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TRIDIAGONAL PAIRS OF TYPE III WITH HEIGHT ONE*

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Abstract. Let $\mathbb{K}$ denote an algebraically closed field with characteristic 0. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension, and let A, A^* denote a tridiagonal pair on V of diameter d. Let $V_0, \ldots, V_d$ denote a standard ordering of the eigenspaces of A on V, and let $\theta_0, \ldots, \theta_d$ denote the corresponding eigenvalues of A. It is assumed that $d \geq 3$. Let ρ_i denote the dimension of V_i. The sequence $\rho_0, \rho_1, \ldots, \rho_d$ is called the *shape* of the tridiagonal pair. It is known that $\rho_0 = 1$ and there exists a unique integer h $(0 \leq h \leq d/2)$ such that $\rho_{i-1} < \rho_i$ for $1 \leq i \leq h$, $\rho_{i-1} = \rho_i$ for $h < i \leq d-h$, and $\rho_{i-1} > \rho_i$ for $d - h < i \leq d$. The integer h is known as the *height* of the tridiagonal pair. In this paper, it is showed that the shape of a tridiagonal pair of type III with height one is either 1, 2, 2, ..., 2, 1 or 1, 3, 3, 1. In each case, an interesting basis is found for V and the actions of A, A^* on this basis are described.

Key words. Tridiagonal pair, Tridiagonal relation, Height, Shape.

AMS subject classifications. 05E30, 05E35, 33C45, 33D45.

1. Introduction. Throughout the paper, $\mathbb{K}$ denotes an algebraically closed field with characteristic 0 and V denotes a vector space over $\mathbb{K}$ with finite positive dimension.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all $\mathbb{K}$-linear transformations from V to V. For $A \in \mathrm{End}(V)$ and for a subspace $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case, θ is the *eigenvalue* of A associated with W. We say A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1. ([1, Definition 1.1]) By a *tridiagonal pair* on V, we mean an ordered pair $A, A^* \in \mathrm{End}(V)$ that satisfy (i)–(iv) below:

(i) Each of A, A^* is diagonalizable on V.
(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \tag{1.1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta), \tag{1.2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

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(iv) There is no subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

We say the pair A, A^* is over $\mathbb{K}$.

Let A, A^* denote a tridiagonal pair on V. By [1, Lemma 4.5], the integers d and δ from Definition 1.1 are equal; we call this common value the *diameter* of A, A^*. We assume that $d \geq 3$. An ordering of the eigenspaces of A (resp., A^*) is said to be *standard* whenever it satisfies (1.1) (resp., (1.2)). By [1, Corollary 5.7], for $0 \leq i \leq d$, the spaces V_i, V_i^* have the same dimension; we denote this common dimension by ρ_i. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of A, A^*. By a *Leonard pair* we mean a tridigonal pair with shape $(1, 1, \ldots, 1)$ [8, Definition 1.1]. See [5, 8, 9] for more information about Leonard pairs. The pair A, A^* is said to be *sharp* whenever $\rho_0 = 1$. By [7, Theorem 1.3], a tridiagonal pair over an algebraically closed field is sharp. See [2, 6] for more information about sharp tridiagonal pairs. By [1, Corollaries 5.7 and 6.6], $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq \frac{d}{2}$. By [3, Theorem 3.3], there exists a unique integer h with $0 \leq h \leq \frac{d}{2}$ such that $\rho_{i-1} < \rho_i$ for $1 \leq i \leq h$, $\rho_{i-1} = \rho_i$ for $h < i \leq d-h$ and $\rho_{i-1} > \rho_i$ for $d - h < i \leq d$. We call h the *height* of A, A^*.

It is known [1, Theorem 10.1] that there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $\mathbb{K}$ such that both

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] = 0,$$
$$[A^*, {A^*}^2A - \beta A^*AA^* + A{A^*}^2 - \gamma^*(A^*A + AA^*) - \varrho^* A] = 0,$$

where $[B, C] = BC - CB$. The sequence is unique if the diameter of the pair is at least 3. The above relations are known as the *tridiagonal relations*. The scalar β is called the *fundamental parameter* of A, A^*. A tridiagonal pair A, A^* is called of *type I, type II, type III* according to $\beta \neq \pm 2$, $\beta = 2$, $\beta = -2$ in the tridiagonal relations, respectively.

In [4], K. Nomura considered tridiagonal pairs A, A^* of type I with height one. He showed that the shape of a tridiagonal pair of height one is either $1, 2, 2, \ldots, 2, 1$ or $1, 3, 3, 1$. And he gave a basis for V and obtained the matrices representing A, A^* with respect to this basis. Motivated by [4], in this paper, we show that the shape of a tridiagonal pair of type III with height one is the same as type I in [4], that is, the shape is either $1, 2, 2, \ldots, 2, 1$ or $1, 3, 3, 1$. Then, in each case, we display a basis for V and give the actions of A, A^* on this basis.

In Section 2, we first recall some basic results from [1] and [3] concerning the split decomposition and the refined split decomposition of the tridiagonal pair. In Section 3, we define some linear transformations $X_i, Y_i, Z_i (0 \leq i \leq d)$, and use them to show that $RL^{(+)}$ (resp., $RL^{(-)}$) is a scalar multiple of $L^{(+)}R$ (resp., $L^{(-)}R$) on each $U^{(r)} \cap U_i$. We also give a linear relation between $RL^{(0)}, L^{(0)}R$ and I (the identity map) on each $U^{(r)} \cap U_i$. In Section 4, we find a basis for V. In Section 5, we obtain the action of L. In Section 6, we determine the shape of the tridiagonal pair of type III with height 1, which is Theorem 6.4. In Section 7, for all cases listed in Theorem 6.4, we display a basis for V and give the actions of A, A^* on this basis.

2. Refined split decomposition of a tridiagonal pair. In this section, we first recall some known facts about the split decomposition and the refined split decomposition of the tridaigonal pair.

With reference to Definition 1.1, for $0 \leq i \leq d$, we set

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d).$$

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By [1, Corollary 5.7], the dimension of U_i is equal to ρ_i for $0 \le i \le d$. And by [1, Theorem 4.6], the space V is decomposed as

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)}. \tag{2.3}$$

The decomposition given in (2.3) is called the *split decomposition* of the tridiagonal pair.

Let $F_i : V \to U_i$ denote the projection with respect to the direct sum (2.3). Then, for $0 \le i, j \le d$,

$$F_0 + F_1 + \cdots + F_d = I \quad \text{and} \quad F_i F_j = \delta_{ij} F_i.$$

The *raising map* R and the *lowering map* L are defined as follows:

$$R = A - \sum_{i=0}^{d} \theta_i F_i \quad \text{and} \quad L = A^* - \sum_{i=0}^{d} \theta_i^* F_i.$$

Lemma 2.1. ([1, Corollary 6.3])

(i) $RU_i \subseteq U_{i+1}$ $(0 \le i < d)$, $RU_d = 0$.
(ii) $LU_i \subseteq U_{i-1}$ $(0 < i \le d)$, $LU_0 = 0$.

Lemma 2.2. ([3, Lemma 2.7]) *Let W denote a subspace of V. Suppose that $RW \subseteq W, LW \subseteq W$ and $F_iW \subseteq W$ for $0 \le i \le d$. Then $W = 0$ or $W = V$.*

Next, we display some results concerning the refined split decomposition of the tridiagonal pair. For the rest of this paper, let h denote the *height* of the tridiagonal pair.

For $0 \le r \le h$ and $r \le i \le d - r$, we set

$$U_i^{(r)} = R^{i-r}(U_r \cap \text{Ker } R^{d-2r+1}).$$

Lemma 2.3. ([3, Lemma 4.1]) *For $0 \le r \le h$, the following hold.*

(i) $U_0^{(0)} = U_0$ *and* $U_d^{(0)} = U_d$,
(ii) $U_i^{(r)} \subseteq U_i$ $(r \le i \le d - r)$,
(iii) $U_r^{(r)} = U_r \cap Ker\ R^{d-2r+1}$,
(iv) $U_i^{(r)} = R^{i-r} U_r^{(r)}$ $(r \le i \le d - r)$,
(v) $RU_i^{(r)} = U_{i+1}^{(r)}$ $(r \le i \le d - r - 1)$, $RU_{d-r}^{(r)} = 0$,
(vi) *The restriction* $R\,|_{U_i^{(r)}} : U_i^{(r)} \to U_{i+1}^{(r)}$ *is a bijection* $(r \le i \le d - r - 1)$.

Lemma 2.4. ([3, Lemma 4.3]) *For $0 \le r \le h$,*

$$\dim U_i^{(r)} = \rho_r - \rho_{r-1} \quad (r \le i \le d - r),$$

where we set $\rho_{-1} = 0$.

Lemma 2.5. ([3, Lemmas 4.7 and 4.8]) *For $0 \le i \le d$,*

$$U_i = \sum_{r=0}^{m} U_i^{(r)} \quad (\textit{direct sum}),$$

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where $m = \min\{i, h, d-i\}$*. And* V *is decomposed as*

$$V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U_i^{(r)} \quad (\textit{direct sum}). \tag{2.4}$$

The decomposition given in (2.4) is called the *refinement split decomposition* of the tridiagonal pair.

For $0 \le r \le h$, we set

$$U^{(r)} = \sum_{i=r}^{d-r} U_i^{(r)}.$$

Lemma 2.6. ([3, Lemma 5.1]) V *is decomposed as*

$$V = \sum_{r=0}^{h} U^{(r)} \quad (\textit{direct sum}).$$

Lemma 2.7. ([3, Lemma 5.3]) *For* $0 \le r \le h$,

$$RU^{(r)} \subseteq U^{(r)}.$$

Let

$$F^{(r)} : V \to U^{(r)} \quad (0 \le r \le h)$$

denote the projection with respect to the direct sum $V = \sum_{r=0}^{h} U^{(r)}$. Then, for $0 \le r \le h$ and $0 \le s \le h$,

$$F^{(0)} + F^{(1)} + \cdots + F^{(h)} = I \quad \text{and} \quad F^{(r)}F^{(s)} = \delta_{rs}F^{(r)}.$$

We set

$$F_i^{(r)} = F_iF^{(r)} \quad (0 \le r \le h,\ 0 \le i \le d).$$

Lemma 2.8. ([3, Lemma 6.1]) *For* $0 \le r \le h$ *and* $0 \le i \le d$,

(i) $F_i^{(r)} = F^{(r)}F_i = F_iF^{(r)}$,
(ii) $F_0^{(0)} = F_0$ *and* $F_d^{(0)} = F_d$,
(iii) $F_i^{(r)} \ne 0$ *if and only if* $r \le i \le d-r$.

