Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 192-208, April 2008

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KREIN SPACES NUMERICAL RANGES AND THEIR COMPUTER GENERATION*

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Abstract. Let J be an involutive Hermitian matrix with signature $(t, n-t)$, $0 \le t \le n$, that is, with t positive and $n-t$ negative eigenvalues. The Krein space numerical range of a complex matrix A of size n is the collection of complex numbers of the form $\frac{\xi^* JA\xi}{\xi^* J\xi}$, with $\xi \in \mathbb{C}^n$ and $\xi^* J\xi \neq 0$. In this note, a class of tridiagonal matrices with hyperbolical numerical range is investigated. A *Matlab* program is developed to generate Krein spaces numerical ranges in the finite dimensional case.

Key words. Krein spaces, Numerical range, Tridiagonal matrices.

AMS subject classifications. 15A60, 15A63.

1. Introduction. Throughout, M_n denotes the algebra of $n \times n$ matrices over the field of complex numbers. Let J be an involutive Hermitian matrix with signature $(t, n-t)$, $0 \le t \le n$, that is, with t positive and $n-t$ negative eigenvalues. Consider $\mathbb{C}^n$ as a Krein space with respect to the indefinite inner product $[\xi, \eta] = \eta^* J\xi,\ \xi, \eta \in \mathbb{C}^n$. The $J-$*numerical range* of $A \in M_n$ is denoted and defined by:

$$W_J(A) = \left\{ \frac{[A\xi, \xi]}{[\xi, \xi]} :\ \xi \in \mathbb{C}^n,\ [\xi, \xi] \neq 0 \right\}.$$

Considering J the identity matrix of order n, I_n, this concept reduces to the well known *classical numerical range*, usually denoted by $W(A)$. The numerical range of an operator defined on an indefinite inner product space is currently being studied (see [11] and references therein). For $W_J(A)$, $A \in M_n$, the following inclusion holds: $\sigma(A) \subset W_J(A)$, where $\sigma(A)$ denotes the set of the eigenvalues of A with $J-$*anisotropic* eigenvectors, that is, eigenvectors with nonvanishing $J-$norm. We denote by $\sigma^{\pm}(A)$ the sets of the eigenvalues of A with associated eigenvectors ξ such that $\xi^* J\xi = \pm 1$. Compactness and convexity are basic properties of the classical numerical range. In

*Received by the editors December 17, 2007. Accepted for publication March 24, 2008. Handling Editor: Peter Lancaster.

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A publication of the International Linear Algebra Society
Volume 17, pp. 192-208, April 2008

contrast with the classical case, $W_J(A)$ may be neither closed nor bounded. On the other hand, $W_J(A)$ may not be convex, but it is the union of two convex sets $W_J(A) = W_J^+(A) \cup W_J^-(A)$, where

$$W_J^+(A) = \left\{ \frac{[A\xi, \xi]}{[\xi, \xi]} : \ \xi \in \mathbb{C}^n, \ [\xi, \xi] > 0 \right\}$$

and

$$W_J^-(A) = \left\{ \frac{[A\xi, \xi]}{[\xi, \xi]} : \ \xi \in \mathbb{C}^n, \ [\xi, \xi] < 0 \right\}.$$

Since the Krein space numerical range is in general neither bounded nor closed, it is difficult to generate an accurate computer plot of this set. For $A \in M_n$ and $n > 2$, the description of $W_J(A)$ is complicated, and so it is of interest to have a code to produce graphical representations. The case $n = 2$ is treated by the Hyperbolical Range Theorem [1] which states the following: if $A \in M_2$ has eigenvalues α_1 and α_2, $J = \operatorname{diag}(1, -1)$ and $2\mathrm{Re}\,(\overline{\alpha}_1 \alpha_2) < \mathrm{Tr}(A^{[*]}A) < |\alpha_1|^2 + |\alpha_2|^2$, where $A^{[*]} = JA^*J$, then $W_J(A)$ is bounded by a nondegenerate hyperbola with foci at α_1 and α_2, and transverse and nontransverse axis of length

$$\sqrt{\mathrm{Tr}(A^{[*]}A) - 2\mathrm{Re}\,(\alpha_1 \bar{\alpha_2})} \qquad \text{and} \qquad \sqrt{|\alpha_1|^2 + |\alpha_2|^2 - \mathrm{Tr}(A^{[*]}A)},$$

respectively. For the degenerate cases, $W_J(A)$ may be a singleton, a line, the union of two half-lines, the whole complex plane, or the complex plane except a line. Independently of the size, certain matrices have a hyperbolical $J-$numerical range.

A matrix $A = (a_{ij}) \in M_n$ is *tridiagonal* if $a_{ij} = 0$ whenever $|i-j| > 1$. Interesting papers have been published on the classical numerical range of tridiagonal matrices [3, 4, 12]. For complex numbers a, b, c, the tridiagonal matrix in M_n with a's on the main diagonal, b's on the first superdiagonal and c's on the first subdiagonal is denoted by $A = \operatorname{tridiag}(c, a, b)$. These matrices are of *Toeplitz* type, because all the entries in each diagonal are equal. Marcus and Shure [12] proved that the numerical range of $\operatorname{tridiag}(0, 0, 1)$ is a circular disc centered at the origin of radius $\cos(\pi/(n+1))$. Eiermann [6] showed that the numerical range of $\operatorname{tridiag}(c, 0, b)$ is the elliptical disc $\{cz + b\overline{z} \ : \ |z| \leq \cos(\pi/(n+1))\}$. Generalizations of Eiermann's results were given by Chien [3, 4], Chien and Nakazato [5], and Brown and Spitkovsky [2]. Likewise, there is interest in studying Krein spaces numerical ranges of these classes.

Motivated by these investigations, in Section 2, we characterize a class of tridiagonal matrices with hyperbolical numerical range. In Section 3, we present an algorithm that allows a computer plot of $W_J(A)$. In Section 4, a *Matlab* program is presented to plot Krein spaces numerical ranges for finite dimensional operators.

