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PARAMETRIZED SOLUTIONS X OF THE SYSTEM $AXA = AEA$ AND $A^kEAX = XAEA^k$ FOR A MATRIX A HAVING INDEX k*

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Abstract. Let A and E be $n \times n$ given complex matrices. This paper provides a necessary and sufficient condition for the solvability to the matrix equation system given by $AXA = AEA$ and $A^kEAX = XAEA^k$, for k being the index of A. In addition, its general solution is derived in terms of a G-Drazin inverse of A. As consequences, new representations are obtained for the set of all G-Drazin inverses; some interesting applications are also derived to show the importance of the obtained formulas.

Key words. Matrix equations, Index, G-Drazin inverses, Generalized inverses.

AMS subject classifications. 15A09, 15A24.

1. Introduction. Matrix equations arise in Matrix Theory as a valuable tool to solve an extensive and varied kind of problems. Sometimes, they appear from stating the model; other times, once the model is stated, after discretizing a continuous-time equation, etc. They have important applications in areas such as physics, mechanics, control theory, and many other fields [25]. For instance, the problem of finding all the solutions of the classical Yang-Baxter matrix equation $AXA = XAX$ is a representative case [14, 17, 21]. On the other hand, some recent results finding algebraic solutions for operator equations such as $AXB = B = BXA$ can be found in [22, 26].

Generalized inverses of matrices are very useful, among other topics, to tackle problems like the aforementioned ones and they play an important role in the study of matrix equations and partial orders [7, 8, 10, 11, 24, 27]. For example, the equation $AX+YB = C$ or the more general version $AXB+CYD = E$ (in the unknowns X and Y) were solved by Baksalary and Kala in [1, 2] by means of projectors associated to generalized inverses.

On the other hand, among other authors, Dinčić investigated the Sylvester equation $AX - XB = C$ under the condition $\sigma(A) \cap \sigma(B) \neq \emptyset$ in [12] by using the Jordan canonical form.

In this paper, we completely solve a related matrix equation system and provide some applications.

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. For $A \in \mathbb{C}^{m\times n}$, let A^*, A^{-1}, $\mathrm{rk}(A)$, and $\mathcal{R}(A)$ denote the conjugate transpose, the inverse ($m = n$), the rank, and the range space of A, respectively. Moreover,

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I_m stands for the $m \times m$ identity matrix and $0_{m\times n}$ denotes the $m \times n$ zero matrix. If the size is clear from the context, it will be directly denoted by I or 0. For $A \in \mathbb{C}^{n\times n}$, the *index* of A is the smallest nonnegative integer k such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$, and is denoted by $k = ind(A)$. Let $A \in \mathbb{C}^{n\times n}$ with $k = ind(A)$. We recall that the *Drazin inverse* of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ such that (2) $XAX = X$, (5) $AX = XA$, and (6) $A^{k+1}X = A^k$ hold, and is denoted by A^D. If A satisfies $ind(A) \leq 1$, then the Drazin inverse of A is called the *group inverse* of A and is denoted by $A^{\#}$. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equation $AXA = A$ is called a $\{1\}$-*inverse* of $A \in \mathbb{C}^{m\times n}$, and it is denoted by A^-. The symbol $A\{1\}$ denotes the set of all $\{1\}$-inverses of A. We also recall that the *Moore-Penrose inverse* of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times m}$ such that (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^* = AX$, and (4) $(XA)^* = XA$ hold, and is denoted by $A^\dagger$. A detailed analysis of all these generalized inverses can be found, for example, in [3].

The Drazin inverse was defined in [13] and it has proved helpful in analyzing Markov chains, difference and differential equations, iterative procedures, etc., as we can see in [4, 5]. Among other applications, this generalized inverse can be used to construct a generalized transfer function for singular descriptor systems [19]. Taking into account its importance, many computational techniques have been developed to calculate it. Using a determinantal technique, another matrix equation problems related to Drazin inverse were investigated by Krychei in [16] by solving (for X) the weighted matrix equations $WAWX = D$, $XWAW = D$, and $W_1AW_1XW_2BW_2 = D$ on quaternions.