Lemma 2.9. ([3, Lemma 6.2]) *For* $0 \le r \le h$ *and* $r \le i \le d-r$, $F_i^{(r)}V = U_i^{(r)}$, *and*

$$F_i^{(r)} : V \to U_i^{(r)}$$

is the projection with respect to the direct sum $V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U_i^{(r)}$.

Lemma 2.10. ([3, Lemma 6.3]) *For* $0 \le r \le h$,

(i) $F^{(r)}R = RF^{(r)}$,
(ii) $F_i^{(r)}R = RF_{i-1}^{(r)} (1 \le i \le d)$,
(iii) $F_r^{(r)}R = 0$.

We set

$$L^{(-)} = \sum_{r=1}^{h} F^{(r-1)}LF^{(r)}, \quad L^{(0)} = \sum_{r=0}^{h} F^{(r)}LF^{(r)}, \quad L^{(+)} = \sum_{r=0}^{h-1} F^{(r+1)}LF^{(r)}. \tag{2.5}$$

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Lemma 2.11. ([3, Lemma 6.5])

$$L = L^{(-)} + L^{(0)} + L^{(+)}.$$

Lemma 2.12. ([3, Lemma 6.7]) *For $0 \le r \le h$, the following hold.*

(i) $L^{(0)} F_r^{(r)} = 0$,
(ii) $L^{(+)} F_r^{(r)} = L^{(+)} F_{r+1}^{(r)} = 0$.

Let A, A^* be a tridiagonal pair with the fundamental parameter β. For the rest of this paper, we always assume that $\beta = -2$.

Lemma 2.13. *For $0 \le i \le d-2$,*

$$\begin{aligned} (R^3L + R^2LR - RLR^2 - LR^3 - \varepsilon_i R^2)F_i = 0, \\ (L^3R + L^2RL - LRL^2 - RL^3 + \varepsilon_i L^2)F_{i+2} = 0, \end{aligned} \tag{2.6}$$

where

$$\varepsilon_i = (\theta_i - \theta_{i+2})(\theta^*_{i+1} - \theta^*_{i+2}) - (\theta^*_{i+2} - \theta^*_i)(\theta_{i+1} - \theta_i). \tag{2.7}$$

Proof. Immediate from [1, Theorem 12.1] and $\beta = -2$. ∎

Lemma 2.14. *The following relations hold.*

$$R^3L^{(+)} + R^2L^{(+)}R - RL^{(+)}R^2 - L^{(+)}R^3 = 0, \tag{2.8}$$

$$R^3L^{(-)} + R^2L^{(-)}R - RL^{(-)}R^2 - L^{(-)}R^3 = 0. \tag{2.9}$$

Proof. Immediate from [3, Theorem 6.8] and $\beta = -2$. ∎

Lemma 2.15. *For $0 \le r \le h$ and $0 \le i \le d-2$,*

$$(R^3L^{(0)} + R^2L^{(0)}R - RL^{(0)}R^2 - L^{(0)}R^3 - \varepsilon_i R^2)F_i^{(r)} = 0. \tag{2.10}$$

Proof. Immediate from [3, Lemma 6.9] and $\beta = -2$. ∎

Lemma 2.16. ([1, Theorem 11.2]) *Let $\{\theta_i\}_{i=0}^d$ (resp., $\{\theta^*_i\}_{i=0}^d$) denote an ordering of the eigenvalues of A (resp., A^*). Then there exist scalars a, a^*, b, b^*, c, c^* in $\mathbb{K}$ such that*

$$\theta_i = a + b(-1)^i + ci(-1)^i \quad (0 \le i \le d), \tag{2.11}$$

$$\theta^*_i = a^* + b^*(-1)^i + c^*i(-1)^i \quad (0 \le i \le d). \tag{2.12}$$

3. Relations between R and $L^{(+)}$ (resp., $L^{(-)}, L^{(0)}$). In this section, we show that $RL^{(+)}$ (resp., $RL^{(-)}$) is a scalar multiple of $L^{(+)}R$ (resp., $L^{(-)}R$) on each $U^{(r)} \cap U_i$. We also give a linear relation between $RL^{(0)}$, $L^{(0)}R$ and I (the identity map) on each $U^{(r)} \cap U_i$.

We first fix an integer r such that $0 \le r \le h-1$, and set

$$X_i := L^{(+)} F_i^{(r)} \quad (0 \le i \le d).$$

Lemma 3.1. $X_r = X_{r+1} = 0$.

Proof. Immediate from Lemma 2.12 (ii). ∎

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Lemma 3.2. *For $r \leq i \leq d-r-3$,*

$$R^3 X_i + R^2 X_{i+1} R - R X_{i+2} R^2 - X_{i+3} R^3 = 0. \tag{3.13}$$

Proof. From (2.8), we have

$$(R^3 L^{(+)} + R^2 L^{(+)} R - R L^{(+)} R^2 - L^{(+)} R^3) F_i^{(r)} = 0.$$

Apply Lemma 2.10 to the above relation to obtain

$$R^3 L^{(+)} F_i^{(r)} + R^2 L^{(+)} F_{i+1}^{(r)} R - R L^{(+)} F_{i+2}^{(r)} R^2 - L^{(+)} F_{i+3}^{(r)} R^3 = 0. \qquad \square$$

Lemma 3.3. *For $r+2 \leq i \leq d-r-1$,*

$$R X_i + X_{i+1} R = 0, \quad \textit{if } i-r \textit{ is even}; \tag{3.14}$$

$$\binom{i-r+1}{2} R X_i + \binom{i-r}{2} X_{i+1} R = 0, \quad \textit{if } i-r \textit{ is odd}. \tag{3.15}$$

Proof. Fix an integer i ($r+2 \leq i \leq d-r-1$). By the definition of X_i and Lemma 2.9, we have

$$R X_i \,|_{U_i^{(r)}} \neq 0, \quad R X_i \,|_{U_j^{(r)}} = 0 \ (i \neq j), \quad X_{i+1} R \,|_{U_i^{(r)}} \neq 0, \quad X_{i+1} R \,|_{U_j^{(r)}} = 0 \ (i \neq j).$$

These imply that (3.14) and (3.15) hold on V if and only if (3.14) and (3.15) hold on $U_i^{(r)}$.

We first show that (3.14) holds using induction on i. Setting $i = r$ in (3.13), and applying Lemma 3.1, we find

$$- R X_{r+2} R^2 - X_{r+3} R^3 = -(R X_{r+2} + X_{r+3} R) R^2 = 0. \tag{3.16}$$

Note that

$$R^2 \,|_{U_r^{(r)}} : U_r^{(r)} \to U_{r+2}^{(r)}$$

is a bijection by Lemma 2.3. Remove the factor $(-1)R^2$ in (3.16) to obtain

$$R X_{r+2} + X_{r+3} R = 0. \tag{3.17}$$

Hence, (3.14) holds at $i = r+2$.

Now suppose (3.14) holds at $i-2$, where $i-2-r$ is a non-negative even integer. We will show that (3.14) holds at i. By the induction hypothesis,

$$R X_{i-2} + X_{i-1} R = 0. \tag{3.18}$$

Replacing i by $i-2$ in (3.13),

$$R^3 X_{i-2} + R^2 X_{i-1} R - R X_i R^2 - X_{i+1} R^3 = 0. \tag{3.19}$$

Multiplying (3.18) by $(-1)R^2$ from the left,

$$- R^3 X_{i-2} - R^2 X_{i-1} R = 0. \tag{3.20}$$

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Adding (3.19) and (3.20),

$$-RX_iR^2 - X_{i+1}R^3 = 0.$$

Since $R^2 \mid_{U_{i-2}^{(r)}} : U_{i-2}^{(r)} \to U_i^{(r)}$ is a bijection by Lemma 2.3, we can remove the factor $(-1)R^2$ to get

$$RX_i + X_{i+1}R = 0.$$

Hence, (3.14) holds.

Next, we show that (3.15) holds using induction on i. Setting $i = r+1$ in (3.13), and applying Lemma 3.1, we obtain

$$R^2X_{r+2}R - RX_{r+3}R^2 - X_{r+4}R^3 = 0.$$

Since $R \mid_{U_{r+1}^{(r)}} : U_{r+1}^{(r)} \to U_{r+2}^{(r)}$ is a bijection by Lemma 2.3, we can remove the factor R to get

$$R^2X_{r+2} - RX_{r+3}R - X_{r+4}R^2 = 0. \tag{3.21}$$

Multiplying (3.17) by $(-1)R$ from the left,

$$-R^2X_{r+2} - RX_{r+3}R = 0. \tag{3.22}$$

Adding (3.21) and (3.22),

$$-2RX_{r+3}R - X_{r+4}R^2 = 0.$$

Since $R \mid_{U_{r+2}^{(r)}} : U_{r+2}^{(r)} \to U_{r+3}^{(r)}$ is a bijection by Lemma 2.3, we may remove the factor R to get

$$\binom{4}{2}RX_{r+3} + \binom{3}{2}X_{r+4}R = 0.$$

Hence, (3.15) holds at $i = r+3$.

