

COROLLARY 2.9. *Let $m, n \in \mathbb{N}^*$, where $n \geq 4$. Let $G : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ be an additive map. Let us suppose that*

$$\sum_{k_1,k_2,k_3=1}^{m} [[G(y^{k_1}), y^{k_2}], y^{k_3}] = 0 \tag{2.7}$$

for all singular matrices $y \in M_n(\mathbb{K})$. If the characteristic of $\mathbb{K}$ is either zero or greater than $3m-2$ then $G(x) = \lambda x + \mu(x)$ for each $x \in M_n(\mathbb{K})$, where $\lambda \in \mathbb{K}$ and μ is a central map.

Proof. Let us denote by $\mathbb{L}$ the prime field of $\mathbb{K}$. Let $\beta \in \mathbb{L}^*$ and $x \in M_n(\mathbb{K})$ be a singular matrix. It is clear that βx is singular. Besides, note that $G(\beta x) = \beta x$, since G is additive. By (2.7), we have

$$\begin{aligned}
&\sum_{k_1,k_2,k_3=1}^{m} [[G(\beta^{k_1}x^{k_1}), \beta^{k_2}x^{k_2}], \beta^{k_3}x^{k_3}] \\
&= \beta^3[[G(x), x], x] + \sum_{\substack{k_1,k_2,k_3=1 \\ k_1+k_2+k_3\geq 4}}^{m} [[G(\beta^{k_1}x^{k_1}), \beta^{k_2}x^{k_2}], \beta^{k_3}x^{k_3}] \\
&= \beta^3[[G(x), x], x] + \beta^4 R_4(x) + \beta^5 R_5(x) + \cdots + \beta^{3m} R_{3m}(x) = 0.
\end{aligned}$$

So,

$$[[G(x), x], x] + \sum_{i=4}^{3m} \beta^{i-3} R_i(x) = 0 \quad \text{for all } \beta \in \mathbb{L}^*.$$

Since, $| \mathbb{L} |> 3m-2$, we can choose $\beta_1, \beta_2, \ldots, \beta_{3m-2} \in \mathbb{L}^*$ pairwise distinct. Hence,

$$\begin{pmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{3(m-1)} \\ 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^{3(m-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \beta_{3m-2} & \beta_{3m-2}^2 & \cdots & \beta_{3m-2}^{3(m-1)} \end{pmatrix} \begin{pmatrix} [[G(x), x], x] \\ R_4(x) \\ R_5(x) \\ \vdots \\ R_{3m}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, from the above system, we can conclude that $[[G(x), x], x] = 0$ for all singular matrices $x \in M_n(\mathbb{K})$. Now, the result follows from Theorem 2.8. ∎

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