Campbell and Meyer introduced in [6] some modifications to the classic Drazin inverse by introducing weak Drazin inverses. A particular case of weak Drazin inverses was defined by H. Wang and X. Liu in [23]. More precisely, for a given $A \in \mathbb{C}^{n\times n}$ of index k, a matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$AXA = A, \quad XA^{k+1} = A^k, \quad \text{and} \quad A^{k+1}X = A^k, \tag{1.1}$$

is called a *G-Drazin inverse* of A, which is, in general, not unique. The symbol $A\{GD\}$ stands for the set of all G-Drazin inverses of A; an element of this set is denoted by A^{GD}. Recently, it was proved [9] that the set of the equations (1.1) is equivalent to the more simplified one given by

$$AXA = A \quad \text{and} \quad A^kX = XA^k. \tag{1.2}$$

The purpose of this paper is to study the matrix equation system given by

$$AXA = AEA \quad \text{and} \quad A^kEAX = XAEA^k, \tag{1.3}$$

for a matrix $A \in \mathbb{C}^{n\times n}$ of index k, and a fixed matrix $E \in \mathbb{C}^{n\times n}$. Clearly, the matrix equation system (1.3) is more general than (1.2) because, for example, by setting $E = A^\dagger$ in (1.3) we get (1.2). We obtain general expressions of solutions of the system (1.3) and derive new formulas for all solutions of the system (1.2). We highlight the importance that the fact of making available a parametrized solution as that provided in this paper gives us the opportunity of using it in further applications, such as occurred with the one quoted in Lemma 2.1 below.

The paper is organized as follows. In Section 2, we provide a necessary and sufficient condition for the solvability to the matrix equation system (1.3) and we find all its solutions by giving a parametrized expression. In Section 3, we derive new representations for the set of all G-Drazin inverses of A. In particular, we give (Subsection 3.1) a more simplified representation for all G-Drazin inverses of a matrix A of index one. Moreover, we obtain (Subsection 3.2) some necessary conditions for any G-Drazin inverse of a matrix B to be a G-Drazin inverse of A.

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2. A parametrized general solution. In this section, we extend the system given by (1.2) to a more general case, by introducing a sort of G-Drazin inverse of A with respect to a fixed matrix E. We solve this new system by finding a parametrized solution.

By mimicking the definition of affine linear variety of a linear system $Ax = b$ (for given $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$), which is given by the set of all its solutions $\{x \in \mathbb{C}^n : Ax = b\} = x_0 + S_H$ (where $Ax_0 = b$ and $S_H = \{x \in \mathbb{C}^n : Ax = 0\}$ is a vector subspace), we introduce the affine linear variety S of all solutions of (1.3):

$$S = \{X \in \mathbb{C}^{n\times n} : X \text{ satisfies equations in (1.3)}\},$$

the corresponding vector subspace

$$S_H = \left\{X \in \mathbb{C}^{n\times n} : AXA = 0 \text{ and } A^kEAX = XAEA^k\right\}$$

associated to the homogeneous case, and a fixed (but arbitrary) element $X_0 \in S$. It is also true that $S = X_0 + S_H$ holds.

The following result will be useful in what follows.

Lemma 2.1. ([3, p. 52]) *Let $A, B, C \in \mathbb{C}^{n\times n}$. The matrix equation $AXB = C$ is consistent if and only if $AA^-CB^-B = C$ for some $A^- \in A\{1\}$ and $B^- \in B\{1\}$, in which case, the general solution of the equation is given by $X = A^-CB^- + Z - A^-AZBB^-$, for arbitrary $Z \in \mathbb{C}^{n\times n}$.*