Now suppose (3.15) holds at $i-2$, where $i-2-r$ is a non-negative odd integer. We will show that (3.15) holds at i. By the induction hypothesis,

$$\binom{i-r-1}{2}RX_{i-2} + \binom{i-r-2}{2}X_{i-1}R = 0. \tag{3.23}$$

Replacing i by $i-2$ in (3.13),

$$R^3X_{i-2} + R^2X_{i-1}R - RX_iR^2 - X_{i+1}R^3 = 0. \tag{3.24}$$

Multiplying (3.24) by $\binom{i-r-1}{2}$,

$$\binom{i-r-1}{2}R^3X_{i-2} + \binom{i-r-1}{2}R^2X_{i-1}R - \binom{i-r-1}{2}RX_iR^2 - \binom{i-r-1}{2}X_{i+1}R^3 = 0. \tag{3.25}$$

Multiplying (3.23) by $(-1)R^2$ from the left,

$$-\binom{i-r-1}{2}R^3X_{i-2} - \binom{i-r-2}{2}R^2X_{i-1}R = 0. \tag{3.26}$$

Adding (3.25) and (3.26), since the restriction $R \mid_{U_{i-2}^{(r)}} : U_{i-2}^{(r)} \to U_{i-1}^{(r)}$ is a bijection by Lemma 2.3, we can remove the factor R to get

$$(i-r-2)R^2X_{i-1} - \binom{i-r-1}{2}RX_iR - \binom{i-r-1}{2}X_{i+1}R^2 = 0. \tag{3.27}$$

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Since $i-1-r$ is even, replacing i by $i-1$ in (3.14), and then multiplying $-(i-r-2)R$ from the left, we obtain

$$-(i-r-2)R^2X_{i-1}-(i-r-2)RX_iR=0. \tag{3.28}$$

Adding (3.27) and (3.28), since $R\,|_{U^{(r)}_{i-1}}: U^{(r)}_{i-1}\to U^{(r)}_i$ is a bijection by Lemma 2.3, we can remove the factor $(-1)R$ to get

$$\binom{i-r+1}{2}RX_i+\binom{i-r}{2}X_{i+1}R=0.$$

Hence, (3.15) holds. ∎

THEOREM 3.4. *For $0\le r\le h-1$ and $r+2\le i\le d-r-1$,*

$$(RL^{(+)}+L^{(+)}R)U^{(r)}_i=0, \quad \textit{if } i-r \textit{ is even}; \tag{3.29}$$

$$\left(RL^{(+)}+\frac{i-r-1}{i-r+1}L^{(+)}R\right)U^{(r)}_i=0, \quad \textit{if } i-r \textit{ is odd}. \tag{3.30}$$

Proof. Immediate from (3.14), (3.15) and the definition of X_i. ∎

Next we fix an integer r such that $1\le r\le h$, and set

$$Y_i:=L^{(-)}F^{(r)}_i\ \ (0\le i\le d) \quad \text{and} \quad Y_{d+1}:=0.$$

LEMMA 3.5. $Y_{d-r+1}=Y_{d-r+2}=0$.

Proof. Immediate from Lemma 2.8 (iii). ∎

LEMMA 3.6. *For $r\le i\le d-r-1$,*

$$R^3Y_i+R^2Y_{i+1}R-RY_{i+2}R^2-Y_{i+3}R^3=0. \tag{3.31}$$

Proof. Similar to the proof of (3.13). ∎

LEMMA 3.7. *For $r\le i\le d-r-1$,*

$$RY_i+Y_{i+1}R=0, \quad \textit{if } d-r-i \textit{ is odd}; \tag{3.32}$$

$$\binom{d-r-i+1}{2}RY_i+\binom{d-r-i+2}{2}Y_{i+1}R=0, \quad \textit{if } d-r-i \textit{ is even}. \tag{3.33}$$

Proof. We first show that (3.32) holds using induction on i. Applying Lemma 3.5, setting $i=d-r-1$ in (3.31) and then removing R^2 from the left, we obtain

$$RY_{d-r-1}+Y_{d-r}R=0. \tag{3.34}$$

Hence, (3.32) holds at $i=d-r-1$.

Now suppose (3.32) holds at $i+2$, where $d-r-i-2$ is a non-negative odd integer. We will show that (3.32) holds at i. By the induction hypothesis,

$$RY_{i+2}+Y_{i+3}R=0. \tag{3.35}$$

Multiplying (3.35) by R^2 from the right, adding (3.31) and then removing R^2 from the left, we can get (3.32).

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Next, we show that (3.33) holds using induction on i. Setting $i = d-r-2$ in (3.31) and applying Lemma 3.5, we obtain

$$R^3 Y_{d-r-2} + R^2 Y_{d-r-1} R - R Y_{d-r} R^2 = 0. \tag{3.36}$$

Multiplying (3.34) by R on the two sides of the equation,

$$R^2 Y_{d-r-1} R + R Y_{d-r} R^2 = 0. \tag{3.37}$$

Adding (3.36) and (3.37), and then removing R^2, we have

$$R Y_{d-r-2} + 2 Y_{d-r-1} R = 0.$$

Hence, (3.33) holds at $i = d-r-2$.

Now suppose (3.33) holds at $i+2$, where $d-r-i-2$ is a non-negative even integer. We will show that (3.33) holds at i. By the induction hypothesis,

$$\binom{d-r-i-1}{2} R Y_{i+2} + \binom{d-r-i}{2} Y_{i+3} R = 0. \tag{3.38}$$

Multiplying (3.31) by $\binom{d-r-i}{2}$, we have

$$\binom{d-r-i}{2} R^3 Y_i + \binom{d-r-i}{2} R^2 Y_{i+1} R - \binom{d-r-i}{2} R Y_{i+2} R^2 - \binom{d-r-i}{2} Y_{i+3} R^3 = 0. \tag{3.39}$$

Multiplying (3.38) by R^2 from the right, adding (3.39), and then removing R, we obtain

$$\binom{d-r-i}{2} R^2 Y_i + \binom{d-r-i}{2} R Y_{i+1} R - (d-r-i-1) Y_{i+2} R^2 = 0. \tag{3.40}$$

Since $d-r-i-1$ is odd, replacing i by $i+1$ in (3.32), and then multiplying $(d-r-i-1)R$ from the right, we have

$$(d-r-i-1) R Y_{i+1} R + (d-r-i-1) Y_{i+2} R^2 = 0. \tag{3.41}$$

Adding (3.40) and (3.41), and then removing R, we can get (3.33). ∎

THEOREM 3.8. *For $1 \le r \le h$ and $r \le i \le d-r-1$,*

$$(RL^{(-)} + L^{(-)} R) U_i^{(r)} = 0, \quad \textit{if } d-r-i \textit{ is odd}; \tag{3.42}$$

$$\left(RL^{(-)} + \frac{d-r-i+2}{d-r-i} L^{(-)} R \right) U_i^{(r)} = 0, \quad \textit{if } d-r-i \textit{ is even}. \tag{3.43}$$

Proof. Immediate from (3.32), (3.33) and the definition of Y_i. ∎

Finally we fix an integer r such that $0 \le r \le h$, and set

$$Z_i := L^{(0)} F_i^{(r)} \quad (0 \le i \le d).$$

LEMMA 3.9. $Z_r = Z_{d-r+1} = 0$.

Proof. Immediate from Lemmas 2.8 (iii) and 2.12 (i). ∎

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Lemma 3.10. *For $r \le i \le d-r-2$,*

$$R^3 Z_i + R^2 Z_{i+1} R - R Z_{i+2} R^2 - Z_{i+3} R^3 - \varepsilon_i R^2 F_i^{(r)} = 0. \tag{3.44}$$

Proof. Immediate from (2.10) and Lemma 2.10. ∎

Lemma 3.11. *For $r+1 \le i \le d-r-1$,*

$$\begin{aligned}&\frac{i-r+1}{2} R^2 Z_i - R Z_{i+1} R - \frac{i-r+1}{2} Z_{i+2} R^2 \\ &- \sum_{j=r}^{i-1} \left[\frac{1}{4} - \frac{1}{4}(-1)^{j-r+1} + \frac{j-r+1}{2}(-1)^{j-r}\right] \varepsilon_j R F_i^{(r)} = 0, \quad \textit{if } i-r \textit{ is odd;}\end{aligned} \tag{3.45}$$

$$\begin{aligned}&\frac{i-r+2}{2} R^2 Z_i - \frac{i-r}{2} Z_{i+2} R^2 + \sum_{j=r}^{i-1} \left[\frac{1}{4} - \frac{1}{4}(-1)^{j-r+1} + \frac{j-r+1}{2}(-1)^{j-r}\right] \varepsilon_j R F_i^{(r)} = 0, \\ &\qquad \textit{if } i-r \textit{ is even.}\end{aligned} \tag{3.46}$$

Proof. We show that the results hold using induction on i. Setting $i = r$ in (3.44), and applying Lemma 3.9, we find

$$R^2 Z_{r+1} R - R Z_{r+2} R^2 - Z_{r+3} R^3 - \varepsilon_r R^2 F_r^{(r)} = 0.$$

Using Lemma 2.10 and removing R, we obtain

$$R^2 Z_{r+1} - R Z_{r+2} R - Z_{r+3} R^2 - \varepsilon_r R F_{r+1}^{(r)} = 0. \tag{3.47}$$

Hence, (3.45) holds at $i = r+1$.

Now suppose the results hold at $i-1$, where $i-1-r$ is a non-negative odd integer. We will show that the results hold at i. By the induction hypothesis,

$$\frac{i-r}{2} R^2 Z_{i-1} - R Z_i R - \frac{i-r}{2} Z_{i+1} R^2 - \sum_{j=r}^{i-2} \left[\frac{1}{4} - \frac{1}{4}(-1)^{j-r+1} + \frac{j-r+1}{2}(-1)^{j-r}\right] \varepsilon_j R F_{i-1}^{(r)} = 0. \tag{3.48}$$

Replacing i by $i-1$ in (3.44), and multiplying by $\frac{i-r}{2}$,

$$\frac{i-r}{2} R^3 Z_{i-1} + \frac{i-r}{2} R^2 Z_i R - \frac{i-r}{2} R Z_{i+1} R^2 - \frac{i-r}{2} Z_{i+2} R^3 - \frac{i-r}{2} \varepsilon_{i-1} R^2 F_{i-1}^{(r)} = 0. \tag{3.49}$$

Multiplying (3.48) by $(-1)R$ from the left, adding (3.49), and then removing R, we can get (3.46).