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2. A class of tridiagonal matrices with hyperbolical numerical range. The proof of the next lemma is similar to the proof of Lemma 3.1 in [2] and is included for the sake of completeness.

LEMMA 2.1. *Let J be $I_1 \oplus -I_1 \oplus \cdots \oplus I_1 \oplus -I_1 \oplus I_1$ or $I_1 \oplus -I_1 \oplus \cdots \oplus I_1 \oplus -I_1$ according to the size of the matrix J being odd or even, respectively. The $J-$numerical range of an $n \times n$ tridiagonal matrix is invariant under interchange of the $(j, j+1)$ and $(j+1, j)$ entries for any $j = 1, \ldots, n-1$.*

Proof. Let

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ c_1 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & a_j & b_j & \ddots & \vdots \\ \vdots & \ddots & c_j & a_{j+1} & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & \cdots & 0 & c_{n-1} & a_n \end{bmatrix}. \tag{2.1}$$

For simplicity we interchange b_1 and c_1. Let $\hat{A}$ be the $n \times n$ tridiagonal matrix that differs from A only by interchanging b_1 and c_1. Consider an arbitrary point $z = z^*JAz \in W_J(A)$, where $z = (z_1, \ldots, z_n)^T \in \mathbb{C}^n$ and $z^*Jz = 1$. We show that there exists $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_n)^T \in \mathbb{C}^n$ such that $z^*JAz = \hat{z}^*J\hat{A}\hat{z}$ and $z^*Jz = \hat{z}^*J\hat{z}$. For the first equality to hold, we require that

$$\bar{z}_1 a_1 z_1 - \bar{z}_2 a_2 z_2 + \bar{z}_1 b_1 z_2 - \bar{z}_2 c_1 z_1 = \bar{\hat{z}}_1 a_1 \hat{z}_1 - \bar{\hat{z}}_2 a_2 \hat{z}_2 + \bar{\hat{z}}_1 c_1 \hat{z}_2 - \bar{\hat{z}}_2 b_1 \hat{z}_1.$$

If $z_1 = 0$, we can choose $\hat{z} = z$. Otherwise, let $|\hat{z}_1| = |z_1|$, $|\hat{z}_2| = |z_2|$, $\arg \hat{z}_1 = -\arg z_1 + \pi$, $\arg \hat{z}_2 = -\arg z_2$. Moreover, we choose $\hat{z}_j = z_j \mathrm{e}^{i\phi}$, $j > 2$, where $\phi = -2 \arg z_j$. By easy calculations, the result follows. ∎

A *supporting line* of a convex set $S \subset \mathbb{C}$ is a line containing a boundary point of S and defining two half-planes, such that one of them does not contain S. The supporting lines of $W_J(A)$ are by definition the supporting lines of the convex sets $W_J^+(A)$ and $W_J^-(A)$. Let

$$H_A = \frac{A + A^{[*]}}{2} \quad \text{and} \quad K_A = \frac{A - A^{[*]}}{2i}. \tag{2.2}$$

be the unique $J-$Hermitian matrices such that $A = H_A + iK_A$. (A matrix A is $J-$Hermitian if it coincides with $A^{[*]}$.) If $ux + vy + w = 0$ is the equation of a supporting line of $W_J(A)$, then $\det(u\,H_A + v\,K_A + w\,I_n) = 0$. The homogeneous polynomial equation $\det(u\,H_A + v\,K_A + w\,I_n) = 0$ can be considered the dual (line) equation of an algebraic curve. The real part of the dual curve is called the *boundary generating curve* of $W_J(A)$.

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We recall that $A \in M_n$ is *essentially* $J-$*Hermitian* if there exist $\zeta_1, \zeta_2 \in \mathbb{C}$ such that $A = \zeta_1 I_n + \zeta_2 A'$ where A' is $J-$Hermitian. In this case, $W_J(A)$ is a line or the union of two half-lines [13]. Next, we assume that $A \in M_n$ is nonessentially $J-$Hermitian.

THEOREM 2.2. *Let $A \in M_n$ be a nonessentially $J-$Hermitian tridiagonal matrix with biperiodic main diagonal, that is, $a_j = a_1$ if j is odd and $a_j = a_2$ if j is even, and with off-diagonal entries b_j, c_j such that either $c_j = k\overline{b}_j$ or $b_j = k\overline{c}_j$ for some $k \in \mathbb{C}$ and $j = 1, \ldots, n-1$. Let J be the diagonal matrix $I_1 \oplus -I_1 \oplus \cdots \oplus I_1 \oplus -I_1 \oplus I_1$ or $I_1 \oplus -I_1 \oplus \cdots \oplus I_1 \oplus -I_1$ according to the size of A being odd or even. Let $\gamma = (\frac{a_1-a_2}{2})^2 + k\lambda_1^2$, where λ_1 is the spectral norm of $C = \mathrm{tridiag}\,(\mathbf{c}, 0, \mathbf{b}) \in M_n$ for $\mathbf{0} = (0, 0, \ldots, 0)$, $\mathbf{b} = (b_1, -b_2, \ldots)$ and $\mathbf{c} = (\overline{b}_1, -\overline{b}_2, \ldots)$. The following holds:*

(i) If $|\gamma| > \frac{1}{2}\lambda_1^2(1 + |k|^2) - \left|\frac{a_1-a_2}{2}\right|^2$, then $W_J(A)$ is bounded by the hyperbola centered at $\frac{a_1+a_2}{2}$, foci at

$$\frac{(a_1 + a_2) \pm \sqrt{(a_1 - a_2)^2 + 4k\lambda_1^2}}{2}$$

and semi-transverse axis of length

$$\alpha = \sqrt{\frac{1}{2}\left|\frac{a_1 - a_2}{2}\right|^2 - \frac{1}{4}\lambda_1^2(1 + |k|^2) + \frac{1}{2}|\gamma|}.$$

(ii) If $|\gamma| = \frac{1}{2}\lambda_1^2(1 + |k|^2) - \left|\frac{a_1-a_2}{2}\right|^2$, then $W_J(A)$ is the whole complex plane except the line with slope $(\arg(\gamma) + \pi)/2$ passing through $(a_1 + a_2)/2$.