It is well known that matrices A^- and B^- in Lemma 2.1 can be arbitrarily chosen as $\{1\}$-inverses of A and B, respectively. Next result will use the notations $P_A := AA^-$ and $Q_A := A^-A$, where A^- is any fixed $\{1\}$-inverse of A.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k and $E \in \mathbb{C}^{n\times n}$ be a fixed matrix. Then the matrix equation system*

$$AXA = AEA \quad \text{and} \quad A^kEAX = XAEA^k, \tag{2.4}$$

is consistent if and only if

$$A^{k+1}(EA)^2 = (AE)^2A^{k+1} \tag{2.5}$$

holds. In this case, if A^{GD} is a fixed G-Drazin inverse of A, the general solution of (2.4) is given by

$$X = A^{GD}AEAA^{GD} + T - Q_{A^kEA}T(I - P_A) - (I - Q_A)TP_{AEA^k} - Q_ATP_A, \tag{2.6}$$

for arbitrary $T \in \mathbb{C}^{n\times n}$.

Proof. Assume that X is any solution of (2.4). Then, pre- and post-multiplying by A the equality $A^kEAX = XAEA^k$, we arrive at $A^{k+1}(EA)^2 = (AE)^2A^{k+1}$, since $AXA = AEA$. Hence, (2.5) holds.

For the converse, let A^{GD} be a fixed G-Drazin inverse of A. If (2.5) holds, then $X_0 := A^{GD}AEAA^{GD}$ is a solution of (2.4). In fact, we first note that $AA^{GD}A = A$ holds. Now,

$$AX_0A = AA^{GD}AEAA^{GD}A = AEA.$$

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Moreover,

$$\begin{aligned}A^kEAX_0 &= A^kEAA^{GD}AEAA^{GD} = A^k(EA)^2A^{GD} = A^{GD}A^{k+1}(EA)^2A^{GD}\\ &= A^{GD}(AE)^2A^{k+1}A^{GD} = A^{GD}(AE)^2A^k = A^{GD}AEAA^{GD}AEA^k\\ &= X_0AEA^k.\end{aligned}$$

Hence, both equations in (2.4) hold. It then follows that $S = A^{GD}AEAA^{GD} + S_H$.

In order to obtain the general solution, it is enough to determine S_H. Applying Lemma 2.1 to the equation $AXA = 0$, its general solution is given by

$$X = Z - Q_AZP_A, \tag{2.7}$$

for arbitrary $Z \in \mathbb{C}^{n\times n}$. Now, by substituting (2.7) in $A^kEAX = XAEA^k$ and by virtue of the equalities $A^kEAQ_A = A^kEA$ and $P_AAEA^k = AEA^k$, we obtain

$$A^kEAZ(I - P_A) = (I - Q_A)ZAEA^k. \tag{2.8}$$

Post-multiplying the equation (2.8) by the projector AA^{GD}, we have

$$(I - Q_A)ZAEA^k = 0. \tag{2.9}$$

Since $I - Q_A$ is a projector, $I - Q_A \in (I - Q_A)\{1\}$ holds. Then, by Lemma 2.1, the general solution (in Z) of (2.9) is given by

$$Z = W - (I - Q_A)^-(I - Q_A)WAEA^k(AEA^k)^- = W - (I - Q_A)WAEA^k(AEA^k)^-, \tag{2.10}$$

for any $(AEA^k)^- \in (AEA^k)\{1\}$ and arbitrary $W \in \mathbb{C}^{n\times n}$. Thus, by using (2.10) and the identity $A(I-Q_A) = 0$, we obtain

$$A^kEAZ(I - P_A) = A^kEAW(I - P_A). \tag{2.11}$$

However, W will be constrained in some sense because (2.8) must be satisfied. In fact, from (2.8), (2.9), and (2.11) we have

$$A^kEAW(I - P_A) = 0. \tag{2.12}$$

Again, by applying Lemma 2.1 to the equation (2.12) (now in the unknown W) and using that $I - P_A \in (I - P_A)\{1\}$, we get