Next, we suppose the results hold at $i-1$, where $i-1-r$ is a non-negative even integer. We will show that the results hold at i. By the induction hypothesis,

$$\frac{i-r+1}{2} R^2 Z_{i-1} - \frac{i-r-1}{2} Z_{i+1} R^2 + \sum_{j=r}^{i-2} \left[\frac{1}{4} - \frac{1}{4}(-1)^{j-r+1} + \frac{j-r+1}{2}(-1)^{j-r}\right] \varepsilon_j R F_{i-1}^{(r)} = 0.$$

Multiplying $(-1)R$ from the left,

$$-\frac{i-r+1}{2} R^3 Z_{i-1} + \frac{i-r-1}{2} R Z_{i+1} R^2 - \sum_{j=r}^{i-2} \left[\frac{1}{4} - \frac{1}{4}(-1)^{j-r+1} + \frac{j-r+1}{2}(-1)^{j-r}\right] \varepsilon_j R^2 F_{i-1}^{(r)} = 0. \tag{3.50}$$

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Replacing i by $i-1$ in (3.44), and multiplying by $\frac{i-r+1}{2}$,

$$\begin{aligned}&\frac{i-r+1}{2}R^3Z_{i-1}+\frac{i-r+1}{2}R^2Z_iR-\frac{i-r+1}{2}RZ_{i+1}R^2-\frac{i-r+1}{2}Z_{i+2}R^3\\&-\frac{i-r+1}{2}\varepsilon_{i-1}R^2F^{(r)}_{i-1}=0.\end{aligned}\tag{3.51}$$

Adding (3.50) and (3.51), and then removing R, we can get (3.45). ∎

Lemma 3.12. *For $r\le i\le d-r-2$,*

$$\begin{aligned}&\frac{d-r-i}{2}R^2Z_i+RZ_{i+1}R-\frac{d-r-i}{2}Z_{i+2}R^2\\&-\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_jRF^{(r)}_i=0,\quad \text{if } d-r-i \text{ is even;}\end{aligned}\tag{3.52}$$

$$\begin{aligned}&\frac{d-r-i-1}{2}R^2Z_i-\frac{d-r-i+1}{2}Z_{i+2}R^2\\&+\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_jRF^{(r)}_i=0,\quad \text{if } d-r-i \text{ is odd.}\end{aligned}\tag{3.53}$$

Proof. We show that the results hold using induction on i. Setting $i=d-r-2$ in (3.44), applying Lemma 3.9 and then removing R, we have

$$R^2Z_{d-r-2}+RZ_{d-r-1}R-Z_{d-r}R^2-\varepsilon_{d-r-2}RF^{(r)}_{d-r-2}=0.\tag{3.54}$$

Hence, (3.52) holds at $i=d-r-2$.

Now suppose the results hold at $i+1$, where $d-r-i-1$ is a non-negative even integer. We will show that the results hold at i. By the induction hypothesis,

$$\begin{aligned}&\frac{d-r-i-1}{2}R^2Z_{i+1}+RZ_{i+2}R-\frac{d-r-i-1}{2}Z_{i+3}R^2\\&-\sum_{j=i+1}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_jRF^{(r)}_{i+1}=0.\end{aligned}\tag{3.55}$$

Multiplying (3.44) by $\frac{d-r-i-1}{2}$,

$$\begin{aligned}&\frac{d-r-i-1}{2}R^3Z_i+\frac{d-r-i-1}{2}R^2Z_{i+1}R-\frac{d-r-i-1}{2}RZ_{i+2}R^2-\frac{d-r-i-1}{2}Z_{i+3}R^3\\&-\frac{d-r-i-1}{2}\varepsilon_iR^2F^{(r)}_i=0.\end{aligned}\tag{3.56}$$

Multiplying (3.55) by $(-1)R$ from the right, adding (3.56), and then removing R, we can get (3.53).

Next, we suppose the results hold at $i+1$, where $d-r-i-1$ is a non-negative odd integer. We will show that the results hold at i. By the induction hypothesis,

$$\frac{d-r-i-2}{2}R^2Z_{i+1}-\frac{d-r-i}{2}Z_{i+3}R^2+\sum_{j=i+1}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_jRF^{(r)}_{i+1}=0.\tag{3.57}$$

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Multiplying (3.44) by $\frac{d-r-i}{2}$,

$$\frac{d-r-i}{2}R^3Z_i+\frac{d-r-i}{2}R^2Z_{i+1}R-\frac{d-r-i}{2}RZ_{i+2}R^2-\frac{d-r-i}{2}Z_{i+3}R^3-\frac{d-r-i}{2}\varepsilon_iR^2F_i^{(r)}=0. \tag{3.58}$$

Multiplying (3.57) by $(-1)R$ from the right, adding (3.58), and then removing R, we can get (3.52). ∎

Lemma 3.13. *For $r+1 \le i \le d-r-1$,*

(i) *If $i-r$ and $d-r-i$ are odd, then*

$$(i-r+1)RZ_i-(d-r-i+1)Z_{i+1}R-e_i^1F_i^{(r)}=0, \tag{3.59}$$

where

$$\begin{aligned} e_i^1 =&(d-r-i+1)\sum_{j=r}^{i-1}\left[\frac{1}{4}-\frac{1}{4}(-1)^{j-r+1}+\frac{j-r+1}{2}(-1)^{j-r}\right]\varepsilon_j \\ &+(i-r+1)\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_j. \end{aligned}$$

(ii) *If $i-r$ is even, $d-r-i$ is odd, then*

$$\frac{d-2r+1}{2}R^2Z_i-\frac{d-2r+1}{2}Z_{i+2}R^2+e_i^2RF_i^{(r)}=0, \tag{3.60}$$

$$(d-2r+1)RZ_i+e_i^{2'}F_i^{(r)}=0, \tag{3.61}$$

where

$$\begin{aligned} e_i^2 =&\sum_{j=r}^{i-1}\left[\frac{1}{4}-\frac{1}{4}(-1)^{j-r+1}+\frac{j-r+1}{2}(-1)^{j-r}\right]\varepsilon_j \\ &+\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_j, \\ e_i^{2'} =&(d-r-i+1)\sum_{j=r}^{i-1}\left[\frac{1}{4}-\frac{1}{4}(-1)^{j-r+1}+\frac{j-r+1}{2}(-1)^{j-r}\right]\varepsilon_j \\ &-(i-r)\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_j. \end{aligned}$$

(iii) *If $i-r$ is odd, $d-r-i$ is even, then*

$$\frac{d-2r+1}{2}R^2Z_i-\frac{d-2r+1}{2}Z_{i+2}R^2-e_i^3RF_i^{(r)}=0, \tag{3.62}$$

$$(d-2r+1)Z_{i+1}R+e_i^{3'}F_i^{(r)}=0, \tag{3.63}$$

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where

$$\begin{aligned} e_i^3 =& \sum_{j=r}^{i-1}\left[\frac{1}{4}-\frac{1}{4}(-1)^{j-r+1}+\frac{j-r+1}{2}(-1)^{j-r}\right]\varepsilon_j \\ &+\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_j, \\ e_i^{3'} =&(d-r-i)\sum_{j=r}^{i-1}\left[\frac{1}{4}-\frac{1}{4}(-1)^{j-r+1}+\frac{j-r+1}{2}(-1)^{j-r}\right]\varepsilon_j \\ &-(i-r+1)\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_j. \end{aligned}$$

(iv) *If $i-r$ and $d-r-i$ are even, then*

$$(d-r-i)RZ_i-(i-r)Z_{i+1}R+e_i^4F_i^{(r)}=0, \tag{3.64}$$

where

$$\begin{aligned} e_i^4 =&(d-r-i)\sum_{j=r}^{i-1}\left[\frac{1}{4}-\frac{1}{4}(-1)^{j-r+1}+\frac{j-r+1}{2}(-1)^{j-r}\right]\varepsilon_j \\ &+(i-r)\sum_{j=i}^{d-r-2}\left[\frac{1}{4}-\frac{1}{4}(-1)^{d-r-j}+\frac{d-r-j}{2}(-1)^{d-r-j}\right]\varepsilon_j. \end{aligned}$$

Proof. (i) Setting $i=d-r-1$ in (3.45), applying Lemma 3.9 and then removing R, we get (3.59) holds at $i=d-r-1$.

Suppose $r+1\le i\le d-r-2$. Eliminating the term of $Z_{i+2}R^2$ from (3.45) and (3.53), and then removing R, we get (3.59).

(ii) Setting $i=d-r-1$ in (3.46) and using Lemma 3.9, we can get (3.60) holds at $i=d-r-1$. Removing R, we get (3.61) holds at $i=d-r-1$.

Suppose $r+1\le i\le d-r-2$. Adding (3.46) and (3.53), we get (3.60). Eliminating the term of $Z_{i+2}R^2$ from (3.46) and (3.53), and then removing R, we get (3.61).

(iii) Eliminating the term of $RZ_{i+1}R$ from (3.45) and (3.52), we get (3.62). Similarly, eliminating the terms of $R^2Z_i, Z_{i+2}R^2$ from (3.45) and (3.52), and then removing R, we get (3.63).

(iv) Eliminating the term of $Z_{i+2}R^2$ from (3.46) and (3.52), and then removing R, we get (3.64). ∎

Lemma 3.14. *With reference to Lemma 2.16, for $0\le i\le d-2$,*

$$\varepsilon_i=4(bc^*+b^*c+2(i+1)cc^*).$$

Proof. Immediate from (2.7) by a conventional calculation. ∎

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LEMMA 3.15. *For $r+1 \leq i \leq d-r-1$, the scalars $e_i^1, e_i^2, e_i^{2'}, e_i^3, e_i^{3'}, e_i^4$ by Lemma 3.13 are given by*

$$\begin{aligned}
e_i^1 &= (d-r-i+1)(i-r+1)(2bc^* + 2b^*c + (d+2i)cc^*),\\
e_i^2 &= (d-2r+1)(d-2i-1)cc^*,\\
e^{2'} &= -(d-r-i+1)(i-r)(d-2r+1)cc^*,\\
e_i^3 &= (d-2r+1)(2bc^* + 2b^*c + (d+2i+1)cc^*),\\
e_i^{3'} &= -(d-r-i)(i-r+1)(d-2r+1)cc^*,\\
e_i^4 &= (d-r-i)(i-r)(2bc^* + 2b^*c + (d+2i)cc^*).
\end{aligned}$$

Proof. Immediate from Lemmas 3.13 and 3.14 by a conventional calculation. □

THEOREM 3.16. *For $0 \leq r \leq h$ and $r+1 \leq i \leq d-r-1$,*

(i) *If $i-r$ and $d-r-i$ are odd, then*

$$\left(RL^{(0)} - \frac{d-r-i+1}{i-r+1} L^{(0)}R - (d-r-i+1)\mu_i I\right) U_i^{(r)} = 0. \tag{3.65}$$

(ii) *If $i-r$ is even, $d-r-i$ is odd, then*

$$(R^2L^{(0)} - L^{(0)}R^2 + 2(d-2i-1)cc^*R)U_i^{(r)} = 0, \tag{3.66}$$

$$(RL^{(0)} - (d-r-i+1)(i-r)cc^*I)U_i^{(r)} = 0. \tag{3.67}$$

(iii) *If $i-r$ is odd, $d-r-i$ is even, then*

$$(R^2L^{(0)} - L^{(0)}R^2 - 2\gamma_i R)U_i^{(r)} = 0, \tag{3.68}$$

$$(L^{(0)}R - (d-r-i)(i-r+1)cc^*I)U_i^{(r)} = 0. \tag{3.69}$$

(iv) *If $i-r$ and $d-r-i$ are even, then*

$$\left(RL^{(0)} - \frac{i-r}{d-r-i} L^{(0)}R + (i-r)\mu_i I\right) U_i^{(r)} = 0, \tag{3.70}$$

where

$$\begin{aligned}
\mu_i &= 2bc^* + 2b^*c + (d+2i)cc^*,\\
\gamma_i &= 2bc^* + 2b^*c + (d+2i+1)cc^*.
\end{aligned}$$

Proof. Immediate from Lemma 3.15 and equations (3.59)–(3.64). □

To end this section, we give the following lemma for the future use.