(iii) If $|\gamma| < \frac{1}{2}\lambda_1^2(1 + |k|^2) - \left|\frac{a_1-a_2}{2}\right|^2$, then $W_J(A)$ is the whole complex plane.

Proof. Let A have even size. According to Lemma 2.1, we may assume without loss of generality that the off-diagonal entries of A are such that $c_j = k\overline{b}_j$, for $j = 1, \ldots, n-1$. Writing $c = (a_1 - a_2)/2$, we have $A = \frac{1}{2}(a_1 + a_2)I_n + B$, where the matrix B is obtained from A replacing the main diagonal by $(c, -c, \ldots, c, -c)$. Without loss of generality, we may assume that $c > 0$.

For $\theta \in [0, 2\pi[$, consider a supporting line of $W_J(B)$ perpendicular to the direction of argument θ. To determine the supporting lines of $W_J(B)$ we search the eigenvalues of the matrix $J\mathrm{Re}\,(\mathrm{e}^{-i\theta} JB)$. By easy computations, we find

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$$\det(\mathrm{Re}\,(\mathrm{e}^{-i\theta}JB)-\mu(\theta)J)=2^{-n}\,|\mathrm{e}^{-i\theta}-\overline{k}\mathrm{e}^{i\theta}|^n$$
$$\times\det\begin{bmatrix} \frac{\mathrm{Re}\,(c\,e^{-i\theta})-\mu(\theta)}{1/2|\mathrm{e}^{-i\theta}-\overline{k}\mathrm{e}^{i\theta}|} & b_1 & 0 & \cdots & 0 \\ \overline{b}_1 & \frac{\mathrm{Re}\,(c\,e^{-i\theta})+\mu(\theta)}{1/2|\mathrm{e}^{-i\theta}-\overline{k}\mathrm{e}^{i\theta}|} & -b_2 & \ddots & \vdots \\ 0 & -\overline{b}_2 & \frac{\mathrm{Re}\,(c\,e^{-i\theta})-\mu(\theta)}{1/2|\mathrm{e}^{-i\theta}-\overline{k}\mathrm{e}^{i\theta}|} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & \overline{b}_{n-1} & \frac{\mathrm{Re}\,(c\,e^{-i\theta})-\mu(\theta)}{1/2|\mathrm{e}^{-i\theta}-\overline{k}\mathrm{e}^{i\theta}|} \end{bmatrix}.$$

For $\frac{(\mathrm{Re}\,(c\,e^{-i\theta}))^2-\mu(\theta)^2}{1/4|\mathrm{e}^{-i\theta}-\overline{k}\mathrm{e}^{i\theta}|^2}=\lambda^2$, the determinant in the right hand side of the above equality coincides with the determinant of $C-\lambda I_n$, where C is the tridiagonal Hermitian matrix with vanishing main diagonal, first superdiagonal and subdiagonal $(b_1,-b_2,\ldots,b_1)$ and $(\overline{b}_1,-\overline{b}_2,\ldots,\overline{b}_1)$, respectively. The eigenvalues of the Hermitian matrix C occur in pairs of symmetric real numbers, and we denote and order them as follows: $\lambda_1>\cdots>\lambda_{\frac{n}{2}}>\cdots>\lambda_n$, with $\lambda_{n-j+1}=-\lambda_j$, $j=1,\ldots,n$. The eigenvalues of $J\mathrm{Re}\,(\mathrm{e}^{-i\theta}JB)$, say $\mu_j(\theta)$, satisfy

$$\mu_j^2(\theta)=(\mathrm{Re}\,(c\,e^{-i\theta}))^2-\frac{\lambda_j^2}{4}|\mathrm{e}^{-i\theta}-\overline{k}\mathrm{e}^{i\theta}|^2\in\mathbb{R},\quad j=1,\ldots,n. \tag{2.3}$$

Let θ be fixed but arbitrary. We analyze the condition for the eigenvalues $\mu_j(\theta)$ to be real. If the smallest of the $\mu_j^2(\theta)$, that is, $\mu_1^2(\theta)$, is nonnegative, then all the $\mu_j(\theta)$ are real. So, we investigate the existence of angles θ such that

$$\mu_1^2(\theta)=P+Q\cos(2\theta)+R\sin(2\theta)\geq 0,$$

where

$$P=\frac{1}{2}c^2-\frac{1}{4}\lambda_1^2(1+|k|^2),$$
$$Q=\frac{1}{2}\mathrm{Re}\,(c^2)+\frac{1}{2}\mathrm{Re}\,(k)\lambda_1^2,$$
$$R=\frac{1}{2}\mathrm{Im}(c^2)+\frac{1}{2}\mathrm{Im}(k)\lambda_1^2.$$

Writing $\gamma=2(Q+iR)=c^2+k\lambda_1^2$ and $\Gamma=\arg(\gamma)$, we have

$$\mu_1^2(\theta)=\alpha\cos^2\left(\frac{\Gamma}{2}-\theta\right)-\beta\sin^2\left(\frac{\Gamma}{2}-\theta\right),$$

where

$$\alpha=\frac{1}{2}c^2-\frac{1}{4}\lambda_1^2(1+|k|^2)+\frac{1}{2}|\gamma|,$$

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and

$$\beta=-\frac{1}{2}c^2+\frac{1}{4}\lambda_1^2(1+|k|^2)+\frac{1}{2}|\gamma|. \tag{2.4}$$

We show that $\beta \geq 0$. In fact, since

$$\frac{|\gamma|}{2}\geq\frac{1}{2}c^2-|k|\frac{\lambda_1^2}{2},$$

from (2.4) we easily conclude that

$$\beta\geq\frac{1}{4}\lambda_1^2(1-|k|)^2\geq 0.$$

It can be easily seen that a tridiagonal matrix A under the hypothesis of the theorem is essentially $J-$Hermitian if and only if $|k|=1$ and $\arg k=2\arg c$. Since A is nonessentially $J-$Hermitian, then $|k|\neq 1$ and to avoid trivial situations we may suppose $\lambda_1\neq 0$. Thus, we may consider $\beta>0$.