$$W = T - (A^kEA)^-A^kEAT(I - P_A), \tag{2.13}$$

for any $(A^kEA)^- \in (A^kEA)\{1\}$ and arbitrary $T \in \mathbb{C}^{n\times n}$. Then, from (2.10) and (2.13), we obtain $Z = T - (A^kEA)^-A^kEAT(I - P_A) - (I - Q_A)TAEA^k(AEA^k)^-$, for arbitrary $T \in \mathbb{C}^{n\times n}$, because $(I - P_A)A = 0$. Finally, by substituting this last expression for Z in (2.7) and by taking into account again the identities $(I - P_A)P_A = 0$ and $Q_A(I - Q_A) = 0$, we arrive at

$$S_H = \{T - Q_{A^kEA}T(I - P_A) - (I - Q_A)TP_{AEA^k} - Q_ATP_A : T \text{ is arbitrary}\}. \tag{2.14}$$

Then, the proof is complete. ∎

Remark 2.3. If the condition (2.5) is not fulfilled in Theorem 2.2, we have that $S = \emptyset$.

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3. Some applications. We can derive some interesting applications, by particularizing the parameter E considered in the statement of the problem (1.3).

3.1. General representations for G-Drazin inverses. In this subsection, we obtain all parametrized solutions for the set of all G-Drazin inverses of a matrix A based on any particular G-Drazin inverse of A. In particular, we obtain more simplified representations for all G-Drazin inverses of a matrix A of index one.

We start with the following auxiliary lemma.

LEMMA 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k. If $X \in A\{GD\}$, then the following assertions are true:*

(i) $A^kX^k = X^kA^k$, (ii) $X^k \in A^k\{GD\}$.

Proof. From $A^kX = XA^k$, it is obvious that (i) holds. If $k \le 1$, the assertion (ii) trivially holds. For $k \ge 2$, $X^k \in A^k\{1\}$ since $A^kX^kA^k = X^kA^kA^k = X^{k-1}(XA^{k+1})A^{k-1} = X^{k-1}A^kA^{k-1} = X^{k-2}(XA^{k+1})A^{k-2} = X^{k-2}A^kA^{k-2} = \cdots = XA^kA = A^k$. As $ind(A^k) = 1$, from (i) we get (ii). ◻

Now, by setting $E = A^\dagger$ in Theorem 2.2, (1.3) reduces to (1.2), and using Lemma 3.1 we can derive an interesting result for G-Drazin inverses; namely, all parametrized solutions of the (equivalent) equation system that define them.

THEOREM 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k and let A^{GD} be a fixed G-Drazin inverse of A. Then the set of all G-Drazin inverses of A is given by*

$$A\{GD\} = \{A^{GD} + (I - P_{A^k})T(I - P_A) + (I - Q_A)T(I - P_{A^k}) - (I - Q_A)T(I - P_A) : T \text{ is arbitrary}\}, \tag{3.15}$$

or equivalently,

$$A\{GD\} = \{A^{GD} + (I - P_{A^k})U(I - P_A) + (I - Q_A)V(I - P_{A^k}) : U, V \text{ are arbitrary}\}, \tag{3.16}$$

where $P_A = AA^{GD}$, $Q_A = A^{GD}A$, and $P_{A^k} = A^k(A^{GD})^k$.