LEMMA 3.17. *Let r denote an integer with $0 \leq r \leq h$. Let Y denote a subspace of $U_r^{(r)}$ such that $L^{(0)}RY \subseteq Y$. We set $W = \sum_{i=0}^{d-2r} R^iY$. Then $L^{(0)}W \subseteq W$.*

Proof. We show

$$L^{(0)}R^iY \subseteq R^{i-1}Y \quad (1 \leq i \leq d-2r). \tag{3.71}$$

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We divide our proof into two cases in term of the parity of diameter d.

First we consider the case that d is odd. If i is even, pick any vector $u \in Y \subseteq U_r^{(r)}$, and observe that $R^{i-1}u$ belongs to $U_{r+i-1}^{(r)}$ by Lemma 2.3. Since $r+i-1-r$ is odd, applying (3.69) to $R^{i-1}u$, we find

$$L^{(0)}R^iu = i(d-2r-i+1)cc^*R^{i-1}u.$$

Hence, $L^{(0)}R^iu \in R^{i-1}Y$. If i is odd, we show (3.71) holds using induction on i. Observe that (3.71) holds at $i=1$ by our assumption. Now suppose $3 \le i \le d-2r$, and suppose (3.71) holds at $i-2$. We will show (3.71) holds at i. Since $R^{i-2}u \in U_{r+i-2}^{(r)}$, and $r+i-2-r$ is odd, applying (3.68) to $R^{i-2}u$, we obtain

$$L^{(0)}R^iu = R^2L^{(0)}R^{i-2}u - 2\gamma_{r+i-2}R^{i-1}u.$$

Note that $L^{(0)}R^{i-2}u \in R^{i-3}Y$ by induction, so $R^2L^{(0)}R^{i-2}u \in R^{i-1}Y$. Hence, $L^{(0)}R^iu \in R^{i-1}Y$.

Next, we consider the case that d is even. We show (3.71) holds using induction on i. Observe that (3.71) holds at $i=1$ by our assumption. Now suppose $2 \le i \le d-2r$, and suppose (3.71) holds at $i-1$. We will show (3.71) holds at i. If i is even, applying (3.65) to $R^{i-1}u$, we find

$$L^{(0)}R^iu = \frac{i}{d-2r-i+2}RL^{(0)}R^{i-1}u - i\mu_{r+i-1}R^{i-1}u.$$

If i is odd, applying (3.70) to $R^{i-1}u$, we find

$$L^{(0)}R^iu = \frac{d-2r-i+1}{i-1}RL^{(0)}R^{i-1}u + (d-2r-i+1)\mu_{r+i-1}R^{i-1}u.$$

These imply that $L^{(0)}R^iu \in \operatorname{span}\{RL^{(0)}R^{i-1}u, R^{i-1}u\}$. By the induction hypothesis, we have $L^{(0)}R^{i-1}u \in R^{i-2}Y$, so $RL^{(0)}R^{i-1}u \in R^{i-1}Y$. Hence, $L^{(0)}R^iY \subseteq R^{i-1}Y$. We have shown (3.71). We also have $L^{(0)}Y \subseteq L^{(0)}U_r^{(r)} = 0$ by Lemma 2.12. Thus, $L^{(0)}W \subseteq W$. ∎

4. The basis for V. In this section, we assume the height of A, A^* is 1. That is $\rho_0 = \rho_d = 1$, $\rho_1 = \rho_2 = \cdots = \rho_{d-1} \ge 2$. Note that by [4, Lemma 4.1], V is decomposed as $V = \sum_{i=0}^{d} U_i^{(0)} + \sum_{i=1}^{d-1} U_i^{(1)}$ (direct sum). Now, we give the bases for $U_i^{(0)} (0 \le i \le d)$ and $U_i^{(1)}$ $(1 \le i \le d-1)$.

Fix a nonzero vector u_0 in U_0, and set $u_i = R^iu_0$ $(1 \le i \le d)$.

LEMMA 4.1. *For $0 \le i \le d$, $\{u_i\}$ is a basis of $U_i^{(0)}$.*

Proof. Immediate from Lemma 2.3. ∎

LEMMA 4.2. *We have*

(i) $L^{(+)}u_0 = 0$, $L^{(+)}u_1 = 0$;
(ii) $L^{(+)}u_2 \neq 0$.

Proof. (i) Immediate from Lemma 2.12.

(ii) Suppose $L^{(+)}u_2 = 0$. Applying Lemma 3.4 to u_i for $2 \le i \le d-1$,

$$L^{(+)}u_{i+1} = L^{(+)}Ru_i = -RL^{(+)}u_i, \quad \text{if } i \text{ is even;}$$

$$L^{(+)}u_{i+1} = L^{(+)}Ru_i = -\frac{i+1}{i-1}RL^{(+)}u_i, \quad \text{if } i \text{ is odd.}$$

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Combining with (i), these imply $L^{(+)}u_i = 0$ for $0 \le i \le d$. So $L^{(+)}U^{(0)} = 0$. Using (2.5) and Lemma 2.11, we have

$$LU^{(0)} = L^{(-)}U^{(0)} + L^{(0)}U^{(0)} + L^{(+)}U^{(0)} = L^{(0)}U^{(0)} \subseteq U^{(0)}.$$

Hence, $U^{(0)}$ is invariant under L. Also we have $RU^{(0)} \subseteq U^{(0)}$ by Lemma 2.7, and $F_iU^{(0)} = U_i^{(0)} \subseteq U^{(0)}(0 \le i \le d)$. From which we have $U^{(0)} = V$ by Lemma 2.2, a contradiction. ∎

We set $v_1 = L^{(+)}u_2,\ v_i = R^{i-1}v_1 \quad (2 \le i \le d-1)$.

Lemma 4.3. *For $1 \le i \le d-1$, $v_i \neq 0$ and v_i lies in $U_i^{(1)}$.*

Proof. Immediate from Lemmas 2.3 and 4.2. ∎

Lemma 4.4. *Suppose $\rho_1 = 2$. Then $\{v_i\}$ is a basis of $U_i^{(1)}$ $(1 \le i \le d-1)$.*

Proof. Immediate from Lemmas 2.5, 4.1 and 4.3. ∎

Lemma 4.5. *For $1 \le i \le d-1$,*

$$L^{(+)}u_{i+1} = b_iv_i, \tag{4.72}$$

where

$$b_i = \frac{i+1}{2}, \quad \text{if } i \text{ is odd}; \tag{4.73}$$

$$b_i = -\frac{i}{2}, \quad \text{if } i \text{ is even}. \tag{4.74}$$

Proof. We show (4.72) holds using induction on i. Observe that (4.72) holds for $i = 1$, so we assume $2 \le i \le d-1$. Now suppose (4.72) holds at $i-1$. We will show (4.72) holds at i. If i is even, applying (3.29) to u_i, then

$$L^{(+)}u_{i+1} = L^{(+)}Ru_i = -RL^{(+)}u_i.$$

By induction,

$$RL^{(+)}u_i = Rb_{i-1}v_{i-1} = b_{i-1}v_i.$$

Hence,

$$L^{(+)}u_{i+1} = -b_{i-1}v_i = -\frac{i}{2}v_i = b_iv_i.$$

If i is odd, applying (3.30) to u_i, then

$$L^{(+)}u_{i+1} = L^{(+)}Ru_i = -\frac{i+1}{i-1}RL^{(+)}u_i.$$

So,

$$L^{(+)}u_{i+1} = -\frac{i+1}{i-1}b_{i-1}v_i = \frac{i+1}{2}v_i = b_iv_i. \qquad ∎$$

Lemma 4.6. *Suppose $\rho_1 \ge 3$. Then $L^{(0)}v_2$ and v_1 are linearly independent.*

Proof. By way of contradiction, we assume $L^{(0)}v_2$ lies in the span Y of $\{v_1\}$. We set $W = \sum_{i=0}^{d-2} R^iY$. Observe that $U^{(0)} + W$ is invariant under R and $F_i(0 \le i \le d)$. We show that $U^{(0)} + W$ is invariant under L.

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Since $h = 1$, we have $W \subseteq U^{(1)}$ and $L^{(+)}U^{(1)} = 0$. So $L^{(+)}W = 0$. Hence, $LW \subseteq L^{(-)}W + L^{(0)}W \subseteq U^{(0)} + L^{(0)}W$. We have $L^{(0)}RY \subseteq Y$ from our assumption $L^{(0)}v_2 \in Y$, and this implies $L^{(0)}W \subseteq W$ by Lemma 3.17. Observe that $L^{(+)}U^{(0)} \subseteq W$ by Lemma 4.5, so that $LU^{(0)} \subseteq L^{(-)}U^{(0)} + L^{(0)}U^{(0)} + L^{(+)}U^{(0)} \subseteq U^{(0)} + W$, since $L^{(-)}U^{(0)} = 0$. So $U^{(0)} + W$ is invariant under L. Therefore, $U^{(0)} + W$ is invariant under L, R and $F_i (0 \leq i \leq d)$. From which we have $W = V$ by Lemma 2.2. This contradicts our assumption $\rho_1 \geq 3$. ∎

When $\rho_1 \geq 3$, we set

$$w_1 = L^{(0)}v_2, \quad w_i = R^{i-1}w_1 \ \ (2 \leq i \leq d-1).$$

LEMMA 4.7. *Suppose $\rho_1 \geq 3$. Then v_i and w_i are linearly independent for $1 \leq i \leq d-1$.*

Proof. Immediate from Lemmas 2.3 and 4.6. ∎

LEMMA 4.8. *Suppose $\rho_1 = 3$. Then $\{v_i, w_i\}$ is a basis of $U_i^{(1)}$ $(1 \leq i \leq d-1)$.*

Proof. Immediate from Lemmas 2.4 and 4.7. ∎

5. The action of L. Recall that $\{u_i\}$ is a basis of $U_i^{(0)}$, $\{v_i\}$ is a basis of $U_i^{(1)}$ if $\rho_1 = 2$, and $\{v_i, w_i\}$ is a basis of $U_i^{(1)}$ if $\rho_1 = 3$. In this section, we consider the action of L on these bases.