If (i) holds, then $|\gamma|>\frac{1}{2}\lambda_1^2(1+|k|^2)-c^2$, and so $\alpha>0$. Thus there exist angles θ such that $\mu_1^2(\theta)\geq 0$. Hence

$$\mu_1(\theta)=\pm\sqrt{\alpha\cos^2\left(\frac{\Gamma}{2}-\theta\right)-\beta\sin^2\left(\frac{\Gamma}{2}-\theta\right)}, \tag{2.5}$$

and all the $\mu_j(\theta)$ are real and pairwise symmetric. It can be easily seen that (2.5) describes a family of hyperbolas, for θ ranging over $[\theta_1,\theta_2]$, $\tan(\Gamma/2-\theta_i)=\alpha/\beta$, $i=1,2$. The parametric equations of the hyperbola generated by $\mu_j(\theta)$ are

$$\begin{cases} x\cos(\theta)+y\sin(\theta)=\mu_j(\theta)\\ -x\sin(\theta)+y\cos(\theta)=\mu_j'(\theta)\end{cases}$$

for $\theta\in[\Gamma/2,\Gamma/2+2\pi[$. Since $0<\mu_1(\theta)<\mu_2(\theta)<\cdots<\mu_{\frac{n}{2}}(\theta)$, these eigenvalues originate a collection of nested hyperbolas. The outer hyperbola is generated by $\mu_1(\theta)$ and its Cartesian equation is

$$\frac{X^2}{\alpha}-\frac{Y^2}{\beta}=1, \tag{2.6}$$

where $X=x\cos(\Gamma/2)-y\sin(\Gamma/2)$ and $Y=x\sin(\Gamma/2)+y\cos(\Gamma/2)$.

Now, we analyze the sign of the $J-$norm of the eigenvectors associated with $\mu_j(\theta)$. We notice that if $\nu_j=\left(x_1^{(j)},\ldots,x_n^{(j)}\right)$ is an eigenvector associated with $\mu_j(\theta)$, then $\nu_{n-j+1}=\left(sx_1^{(j)},x_2^{(j)},sx_3^{(j)},\ldots,x_n^{(j)}\right)$ is an eigenvector associated with $-\mu_j(\theta)$, where

$$s=\frac{c\cos\theta-\mu_j(\theta)}{c\cos\theta+\mu_j(\theta)}.$$

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Easy calculations show that the $J-$norm of ν_j is

$$\left(|x_1^{(j)}|^2 + |x_3^{(j)}|^2 + \cdots + |x_n^{(j)}|^2\right)(1-s),$$

while the $J-$norm of ν_{n-j+1} is

$$\left(|x_1^{(j)}|^2 + |x_3^{(j)}|^2 + \cdots + |x_n^{(j)}|^2\right)s(s-1).$$

Therefore, $\mu_j(\theta) \in \sigma^+(J\mathrm{Re}\,(\mathrm{e}^{-i\theta}JB))$, $j = 1, \ldots, \frac{n}{2}$ and $\mu_{n-j+1}(\theta) \in \sigma^-(J\mathrm{Re}\,(\mathrm{e}^{-i\theta}$ $JB))$, $j = \frac{n}{2}+1, \ldots, n$. Hence,

$$W_J^+(J\mathrm{Re}\,(\mathrm{e}^{-i\theta}JB)) = [\mu_1(\theta), +\infty[\ ; \quad W_J^-(J\mathrm{Re}\,(\mathrm{e}^{-i\theta}JB)) =]-\infty, -\mu_1(\theta)]. \tag{2.7}$$

Thus, $\partial W_J(A)$ is the asserted hyperbola.

Now, we consider that A has odd size. The situation is similar to the one treated above, with the eigenvalues of $J\mathrm{Re}\,(\mathrm{e}^{-i\theta}J\,B)$ occurring in pairs of symmetric real numbers, being $\mathrm{Re}\,(e^{-i\theta}c)$ an eigenvalue with associated eigenvector $(x_1^{(j)}, 0, x_3^{(j)}, \ldots, x_n^{(j)})$ of positive $J-$norm. For $\theta = 0$, we have

$$\alpha \leq c^2 - \frac{1}{2}\lambda_1^2(1-|k|^2).$$

Therefore, $\sqrt{\alpha} \leq c$, so the point c lies inside the hyperbola (2.6). Thus, (i) follows.

(ii) If $|\gamma| = \frac{1}{2}\lambda_1^2(1+|k|^2) - c^2$, then $\alpha = 0$ and $\mu_1^2(\theta) < 0$ for all $\theta \neq \Gamma/2$. The matrix $J\mathrm{Re}\,(J\mathrm{e}^{-i\theta}B)$ has complex eigenvalues in all directions, except in the direction $\theta = \Gamma/2$. Thus, the projection of $W_J(A)$ in all the directions is the whole line, possibly except in the direction $\theta = \Gamma/2$. Thus, we may conclude that $W_J(A)$ is the complex plane possibly without one line. Now, we show that the line with slope $(\Gamma+\pi)/2$ and passing through $(a_1+a_2)/2$ is not contained in $W_J(A)$.

Since $\mu_1^2(\Gamma/2) = 0$, 0 is a double eigenvalue of $B^{\Gamma/2} := J\mathrm{Re}\,(J\mathrm{e}^{-i\Gamma/2}B)$ and we use a perturbative method. Let $B_\epsilon^{\Gamma/2} = J\mathrm{Re}\,(\mathrm{e}^{-i\Gamma/2}JB_\epsilon)$, where B_ϵ is obtained from B replacing c by $c+\epsilon$, with $\epsilon \in \mathbb{R}$ chosen as follows. We have $\alpha(\epsilon) = \alpha + \epsilon\, M + O(\epsilon^2)$, where M is real and nonzero. Choosing ϵ such that $\epsilon\, M$ is positive, then $\alpha(\epsilon) > 0$ and by (2.7), we may conclude that $W_J^+(B_\epsilon^{\Gamma/2}) = \left[\sqrt{\alpha(\epsilon)}, +\infty\right[$ and $W_J^-(B_\epsilon^{\Gamma/2}) = \left]-\infty, -\sqrt{\alpha(\epsilon)}\right]$. If $\epsilon \to 0$, then $\alpha(\epsilon) \to 0$ and we find that $]-\infty, 0[\ \cup\ ]0, +\infty[$ is contained in $W_J(B^{\Gamma/2})$.