Proof. Set $E = A^\dagger$. We recall that the solution set of the system (1.3) is given by $S = X_0 + S_H$ for some (fixed but arbitrary) X_0. For a fixed $A^{GD} \in A\{GD\}$, notice that clearly $X_0 := A^{GD}$ is a solution of (1.2). In particular, $A^{GD} \in A\{1\}$. Now, by Lemma 3.1, we have $(A^{GD})^k \in A^k\{1\}$. Setting $A^- = A^{GD}$ and $(AEA^k)^- = (A^k)^- = (A^{GD})^k$ in Theorem 2.2, projectors become $P_A = AA^{GD}$, $Q_A = A^{GD}A$, $P_{A^k} = A^k(A^{GD})^k$, and $Q_{A^k} = (A^{GD})^kA^k$. By Lemma 3.1, it follows that $P_{A^k} = Q_{A^k}$. Since $-Q_{A^k} = -P_{A^k} = (I - P_{A^k}) - I$, we have $-Q_{A^k}T(I - P_A) = (I - P_{A^k})T(I - P_A) - T(I - P_A)$. Similarly, $(I - Q_A)T(-P_{A^k}) = (I - Q_A)T(I - P_{A^k}) - T + Q_AT$. Now, some computations in (2.14) lead to (3.15). In order to see the equality between expressions in (3.15) and (3.16), we first choose $T = U(I - P_A) + (I - Q_A)V$, with arbitrary U and V to be substituted in (3.15) and by using the facts that $ATA = 0$ and $P_AP_{A^k} = P_{A^k}Q_A = P_{A^k}$, some calculations yield (3.16). For the other inclusion, by setting $U = T$ and $V = (I - Q_A)T - T(I - P_A)$, with arbitrary T, (3.15) can be easily deduced from (3.16). ◻

REMARK 3.3. Observe that while expression (3.16) for $A\{GD\}$ is expressed in terms of two parameters (namely, U and V), expression (3.15) requires only one (namely, T).

COROLLARY 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index one and let A^{GD} be a fixed G-Drazin inverse of A. Then the set of all G-Drazin inverses of A is given by*

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$$A\{GD\} = \{A^{GD} + (I - A^{GD}A)T(I - AA^{GD}) : T \text{ is arbitrary}\}.$$

Corollary 3.5. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k. Then the general solution of the system*

$$AXA = AA^DA \quad \text{and} \quad A^kX = XA^k \tag{3.17}$$

is given by

$$X = A^D + T - AA^DT + AA^DTP_A - TAA^D + Q_ATAA^D - Q_ATP_A,$$

for arbitrary $T \in \mathbb{C}^{n\times n}$.

Proof. By setting $E = A^D$ in Theorem 2.2, (1.3) reduces to (3.17). Since AA^D is a projector and $AA^D = A^DA$, it is easy to see that $A^{GD}AA^D = A^DAA^{GD} = A^D$ and $(A^D)^k$ is a $\{1\}$-inverse of A^k. Thus, the particular solution has the form $A^{GD}AA^DAA^{GD} = A^D$. Now, applying Theorem 2.2 with $(A^k)^- = (A^D)^k$, we obtain the result. ∎

Next, as a consequence, we derive the following result about commuting $\{1\}-$inverses which was proved in [20, Lemma 4.5.2].

Corollary 3.6. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index 1. Then the general solution of the system $AXA = A$ and $AX = XA$ is given by $X = A^\# + (I - A^\#A)T(I - AA^\#)$, for arbitrary $T \in \mathbb{C}^{n\times n}$.*

3.2. What about range spaces of A and B whenever $B\{GD\} \subseteq A\{GD\}$ holds?. This subsection gives necessary conditions for any G-Drazin inverse of a matrix B to be a G-Drazin inverse of A. Next lemma plays an important role in the subsequent result.

Lemma 3.7. *Let $A, B \in \mathbb{C}^{n\times n}$. If $AXB = 0$ for all $X \in \mathbb{C}^{n\times n}$, then $A = 0$ or $B = 0$.*

Proof. Let $E_{i,j}$ be the matrix whose entries are all 0's except its (i, j)-entry which is 1. Then, we can consider $A = [a_{ij}]$, and note that $0 = E_{kj}AE_{ik}B = a_{ij}E_{kk}B$ for all integer $1 \le k \le n$. Hence, $a_{ij}B = 0$ for all integer $1 \le i, j \le n$ from which we conclude that $A = 0$ or $B = 0$. ∎

In [23], the condition $B\{GD\} \subseteq A\{GD\}$ was studied related to the $\mathcal{G}$-based partial order that the G-Drazin determines. Next result goes more deeply in this sense.