By Lemma 4.1, we know that $L^{(0)}u_{i+1}$ is a scalar multiple of u_i. We set

$$L^{(0)}u_{i+1} = a_i u_i \quad (0 \leq i \leq d-1) \tag{5.75}$$

for some scalars $a_0, a_1, \ldots, a_{d-1}$.

Clearly, $L^{(-)}v_{i+1}$ is a scalar multiple of u_i by Lemmas 4.1 and 4.3. So, we set

$$L^{(-)}v_{i+1} = e_i u_i \quad (0 \leq i \leq d-2) \tag{5.76}$$

for some scalars $e_0, e_1, \ldots, e_{d-2}$.

When $\rho_1 = 2$, by Lemma 4.4,

$$L^{(0)}v_1 = 0, \quad L^{(0)}v_{i+1} = c_i v_i \ \ (1 \leq i \leq d-2) \tag{5.77}$$

for some scalars $c_1, \ldots, c_{d-2}$.

When $\rho_1 \geq 3$, by Lemma 4.1, we set

$$L^{(-)}w_{i+1} = f_i u_i \quad (0 \leq i \leq d-2) \tag{5.78}$$

for some scalars $f_0, f_1, \ldots, f_{d-2}$.

Similarly, when $\rho_1 = 3$, by Lemma 4.8,

$$L^{(0)}w_{i+1} = s_i v_i + t_i w_i \quad (1 \leq i \leq d-2) \tag{5.79}$$

for some scalars s_i, t_i.

Now we determine the parameters $a_i, e_i, c_i, f_i, m_i, n_i, s_i, t_i$. We divide the arguments into two cases in term of the parity of diameter d.

Case I: d is odd. By equations (5.75)–(5.79), we have the following lemmas.

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Lemma 5.1. *For* $0 \le i \le d-1$,

$$a_i = (i+1)(d-i)cc^*, \quad \textit{if } i \textit{ is odd}; \tag{5.80}$$

$$a_i = a_0 - i(2bc^* + 2b^*c + (d+i+1)cc^*), \quad \textit{if } i \textit{ is even}. \tag{5.81}$$

Proof. We first show (5.80) holds. Applying (3.69) to u_i,

$$L^{(0)}Ru_i = (i+1)(d-i)cc^*u_i.$$

Note that $L^{(0)}Ru_i = L^{(0)}u_{i+1} = a_iu_i$. Hence, $a_i = (i+1)(d-i)cc^*$.

Next, we show (5.81) holds using induction on i. Clearly (5.81) holds for $i = 0$, so we assume $2 \le i \le d-1$. Now suppose (5.81) holds at $i-2$. We will show (5.81) holds at i. Applying (3.68) to u_{i-1},

$$L^{(0)}R^2u_{i-1} = R^2L^{(0)}u_{i-1} - 2\gamma_{i-1}Ru_{i-1}.$$

Note that $L^{(0)}R^2u_{i-1} = L^{(0)}u_{i+1} = a_iu_i$, and $R^2L^{(0)}u_{i-1} = R^2a_{i-2}u_{i-2} = a_{i-2}u_i$. So $a_i = a_{i-2} - 2\gamma_{i-1}$. By induction,

$$a_{i-2} = a_0 - (i-2)(2bc^* + 2b^*c + (d+i-1)cc^*).$$

Hence, (5.81) holds. □

Lemma 5.2. *For* $0 \le i \le d-2$,

$$e_i = -\frac{d-i}{d-1}e_0, \quad \textit{if } i \textit{ is odd}; \tag{5.82}$$

$$e_i = \frac{d-i-1}{d-1}e_0, \quad \textit{if } i \textit{ is even}. \tag{5.83}$$

Proof. We show that the results hold using induction on i. Observe that the results hold at $i = 0$. Now suppose $1 \le i \le d-2$, and suppose the results hold at $i-1$. We will show that the results hold at i. If i is odd, applying (3.42) to v_i, then

$$L^{(-)}Rv_i = -RL^{(-)}v_i.$$

Note that $L^{(-)}Rv_i = L^{(-)}v_{i+1} = e_iu_i$, and $RL^{(-)}v_i = Re_{i-1}u_{i-1} = e_{i-1}u_i$. So $e_i = -e_{i-1}$. By induction,

$$e_{i-1} = \frac{d-i}{d-1}e_0.$$

Hence, (5.82) holds. If i is even, applying (3.43) to v_i, then

$$L^{(-)}Rv_i = -\frac{d-i-1}{d-i+1}RL^{(-)}v_i.$$

So,

$$e_i = -\frac{d-i-1}{d-i+1}e_{i-1}.$$

By induction,

$$e_{i-1} = -\frac{d-i+1}{d-1}e_0.$$

Hence, (5.83) holds. □

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LEMMA 5.3. *Suppose $\rho_1 = 2$. Then, for $1 \leq i \leq d-2$,*

$$c_i = c_1 - (i-1)(2bc^* + 2b^*c + (d+i+2)cc^*), \quad \textit{if } i \textit{ is odd}; \tag{5.84}$$

$$c_i = i(d-i-1)cc^*, \quad \textit{if } i \textit{ is even}. \tag{5.85}$$

Proof. We first show (5.84) holds using induction on i. Observe that (5.84) holds for $i = 1$, so we assume $3 \leq i \leq d-2$. Now suppose (5.84) holds at $i-2$. We will show (5.84) holds at i. Applying (3.68) to v_{i-1},

$$L^{(0)}R^2 v_{i-1} = R^2 L^{(0)} v_{i-1} - 2\gamma_{i-1} R v_{i-1}.$$

Note that $L^{(0)}R^2 v_{i-1} = L^{(0)} v_{i+1} = c_i v_i$, and $R^2 L^{(0)} v_{i-1} = R^2 c_{i-2} v_{i-2} = c_{i-2} v_i$. Thus, $c_i = c_{i-2} - 2\gamma_{i-1}$. By induction,

$$c_{i-2} = c_1 - (i-3)(2bc^* + 2b^*c + (d+i)cc^*).$$

Hence, (5.84) follows.

Next, we will show (5.85) holds. Applying (3.69) to v_i,

$$L^{(0)} R v_i = i(d-i-1)cc^* v_i.$$

Note that $L^{(0)} R v_i = L^{(0)} v_{i+1} = c_i v_i$. Hence, (5.85) holds. ∎

LEMMA 5.4. *Suppose $\rho_1 \geq 3$. Then, for $0 \leq i \leq d-2$,*

$$f_i = -\frac{d-i}{d-1} f_0, \quad \textit{if } i \textit{ is odd}; \tag{5.86}$$

$$f_i = \frac{d-i-1}{d-1} f_0, \quad \textit{if } i \textit{ is even}. \tag{5.87}$$

Proof. Similar to the proof of Lemma 5.2. ∎

LEMMA 5.5. *Suppose $\rho_1 \geq 3$. Then*

$$L^{(0)} v_{i+1} = m_i v_i + n_i w_i \quad (1 \leq i \leq d-2), \tag{5.88}$$

where

$$m_i = -(i-1)(2bc^* + 2b^*c + (d+i+2)cc^*), \; n_i = 1, \quad \textit{if } i \textit{ is odd}; \tag{5.89}$$

$$m_i = i(d-i-1)cc^*, \; n_i = 0, \quad \textit{if } i \textit{ is even}. \tag{5.90}$$

Proof. We divide our proof into two cases in term of the parity of i. First suppose i is odd. We will show (5.88) holds using induction on i. Observe that (5.88) holds for $i = 1$ with $m_1 = 0, n_1 = 1$, since $L^{(0)} v_2 = w_1$. Now we assume $3 \leq i \leq d-2$, and suppose (5.88) holds at $i-2$. We will show (5.88) holds at i. Applying (3.68) to v_{i-1},

$$L^{(0)}R^2 v_{i-1} = R^2 L^{(0)} v_{i-1} - 2\gamma_{i-1} R v_{i-1}.$$

Note that $L^{(0)}R^2 v_{i-1} = L^{(0)} v_{i+1}$, and $R v_{i-1} = v_i$. By induction, we have

$$R^2 L^{(0)} v_{i-1} = R^2 (m_{i-2} v_{i-2} + n_{i-2} w_{i-2}) = m_{i-2} v_i + n_{i-2} w_i.$$

Hence,

$$\begin{aligned} L^{(0)} v_{i+1} &= (-(i-3)(2bc^* + 2b^*c + (d+i)cc^*) - 2(2bc^* + 2b^*c + (d+2i-1)cc^*)) v_i + w_i \\ &= -(i-1)(2bc^* + 2b^*c + (d+i+2)cc^*) v_i + w_i. \end{aligned}$$

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Next suppose i is even. Applying (3.67) to v_{i+1} to get

$$RL^{(0)}v_{i+1} = i(d-i-1)cc^*v_{i+1}.$$

Removing R, we find

$$L^{(0)}v_{i+1} = i(d-i-1)cc^*v_i = m_iv_i + n_iw_i. \qquad \square$$

LEMMA 5.6. *Suppose $\rho_1 = 3$. Then, for $1 \le i \le d-2$,*

$$s_i = s_1,\ t_i = t_1 - (i-1)(2bc^* + 2b^*c + (d+i+2)cc^*), \quad \text{if } i \text{ is odd}; \tag{5.91}$$

$$s_i = 0,\ t_i = i(d-i-1)cc^*, \quad \text{if } i \text{ is even}. \tag{5.92}$$

Proof. Similar to the proof of Lemma 5.5. $\square$

LEMMA 5.7. *When $\rho_1 \ge 3$, the following hold with the values of $a_i, b_i, e_i, f_i, m_i, n_i$ given by (5.80), (5.81), (4.73), (4.74), (5.82), (5.83), (5.86), (5.87), (5.89) and (5.90).*

(i) $Lu_0 = 0$, $Lu_1 = a_0u_0$, $Lu_{i+1} = a_iu_i + b_iv_i$ $(1 \le i \le d-1)$,

(ii) $Lv_1 = e_0u_0$, $Lv_{i+1} = e_iu_i + m_iv_i + n_iw_i$ $(1 \le i \le d-2)$,

(iii) $L^{(-)}w_{i+1} = f_iu_i$ $(0 \le i \le d-2)$.