We show that the origin is not an element of $W_J(B^{\Gamma/2})$. Firstly, we assume that n is even and we use a perturbative method. Let the eigenvalues of $B_\epsilon^{\Gamma/2}$ be $\mu_1(\epsilon), \ldots, \mu_{n/2}(\epsilon) \in \sigma^+(B_\epsilon^{\Gamma/2})$ and $\mu_{n/2+1}(\epsilon), \ldots, \mu_n(\epsilon) \in \sigma^-(B_\epsilon^{\Gamma/2})$ with associated eigenvectors $\nu_1(\epsilon), \nu_2(\epsilon), \ldots,\ \nu_{n/2}(\epsilon), \nu_{n/2+1}(\epsilon), \ldots, \nu_n(\epsilon)$. Assume that $0 < \mu_1(\epsilon) <$

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$\cdots < \mu_{n/2}(\epsilon)$. Consider the basis $\mathcal{B}(\epsilon)$ obtained from the above eigenbasis replacing the vectors $\nu_1(\epsilon)$ and $\nu_n(\epsilon)$, respectively, by $v_1(\epsilon)$ and $v_n(\epsilon)$, with $J-$norms 1 and -1, so that the matrix $B_\epsilon^{\Gamma/2}$ is represented in $\mathcal{B}(\epsilon)$ in the form $B_1(\epsilon) \oplus B_2(\epsilon) \oplus B_3(\epsilon)$, where $B_1(\epsilon) = \mathrm{diag}(\mu_{\frac{n}{2}}(\epsilon), \ldots, \mu_2(\epsilon))$,

$$B_2(\epsilon) = \begin{bmatrix} 1 & \sqrt{1-\alpha^2(\epsilon)} \\ -\sqrt{1-\alpha^2(\epsilon)} & -1 \end{bmatrix},$$

and $B_3(\epsilon) = \mathrm{diag}(\mu_{n+1}(\epsilon), \ldots, \mu_{n/2+1}(\epsilon))$. Clearly,

$$B_2(\epsilon) = \begin{bmatrix} 1 & \sqrt{1-\alpha^2(\epsilon)} \\ -\sqrt{1-\alpha^2(\epsilon)} & -1 \end{bmatrix} \xrightarrow[\epsilon\to 0]{} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Taking the limit of each element of $\mathcal{B}(\epsilon)$ as $\epsilon \to 0$, we obtain the basis denoted by $\mathcal{B}$. Let $v = x_1 v_1 + x_n v_n + \sum_{i=2}^{n-1} x_i \nu_i$ be an arbitrary anisotropic vector of $\mathbb{C}^n$ expressed in $\mathcal{B}$. So

$$\frac{v^*JBv}{v^*Jv} = \frac{|x_1+x_n|^2 + \mu_2(|x_2|^2+|x_{n-1}|^2) + \cdots + \mu_{\frac{n}{2}}(|x_{\frac{n}{2}}|^2 + |x^2_{\frac{n}{2}+1}|)}{|x_1|^2 - |x_n|^2 + |x_2|^2 - |x_{n-1}|^2 + \cdots + |x_{\frac{n}{2}}|^2 - |x_{\frac{n}{2}+1}|^2} \in W_J^+(B^{\Gamma/2}).$$

Since v is anisotropic, the denominator is nonzero and the numerator vanishes if and only if $x_2 = x_{n-1} = \cdots = x_{\frac{n}{2}} = x_{\frac{n}{2}+1} = 0$ and $x_n + x_1 = 0$, which is impossible. If n is odd an analogous argument holds.

(iii) If $|\gamma| < \frac{1}{2}\lambda_1^2(1+|k|^2) - |c|^2$, then $\mu_1^2(\theta)$ is negative and so $\mu_1(\theta)$ is imaginary. Thus, the projection of $W_J(A)$ in each direction is the whole line, and as a consequence, $W_J(A) = \mathbb{C}$. ∎

Corollary 2.3. *Let J be the infinite diagonal matrix $I_1 \oplus -I_1 \oplus \cdots$ and let A be the infinite tridiagonal matrix with biperiodic main diagonal, that is, $a_j = a_1$ if j is odd and $a_j = a_2$ if j is even, and with off-diagonal entries b_j, c_j such that either $c_j = k\overline{b}_j$ or $b_j = k\overline{c}_j$ $k \in \mathbb{C}$, $j = 1, 2, \ldots$. Let $\gamma = (\frac{a_1-a_2}{2})^2 + k(|b_1|+|b_2|)^2$. Then:*

(i) If $|\gamma| > \frac{1}{2}(|b_1|+|b_2|)^2(1+|k|^2) - \left|\frac{a_1-a_2}{2}\right|^2$, then $W_J(A)$ is the open region bounded by the hyperbola centered at $\frac{a_1+a_2}{2}$, foci at

$$\frac{(a_1+a_2) \pm \sqrt{(a_1-a_2)^2 + 4k(|b_1|+|b_2|)^2}}{2}$$

and semi-transverse axis of length

$$\sqrt{\frac{1}{2}\left|\frac{a_1-a_2}{2}\right|^2 - \frac{1}{4}(|b_1|+|b_2|)^2(1+|k|^2) + \frac{1}{2}|\gamma|}\,.$$

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(ii) If $|\gamma| = \frac{1}{2}(|b_1|+|b_2|)^2(1+|k|^2) - \left|\frac{a_1-a_2}{2}\right|^2$*, then* $W_J(A)$ *is the whole complex plane except the line with slope* $\frac{\arg(\gamma)+\pi}{2}$ *passing through* $\frac{a_1+a_2}{2}$*.*