Theorem 3.8. *Let $A, B \in \mathbb{C}^{n\times n}$ be such that $B\{GD\} \subseteq A\{GD\}$. Let $k = ind(A)$ and $\ell = ind(B)$. Then the following properties hold:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ *or* $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$, (ii) $\mathcal{R}(A^k) \subseteq \mathcal{R}(B^\ell)$ *and* $\mathcal{R}((A^k)^*) \subseteq \mathcal{R}((B^\ell)^*)$.

Proof. If $\ell = 0$, items (i) and (ii) are obvious. In order to show (i) and (ii) for $\ell \ge 1$, we consider a fixed G-Drazin inverse B^{GD} of B. By using the projectors $P_B = BB^{GD}$, $Q_B = B^{GD}B$, and $P_{B^\ell} = B^\ell(B^{GD})^\ell$, from Theorem 3.2, it follows that the formula

$$X = B^{GD} + (I - P_{B^\ell})U(I - P_B) + (I - Q_B)V(I - P_{B^\ell}), \tag{3.18}$$

provides all G-Drazin inverses of B for arbitrary $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{n\times n}$.

(i) Since $X, B^{GD} \in B\{GD\} \subseteq A\{GD\}$, the equalities $AXA = AB^{GD}A = A$ hold. Hence, by setting $U = 0$ in (3.18), it follows that $A(I - Q_B)V(I - P_{B^\ell})A = 0$ for arbitrary V. Therefore, from Lemma 3.7 we get $(I - P_{B^\ell})A = 0$ or $A(I - Q_B) = 0$, which is equivalent to $\mathcal{R}(A) \subseteq \mathcal{R}(B^\ell) \subseteq \mathcal{R}(B)$ or $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$, respectively.

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(ii) Since $X, B^{GD} \in B\{GD\} \subseteq A\{GD\}$, we have $XA^{k+1} = B^{GD}A^{k+1} = A^k$. By setting $U = 0$ in (3.18) we get $(I - Q_B)V(I - P_{B^\ell})A^k = (I - Q_B)V(I - P_{B^\ell})A^{k+1}A^D = 0$, for arbitrary V. Now, Lemma 3.7 implies $(I - P_{B^\ell})A^k = 0$ because $\ell \neq 0$. Hence, $\mathcal{R}(A^k) \subseteq \mathcal{R}(B^\ell)$. Similarly, by setting $V = 0$ in (3.18) we arrive at $A^k(I - P_{B^\ell})U(I - P_B) = 0$, for arbitrary U. Then $A^k(I - P_{B^\ell}) = 0$, i.e., $A^k = A^kB^\ell(B^{GD})^\ell = A^k(B^{GD})^\ell B^\ell$ by Lemma 3.1. Thus, $\mathcal{R}((A^k)^*) \subseteq \mathcal{R}((B^\ell)^*)$. ◻

Remark 3.9. We point out that most of the results provided in this paper are also valid for Hilbert space operators, under an adequate interpretation. Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on $\mathcal{H}$. Now, the matrix equation system (1.3) should be interpreted as operator equations, where $A \in \mathcal{B}(H)$ has closed range and $k = \text{ind}(A)$ is the Drazin index of A (see [15]). The crucial fact for solving these operator equations is that Lemma 2.1 remains valid. In fact, if $A, B \in \mathcal{B}(\mathcal{H})$ have closed ranges, then $AXB = C$ has a solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $AA^-CB^-B = C$. In this case, the general solution is given by $X = A^-CB^- + Z - A^-AZB^-B$, for arbitrary $Z \in \mathcal{B}(\mathcal{H})$ (see [10]). Finally, we would like to highlight that is not necessary to use Hilbert space operator techniques for solving the new system. It is enough to apply strictly algebraic techniques such as those used in the proof of Theorem 2.2. In consequence, a natural further question is to ask about considering our results in rings.

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