Proof. Immediate from Lemma 2.11, and equations (4.72), (5.75), (5.76), (5.78) and (5.88). $\square$

Case II: d is even. Similarly, by equations (5.75)–(5.79), we have the following lemmas.

LEMMA 5.8. *For $0 \le i \le d-1$,*

$$a_i = (i+1)\left(\frac{a_0}{d} + \sum_{k=1}^{i}(-1)^k\mu_k\right), \quad \text{if } i \text{ is odd}; \tag{5.93}$$

$$a_i = (d-i)\left(\frac{a_0}{d} + \sum_{k=1}^{i}(-1)^k\mu_k\right), \quad \text{if } i \text{ is even}. \tag{5.94}$$

Proof. We show that the results hold using induction on i. Observe that the results hold for $i = 0$, so we assume $1 \le i \le d-1$, and suppose the results hold at $i-1$. We will show that the results hold at i. If i is odd, applying (3.65) to u_i, then

$$L^{(0)}Ru_i = \frac{i+1}{d-i+1}RL^{(0)}u_i - (i+1)\mu_iu_i.$$

Note that $L^{(0)}Ru_i = L^{(0)}u_{i+1} = a_iu_i$, and $RL^{(0)}u_i = R(a_{i-1}u_{i-1}) = a_{i-1}Ru_{i-1} = a_{i-1}u_i$. Hence,

$$a_i = \frac{i+1}{d-i+1}a_{i-1} - (i+1)\mu_i.$$

By induction,

$$a_{i-1} = (d-i+1)\left(\frac{a_0}{d} + \sum_{k=1}^{i-1}(-1)^k\mu_k\right).$$

Now (5.93) holds. If i is even, applying (3.70) to u_i, then

$$L^{(0)}Ru_i = \frac{d-i}{i}RL^{(0)}u_i + (d-i)\mu_iu_i.$$

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So

$$a_i = \frac{d-i}{i} a_{i-1} + (d-i)\mu_i.$$

By induction,

$$a_{i-1} = i \left(\frac{a_0}{d} + \sum_{k=1}^{i-1} (-1)^k \mu_k \right).$$

Now (5.94) holds. ∎

LEMMA 5.9. *For* $0 \le i \le d-2$,

$$e_i = -\frac{d-i-1}{d} e_0, \quad \textit{if } i \textit{ is odd}; \tag{5.95}$$

$$e_i = \frac{d-i}{d} e_0, \quad \textit{if } i \textit{ is even}. \tag{5.96}$$

Proof. Similar to the proof of Lemma 5.2. ∎

LEMMA 5.10. *Suppose* $\rho_1 = 2$. *Then, for* $1 \le i \le d-2$,

$$c_i = (d-i-1) \left(\frac{c_1}{d-2} - \sum_{k=2}^{i} (-1)^k \mu_k \right), \quad \textit{if } i \textit{ is odd}; \tag{5.97}$$

$$c_i = i \left(\frac{c_1}{d-2} - \sum_{k=2}^{i} (-1)^k \mu_k \right), \quad \textit{if } i \textit{ is even}. \tag{5.98}$$

Proof. We show that the results hold using induction on i. Observe that the results hold for $i = 1$. So we assume $2 \le i \le d-2$. Now suppose the results hold at $i-1$. We will show that the results hold at i. If i is odd, applying (3.70) to v_i, then

$$L^{(0)} R v_i = \frac{d-i-1}{i-1} R L^{(0)} v_i + (d-i-1)\mu_i v_i.$$

Note that $L^{(0)} R v_i = L^{(0)} v_{i+1} = c_i v_i$, and $R L^{(0)} v_i = R c_{i-1} v_{i-1} = c_{i-1} v_i$. Hence,

$$c_i = \frac{d-i-1}{i-1} c_{i-1} + (d-i-1)\mu_i.$$

By induction,

$$c_{i-1} = (i-1) \left(\frac{c_1}{d-2} - \sum_{k=2}^{i-1} (-1)^k \mu_k \right).$$

Now (5.97) holds. If i is even, applying (3.65) to v_i, then

$$L^{(0)} R v_i = \frac{i}{d-i} R L^{(0)} v_i - i\mu_i v_i.$$

So

$$c_i = \frac{i}{d-i} c_{i-1} - i\mu_i.$$

By induction,

$$c_{i-1} = (d-i) \left(\frac{c_1}{d-2} - \sum_{k=2}^{i-1} (-1)^k \mu_k \right).$$

Now (5.98) holds. ∎

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LEMMA 5.11. *Suppose $\rho_1 \geq 3$. Then, for $0 \leq i \leq d-2$,*

$$f_i = -\frac{d-i-1}{d} f_0, \quad \text{if } i \text{ is odd}; \tag{5.99}$$

$$f_i = \frac{d-i}{d} f_0, \quad \text{if } i \text{ is even}. \tag{5.100}$$

Proof. Similar to the proof of Lemma 5.2. ∎

LEMMA 5.12. *Suppose $\rho_1 \geq 3$. Then*

$$L^{(0)} v_{i+1} = m_i v_i + n_i w_i \quad (1 \leq i \leq d-2), \tag{5.101}$$

where

$$m_i = -(d-i-1)\sum_{k=2}^{i}(-1)^k \mu_k,\ n_i = \frac{d-i-1}{d-2}, \quad \text{if } i \text{ is odd}; \tag{5.102}$$

$$m_i = -i\sum_{k=2}^{i}(-1)^k \mu_k,\ n_i = \frac{i}{d-2}, \quad \text{if } i \text{ is even}. \tag{5.103}$$

Proof. We show (5.101) holds using induction on i. Obverse that (5.101) holds for $i = 1$ with $m_1 = 0, n_1 = 1$, since $L^{(0)} v_2 = w_1$. Now we assume $2 \leq i \leq d-2$, and suppose (5.101) holds at $i-1$. We will show (5.101) holds at i. If i is odd, applying (3.70) to v_i, then

$$L^{(0)} R v_i = \frac{d-i-1}{i-1} R L^{(0)} v_i + (d-i-1)\mu_i v_i.$$

Note that $L^{(0)} R v_i = L^{(0)} v_{i+1}$. By induction,

$$R L^{(0)} v_i = R(m_{i-1} v_{i-1} + n_{i-1} w_{i-1}) = m_{i-1} v_i + n_{i-1} w_i.$$

Hence,

$$\begin{aligned} L^{(0)} v_{i+1} =& \frac{d-i-1}{i-1}\left(-(i-1)\sum_{k=2}^{i-1}(-1)^k \mu_k v_i + \frac{i-1}{d-2} w_i\right) + (d-i-1)\mu_i v_i \\ =& -(d-i-1)\sum_{k=2}^{i}(-1)^k \mu_k v_i + \frac{d-i-1}{d-2} w_i. \end{aligned}$$

If i is even, applying (3.65) to v_i, then

$$L^{(0)} R v_i = \frac{i}{d-i} R L^{(0)} v_i - i\mu_i v_i.$$

Hence

$$\begin{aligned} L^{(0)} v_{i+1} =& \frac{i}{d-i}\left(-(d-i)\sum_{k=2}^{i-1}(-1)^k \mu_k v_i + \frac{d-i}{d-2} w_i\right) - i\mu_i v_i \\ =& -i\sum_{k=2}^{i}(-1)^k \mu_k v_i + \frac{i}{d-2} w_i. \qquad \square \end{aligned}$$

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Lemma 5.13. *Suppose $\rho_1 = 3$. Then, for $1 \le i \le d-2$,*

$$s_i = \frac{d-i-1}{d-2}s_1, \quad t_i = (d-i-1)\left(\frac{t_1}{d-2} - \sum_{k=2}^{i}(-1)^k\mu_k\right), \quad \textit{if } i \textit{ is odd;} \tag{5.104}$$

$$s_i = \frac{i}{d-2}s_1, \quad t_i = i\left(\frac{t_1}{d-2} - \sum_{k=2}^{i}(-1)^k\mu_k\right), \quad \textit{if } i \textit{ is even.} \tag{5.105}$$

Proof. Similar to the proof of Lemma 5.12. ∎

Lemma 5.14. *When $\rho_1 \ge 3$, the following hold with the values of $a_i, b_i, e_i, f_i, m_i, n_i$ given by (5.93), (5.94), (4.73), (4.74), (5.95), (5.96), (5.99), (5.100), (5.102) and (5.103).*

(i) $Lu_0 = 0$, $Lu_1 = a_0u_0$, $Lu_{i+1} = a_iu_i + b_iv_i$ $(1 \le i \le d-1)$,

(ii) $Lv_1 = e_0u_0$, $Lv_{i+1} = e_iu_i + m_iv_i + n_iw_i$ $(1 \le i \le d-2)$,

(iii) $L^{(-)}w_{i+1} = f_iu_i$ $(0 \le i \le d-2)$.

Proof. Similar to the proof of Lemma 5.7. ∎

6. Determining the shape of the tridiagonal pair. Recall that $u_0 \in U_0, u_i = R^iu_0$ $(1 \le i \le d)$, $v_1 = L^{(+)}u_2, v_i = R^{i-1}v_2$ $(2 \le i \le d-1)$, $w_1 = L^{(0)}v_2$, and $w_i = R^{i-1}w_1$ $(2 \le i \le d-1)$. In this section, we assume the height of A, A^* is 1, and determine the shape of A, A^*.

Lemma 6.1. *Suppose $\rho_1 \ge 4$. Then the vectors $v_1, w_1, L^{(0)}w_2$ are linearly independent.*

Proof. By way of contradiction, we suppose $v_1, w_1, L^{(0)}w_2$ are linearly dependent. Since v_1, w_1 are linearly independent by Lemma 4.7 and by our assumption, $L^{(0)}w_2$ lies in $Y = \operatorname{span}\{v_1, w_1\}$. We set $W = \sum_{i=0}^{d-2} R^iY$. Observe that $U^{(0)} + W$ is invariant under R and $F_i (0 \le i \le d)$. We show $U^{(0)} + W$ is invariant under L.