(iii) If $|\gamma| < \frac{1}{2}(|b_1|+|b_2|)^2(1+|k|^2) - \left|\frac{a_1-a_2}{2}\right|^2$*, then* $W_J(A)$ *is the whole complex plane.*

Proof. The corollary is a simple consequence of the last theorem, by taking limits as the size of $A \in M_n$ tends to infinity. Having in mind that the eigenvalues of the matrix $C \in M_n$ in Theorem 2.2 are (cfr. [8])

$$\lambda = 0 \quad \text{and} \quad \lambda_r = \pm\sqrt{|b_1|^2+|b_2|^2+2|b_1||b_2|\cos\left(\frac{r\pi}{m+1}\right)}, \quad r = 1,\dots,m$$

for $n = 2m+1$ and

$$\lambda_r = \pm\sqrt{|b_1|^2+|b_2|^2+2|b_1||b_2|Q_r}, \quad r = 1,\dots,m$$

for $n = 2m$, where Q_r, $r = 1,\dots m$, are the roots of the polynomial $q_m(\mu)$ recurrently defined by

$$q_0(\mu) = 1, \quad q_1(\mu) = 1+\beta, \quad \beta^2 = \frac{|b_2|^2}{|b_1|^2}, \quad q_{m+1}(\mu) = \mu q_m(\mu) - q_{m-1}(\mu), \quad m > 1.$$

As n tends to infinity we can easily show that $\lambda_1^2 = (|b_1|+|b_2|)^2$. In the infinite case, the half-rays (2.7) are open. In fact, if their origins were attained, they would be eigenvalues of the infinite matrix $J\mathrm{Re}\,(e^{-i\theta}JB)$, which is impossible since under the hypothesis the matrix is non scalar. Thus, $W_J(A)$ is the open region bounded by the asserted hyperbola in (i). Now, the corollary straightforwardly follows. ◻

Remark 2.4. Given $A = H_A + i\,K_A \in M_n$, with H_A and K_A as in (2.2), the $J-$*generalized Levinger transform* of A (see [7]) is defined by

$$\mathcal{L}_J(A,\alpha,\beta) = \alpha H_A + \beta K_A, \quad \text{with} \quad \alpha,\beta \in \mathbb{R}.$$

For every $\alpha,\beta \in \mathbb{R}$, we clearly have

$$H_{\mathcal{L}(A,\alpha,\beta)} = \alpha H_A \;\text{ and }\; K_{\mathcal{L}(A,\alpha,\beta)} = -i\beta K_A.$$

Thus, we may write

$$W_J(\mathcal{L}_J(A,\alpha,\beta)) = \{\alpha\, x + i\,\beta\, y :\; x,y \in \mathbb{R},\; x+i\,y \in W_J(A)\}.$$

There is a relation between $W_J(A)$ and $W_J(\mathcal{L}_J(A,\alpha,\beta))$, in case the sets are hyperbolical. In fact, supposing that the boundary of $W_J(A)$ in the plane (u,v) has equation

$$\frac{u^2}{M^2} - \frac{v^2}{N^2} = 1, \quad M,N > 0,$$

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for $\alpha \neq 0$ and $\beta > 0$, changing the variables $u = \alpha^{-1}X$ and $v = \beta^{-1}Y$, then the boundary of $W_J(A)$ is the hyperbola

$$\frac{X^2}{\alpha^2 M^2} - \frac{Y^2}{\beta^2 N^2} = 1.$$

3. Algorithm and Examples. As an heuristic tool, it is convenient to have a code to produce the plot of $W_J(A)$. In [10], a *Matlab* program for plotting $W_J^+(A)$, $A \in M_n$, was presented and the authors mention that there is place to improvement. We would like to observe that in some cases, such as in Example 3.4 (Figure 3.3), this program fails. We include a *Matlab* program to generate Krein spaces numerical ranges of arbitrary complex matrices that treats the degenerate cases and represents the boundary generating curves. As an essential complement, an algorithm is given for computing the pseudoconvex hull of a finite number of points. The accuracy of our program is quite good because a routine, namely *rounding.m*, was implemented to remove the rounding errors in the program. Its speed is equivalent to the one of the program in [10]. We emphazise that our program plots the boundary generating curves and their pseudoconvex hull. Moreover, it also works for Hilbert spaces numerical ranges.

Our approach uses the elementary idea that the boundary, $\partial W_J(A)$, may be traced by computing the extreme eigenvalues (as specified below) of $J\mathrm{Re}\left(e^{-i\theta}JA\right)$ in $\sigma^+(J\mathrm{Re}\left(e^{-i\theta}JA\right))$ and in $\sigma^-(J\mathrm{Re}\left(e^{-i\theta}JA\right))$, and the associated eigenvectors ν_θ^+ and ν_θ^-, for θ running over a finite discretization of $0 \leq \theta < \pi$. The points

$$\frac{\nu_\theta^{+*} JA\nu_\theta^+}{\nu_\theta^{+*} J\nu_\theta^+} \quad \text{and} \quad \frac{\nu_\theta^{-*} JA\nu_\theta^-}{\nu_\theta^{-*} J\nu_\theta^-}$$

are boundary points of $W_J^+(A)$ and $W_J^-(A)$, respectively.

To describe the algorithm, we recall the concepts of noninterlacing eigenvalues and of pseudoconvex hull of a set of points.

Let H be a $J-$Hermitian matrix whose eigenvalues are all real and $\alpha_1 \geq \cdots \geq \alpha_r \in \sigma^+(H)$ and $\alpha_{r+1} \geq \cdots \geq \alpha_n \in \sigma^-(H)$. If $\alpha_r > \alpha_{r+1}$ or $\alpha_n > \alpha_1$, we say that the eigenvalues of H *do not interlace.*

Consider a set of points $P = \{p_1, \ldots, p_k\} \subset \mathbb{R}^k$ with associated signs $\{\epsilon_1, \ldots, \epsilon_k\}$, where $\epsilon_j = \pm 1$, $j = 1, \ldots, k$. The *pseudoconvex hull* of P is the set of the *pseudo convex* combinations of points, that is, the set of the form

$$\left\{ \frac{\sum_{j=1}^k t_j \epsilon_j p_j}{\sum_{j=1}^k \epsilon_j t_j} : t_j \geq 0,\ j = 1, 2, \ldots, k,\ \ \sum_{j=1}^k \epsilon_j t_j \neq 0 \right\}.$$

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If the ϵ_j are all equal, then the pseudoconvex hull reduces to the convex hull.

Step 1: For an arbitrary complex matrix A of order n, compute the eigenvalues of the matrix $J\mathrm{Re}\left(e^{-i\theta_r}JA\right)$, with

$$\theta_r = \frac{\pi(r-1)}{2m}, \quad r = 1, \ldots, 2m$$

for some positive integer m and for some involutive Hermitian matrix J. Construct the vector formed by all the values of r such that the matrix $J\mathrm{Re}\left(e^{-i\theta_r}JA\right)$ has at least one real eigenvalue. For each choice of r, test the multiplicity of the eigenvalues. If there exists at least one multiple eigenvalue, perturb the direction θ_r. If the above mentioned vector is nonempty and there exists at least a value of r such that the eigenvalues of the matrix $J\mathrm{Re}\left(e^{-i\theta_r}JA\right)$ are all real with anisotropic eigenvectors, go to Step 2. Otherwise, go to Step 5 and we have a degenerate case.

Step 2: For each θ_r described above, compute the eigenvalues of the matrix $J\mathrm{Re}\left(e^{-i\theta_r}JA\right)$, and the associated eigenvectors $\xi_i(r), i = 1, \ldots, n$. Evaluate

$$\rho_i(r) = \frac{\xi_i(r)^* JA\xi_i(r)}{\xi_i(r)^* J\xi_i(r)}, \ i = 1, \ldots, n$$

and construct two vectors formed by the elements $\rho_i(r)$ such that the sign of the scalar $\xi_i(r)^* J\xi_i(r)$ is $+1$ and -1, respectively. The components of these vectors produce points of the boundary generating curves of $W_J(A)$.

Step 3: Investigate the existence of directions for which the eigenvalues of the matrix $J\mathrm{Re}\left(e^{-i\theta_r}JA\right)$ do not interlace. If they exist, go to the next step. Otherwise, follow to Step 5.

Step 4: Compute the pseudoconvex hull of the boundary generating curves of $W_J(A)$.

Step 5: Compute $\dfrac{\zeta_i^* JA\zeta_i}{\zeta_i^* J\zeta_i}$ for a sample of anisotropic vectors ζ_i randomly chosen. The distribution of these points allows to conclude whether $W_J(A)$ is the complex plane (possibly the complex plane without a line) or a line (possibly without a point).

Remark 3.1. Obviously, if $J = I, -I$, then $W_J(A)$ reduces to the classical numerical range. In this case, the algorithm consists of the Steps 1, 2, 4, where the pseudoconvex hull gives rise to the convex hull. Thus, our algorithm and program also work for the classical case.

We illustrate Theorem 2.2 with the following example. The points represented by a cross are the eigenvalues of the matrix A. The $J-$numerical ranges presented in this note were generated by our *Matlab* program.

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EXAMPLE 3.2. Let $A \in M_6$ be the tridiagonal matrix with $a_1 = 2$, $a_2 = -2$, $b_j = i$ and $c_j = -2i$, $j = 1, \ldots, 5$. According to Theorem 2.2, $W_J(A)$ is bounded by the hyperbola centered at $(0,0)$ and with semi-transverse axis of length approximately 1.79. For $\sigma^+(H_A) = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\sigma^+(A) = \{\gamma_1, \gamma_2, \gamma_3\}$ increasingly ordered and $\beta_j = \sqrt{\gamma_j^2 - \alpha_j^2}$, $j = 1, \ldots, 6$, the line equation of the boundary generating curve is

$$(w^2 - \alpha_1^2 u^2 + \beta_1^2 v^2)(w^2 - \alpha_2^2 u^2 + \beta_2^2 v^2)(w^2 - \alpha_3^2 u^2 + \beta_3^2 v^2) = 0.$$

The foci of the hyperbolas in Figure 3.1 are the eigenvalues of A.

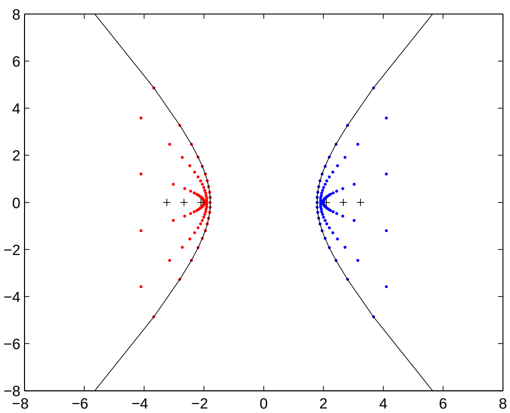

FIG. 3.1. $W_J(A)$ *and boundary generating curves for the matrix of Example 3.2.*

Observe that the boundary of the $J-$numerical range of a tridiagonal matrix with biperiodic main diagonal may not be hyperbolic if the super and subdiagonals do not satisfy the conditions in Theorem 2.2.

EXAMPLE 3.3. Let $J = I_1 \oplus -I_1 \oplus I_1 \oplus -I_1 \oplus I_1 \oplus -I_1$ and let $A \in M_6$ be the tridiagonal matrix with $a_1 = 2$, $a_2 = -2$, $b_j = 1$, $c_j = -1$ for j odd and $c_j = 1$ for j even. There are two flat portions on the boundary, namely the line segments $[\sqrt{3} + i\sqrt{6}/6, \sqrt{3} - i\sqrt{6}/6]$ and $[-\sqrt{3} + i\sqrt{6}/6, -\sqrt{3} - i\sqrt{6}/6]$. The line equation of the boundary generating curve is

$$-27u^6 + w^2(v^2 + w^2)^2 + 3u^4(8v^2 + 9w^2) - u^2(4v^4 + 14v^2w^2 + 9w^4) = 0$$

and it is not factorizable (cf. Figure 3.2).

We illustrate the algorithm with another example.

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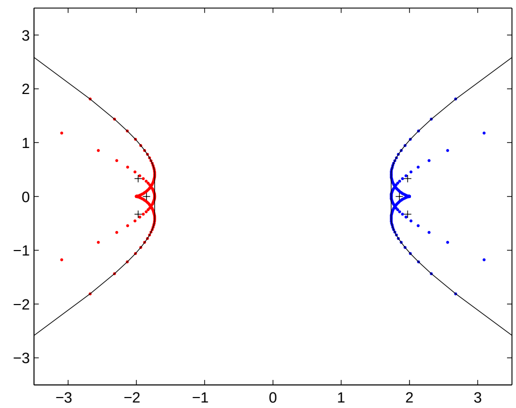

FIG. 3.2. *$W_J(A)$ and boundary generating curves for the matrix of Example 3.3.*

EXAMPLE 3.4. Let

$$A = \begin{bmatrix} 4 & 0 & -1 & 0 & 0 & 0 \\ 0 & -4 & 0 & -1 & 0 & 0 \\ 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4+2\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -4+2\sqrt{2} \end{bmatrix}.$$

The $J-$numerical range of A has one flat portion on the boundary, namely the line segment $[4+i, 4-i]$ (cf. Figure 3.3).

4. *Matlab* program.

In this section, we present the code for plotting the points defining the boundary generating curves of the $J-$numerical range of an arbitrary complex matrix. The program is listed below and is also available at the following website:

http://www.mat.uc.pt/∼bebiano

The below mentioned routines and a routine for plotting the pseudoconvex hull of the boundary generating curves can also be found there.

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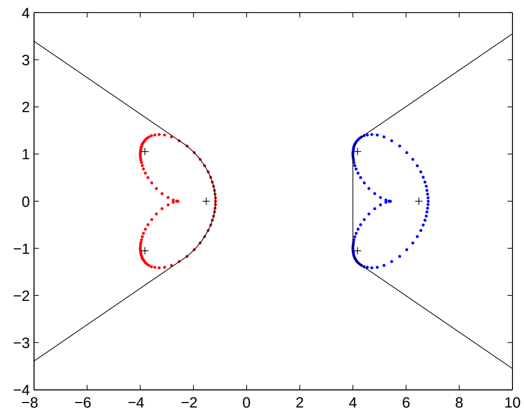

FIG. 3.3. *$W_J(A)$ and boundary generating curves for the matrix of Example 3.4.*

MATLAB PROGRAM FOR PLOTTING THE BOUNDARY GENERATING CURVE OF THE $J-$NUMERICAL RANGE OF A COMPLEX MATRIX

```
%
%boundary_curve(A,J,m,Tol), where J is the J-Hermitian matrix that defines
%the indefinite inner product, A is the Krein space matrix for which
%the program computes points in the Krein space numerical range, 2m
%is the number of directions and Tol>0 is the considered tolerance.
%
function [X1_round,X2_round,degenerate1,degenerate2]=boundary_curve(A,J,m,Tol)
%
global definite
X1_round=[]; X2_round=[];
%
if ~isequal(size(J,1),size(J,2))
    error('J must be a square matrix');
end
if ~isequal(size(A,1),size(A,2))
    error('A must be a square matrix');
end
if ~isequal(size(J),size(A))
    error('A and J must have the same size');
end
for r=1:size(J,2)
    for s=1:size(J,2)
        if r~=s && J(r,s)~=0 || J(r,r)~=1 && J(r,r)~=-1
            error('J must be an involutive Hermitian matrix');
```

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ELA


```
        end
    end
end
%
% Evaluation of the vector described in Step 1
[degenerate1,direc,eig_real]=directions(A,J,m,Tol);
%
% Evaluation of the points of the boundary generating
degenerate2=0;
if degenerate1~=1
    row1=1; row2=1; X1=[]; X2=[]; vec=[];
    for t=1:size(direc,2)
        D1=[]; D2=[];
        w=direc(t);
        T=(exp(pi*i*(w-1)/(2*m))*A+ exp(-pi*i*(w-1)/(2*m))*J*A'*J)/2;
        [U,D]=eig(T);
        for s=1:size(U,2)
            u=U(:,s);
            if abs(real(u'*J*u))>=Tol %no null J-norm
                z2=real(u'*J*A*u)/real(u'*J*u);
                z3=imag(u'*J*A*u)/real(u'*J*u);
                if real(u'*J*u)>0 %positive J-norm
                    X1(row1)=z2+i*z3;
                    row1=row1+1;
                    D1=[D1 D(s,s)];
                else
                    X2(row2)=z2+i*z3;
                    row2=row2+1;
                    D2=[D2 D(s,s)];
                end
            end
        end
        for r=1:size(eig_real,2)
            if eig_real(r)==w
               [interla]=interlacing(D1,D2);
               vec=[vec interla];
            end
        end
    end
    %
    %Cheking if there exists at least one direction
    %with noninterlacing eigenvalues
    aux=0;
    for t=1:size(vec,2)
```

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```
        if vec(t)==2
            aux=1; % Noninterlacing eigenvalues in the direction w=eig_real(t)
            break;
        end
    end
    %
    if aux==1 || definite==1
        [X1_round]=rounding(X1); %Remove of rounding errors of X1
        if definite==0
            [X2_round]=rounding(X2); %Remove of rounding errors of X2
        else
            X2_round=[]; %Definite case
        end
        %
        %Plot of the boundary generating curves
        plot(real(X1_round),imag(X1_round),'.b');
        hold on;
        plot(real(X2_round),imag(X2_round),'.r');
        hold on;
    else
        degenerate2=1; %Degenerate cases.
        return;
    end
end
```

Acknowledgment: The authors are very grateful to the referee for valuable suggestions.

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