Obverse that $RY = \operatorname{span}\{v_2, w_2\}$ and $L^{(0)}v_2 = w_1 \in Y$, so that $L^{(0)}RY \subseteq Y$. Hence, W is invariant under $L^{(0)}$ by Lemma 3.17. This implies $LW \subseteq L^{(-)}W + L^{(0)}W \subseteq U^{(0)} + W$. Moreover, $L^{(+)}U^{(0)} \subseteq W$ by Lemma 4.5. So, $U^{(0)}+W$ is invariant under L. Therefore, $U^{(0)}+W$ is invariant under L, R and $F_i(0 \le i \le d)$. From which we have $U^{(0)} + W = V$ by Lemma 2.2. It follows that $U_1 = \{u_1, v_1, w_1\}$, contradicting our assumption $\rho_1 \ge 4$. ∎

Lemma 6.2. *Suppose $\rho_1 \ge 4$, and we set $L^{(0)}w_2 = x_1$. Then*

(i) *If $d = 3$, we have*

$$\begin{aligned}
L^3Ru_3 &= 0,\\
L^2RLu_3 &= (a_1a_2a_2 + e_1a_2b_2)u_1 + a_2a_2v_1 + a_2b_2w_1,\\
LRL^2u_3 &= (a_1(a_1a_2 + e_1b_2) + e_1a_2 + f_1b_2)u_1 + (a_1a_2 + e_1b_2)v_1 + a_2w_1 + b_2x_1,\\
RL^3u_3 &\in \operatorname{span}\{u_1\},\\
L^2u_3 &= (a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1.
\end{aligned}$$

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(ii) *If $d \geq 4$, we have*

$$\begin{aligned}
L^3Ru_3 =&(a_2a_3+e_2b_3)(a_1u_1+v_1)+(b_2a_3+m_2b_3)(e_1u_1+w_1)+n_2b_3(f_1u_1+x_1),\\
L^2RLu_3 =&(a_2a_2+b_2e_2)(a_1u_1+v_1)+(a_2b_2+b_2m_2)(e_1u_1+w_1)+b_2n_2(f_1u_1+x_1),\\
LRL^2u_3 =&(a_1(a_1a_2+e_1b_2)+e_1a_2+f_1b_2)u_1+(a_1a_2+e_1b_2)v_1+a_2w_1+b_2x_1,\\
RL^3u_3 \in& \operatorname{span}\{u_1\},\\
L^2u_3 =&(a_1a_2+e_1b_2)u_1+a_2v_1+b_2w_1.
\end{aligned}$$

Proof. Immediate from Lemmas 2.3, 2.11, 5.7 and 5.14, and (5.78) by a routine computation. ∎

Lemma 6.3. $\rho_1 \leq 3$.

Proof. By way of contradiction, we suppose $\rho_1 \geq 4$. Observe that u_1, v_1, w_1, x_1 are linearly independent by Lemmas 2.6 and 6.1. Applying (2.6) to u_3,

$$L^3Ru_3 + L^2RLu_3 - LRL^2u_3 - RL^3u_3 + \varepsilon_1 L^2u_3 = 0. \tag{6.106}$$

According to the parity of diameter d and the range of diameter d, we divide our computation into three cases: (I) $d = 3$; (II) d is odd and $d \geq 5$; (III) d is even.

Case (I) $d = 3$.

Obverse that the coefficient of x_1 in (6.106) becomes $-b_2$ by Lemma 6.2, so that $-b_2 = 0$, contradicting our convention.

Case (II) d is odd and $d \geq 4$.

Looking at the coefficients of x_1 in (6.106), we have $n_2b_3 + b_2n_2 - b_2 = 0$ by Lemma 6.2, so that 1=0, contradicting our convention.

Case (III) d is even.

Looking at the coefficients of x_1 in (6.106) by Lemma 6.2,

$$\frac{2}{d-2}\cdot\frac{3+1}{2}+(-1)\cdot\frac{2}{d-2}-(-1)=\frac{d}{d-2}=0,$$

so that $d = 0$, contradicting to the assumption. These complete the proof of $\rho_1 \leq 3$. ∎

Theorem 6.4. *One of the following holds.*

(i) $\rho_0 = 1$, $\rho_1 = \rho_2 = \cdots = \rho_{d-1} = 2$, $\rho_d = 1$,

(ii) $d = 3$, $\rho_0 = 1$, $\rho_1 = \rho_2 = 3$, $\rho_3 = 1$.

Proof. Since $\mathbb{K}$ is an algebraically closed field, by [7, Theorem 1.3], $\rho_0 = 1$. And by Lemma 6.3, $\rho_1 \leq 3$. So, $\rho_1 = 2$ or $\rho_1 = 3$, since $h = 1$. If $\rho_1 = 2$, then (i) holds. If $\rho_1 = 3$, we need to show $d = 3$. Applying (2.6) to v_3,

$$L^3Rv_3 + L^2RLv_3 - LRL^2v_3 - RL^3v_3 + \varepsilon_1 L^2v_3 = 0. \tag{6.107}$$

From Lemmas 2.11, 3.14, 5.7 and 5.14, we can compute each term of (6.107). The term of L^3Rv_3 vanishes

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when $d=4$. When $d\geq 5$, it becomes

$$\begin{aligned}
L^3Rv_3 =&(a_1(a_2e_3+e_2m_3+f_2n_3)+e_1(b_2e_3+m_2m_3+s_2n_3)+f_1(n_2m_3+t_2n_3))u_1\\
&+((a_2e_3+e_2m_3+f_2n_3)+s_1(n_2m_3+t_2n_3))v_1\\
&+((b_2e_3+m_2m_3+s_2n_3)+t_1(n_2m_3+t_2n_3))w_1,\\
L^2RLv_3 =&(a_1(a_2e_2+e_2m_2+f_2n_2)+e_1(b_2e_2+m_2m_2+n_2s_2)+f_1(m_2n_2+n_2t_2))u_1\\
&+((a_2e_2+e_2m_2+f_2n_2)+s_1(m_2n_2+n_2t_2))v_1\\
&+((b_2e_2+m_2m_2+n_2s_2)+t_1(m_2n_2+n_2t_2))w_1,\\
LRL^2v_3 =&(a_1(a_1e_2+e_1m_2+f_1n_2)+e_1(e_2+s_1n_2)+f_1(m_2+n_2t_2))u_1\\
&+((a_1e_2+e_1m_2+f_1n_2)+s_1(m_2+t_1n_2))v_1+((e_2+s_1n_2)+t_1(m_2+t_1n_2))w_1,\\
RL^3v_3 \in&\operatorname{span}\{u_1\},\\
L^2v_3 =&(a_1e_2+e_1m_2+f_1n_2)u_1+(e_2+s_1n_2)v_1+(m_2+t_1n_2)w_1.
\end{aligned}$$

According to the parity of diameter d and the range of diameter d, we divide the arguments into four cases: (I) $d=3$; (II) d is odd and $d\geq 5$; (III) $d=4$; (IV) d is even and $d\geq 6$. To show (I) holds, we show that (II), (III), and (IV) do not occur.

Suppose case (II) holds for a contradiction.

By a routine computation, the coefficient of w_1 in (6.107) becomes

$$\begin{aligned}
&b_2e_3+m_2m_3+s_2n_3+t_1(n_2m_3+t_2n_3)+b_2e_2+m_2m_2+n_2s_2+t_1(m_2n_2+n_2t_2)\\
&-(e_2+s_1n_2+t_1(m_2+t_1n_2))+\varepsilon_1(m_2+t_1n_2)=-\frac{d-3}{d-1}e_0=0.
\end{aligned}$$

It follows that $e_0=0$. And this implies $e_i=0$ $(0\leq i\leq d-2)$. The coefficient of v_1 becomes

$$\begin{aligned}
&f_2n_3+s_1(n_2m_3+t_2n_3)+f_2n_2+s_1(m_2n_2+n_2t_2)-(f_1n_2+s_1(m_2+t_1n_2))+\varepsilon_1s_1n_2\\
&=\frac{d-3}{d-1}f_0=0.
\end{aligned}$$

It follows that $f_0=0$. And this implies $f_i=0$ $(0\leq i\leq d-2)$. Since $L^{(-)}w_{i+1}=f_iu_i$, $L^{(-)}v_{i+1}=e_iv_i$ $(0\leq i\leq d-2)$, we have $L^{(-)}U^{(1)}=0$ by Lemma 4.8. Hence, $LU^{(1)}\subseteq U^{(1)}$. Since $U^{(1)}$ is invariant under R and F_i $(0\leq i\leq d)$, we get $U^{(1)}=V$ by Lemma 2.2, a contradiction.

Suppose case (III) holds for a contradiction.

The coefficient of w_1 in (6.107) becomes

$$b_2e_2+m_2m_2+n_2s_2+t_1(m_2n_2+n_2t_2)-(e_2+s_1n_2+t_1(m_2+t_1n_2))+\varepsilon_1(m_2+t_1n_2)=-e_0=0,$$

It implies $e_i=0$ $(0\leq i\leq d-2)$. The coefficient of v_1 becomes

$$f_2n_2+s_1(m_2n_2+n_2t_2)-(f_1n_2+s_1(m_2+t_1n_2))+\varepsilon_1s_1n_2=f_0=0,$$

It implies $f_i=0$ $(0\leq i\leq d-2)$. Hence, $LU^{(1)}\subseteq U^{(1)}$. Since $U^{(1)}$ is invariant under R and F_i $(0\leq i\leq d)$, we get $U^{(1)}=V$ by Lemma 2.2, a contradiction.

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Suppose case (IV) holds for a contradiction.

By a routine computation, the coefficient of w_1 in (6.107) becomes

$$\begin{aligned}&b_2e_3+m_2m_3+s_2n_3+t_1(n_2m_3+t_2n_3)+b_2e_2+m_2m_2+n_2s_2+t_1(m_2n_2+n_2t_2)\\&-(e_2+s_1n_2+t_1(m_2+t_1n_2))+\varepsilon_1(m_2+t_1n_2)=-e_0=0,\end{aligned}$$

It follows that $e_0=0$. And this implies $e_i=0$ $(0\le i\le d-2)$. The coefficient v_1 becomes

$$\begin{aligned}&f_2n_3+s_1(n_2m_3+t_2n_3)+f_2n_2+s_1(m_2n_2+n_2t_2)-(f_1n_2+s_1(m_2+t_1n_2))+\varepsilon_1s_1n_2\\&=f_0=0.\end{aligned}$$

It follows that $f_0=0$. And this implies $f_i=0$ $(0\le i\le d-2)$. So $L^{(-)}U^{(1)}=0$. Hence, $LU^{(1)}\subseteq U^{(1)}$. Since $U^{(1)}$ is invariant under R and F_i $(0\le i\le d)$, we get $U^{(1)}=V$ by Lemma 2.2, a contradiction. These complete the proof of Theorem 6.4. □

7. Determine the structure of tridiagonal pairs A, A^* of height 1.

In Section 6, we have already displayed the shape of tridiagonal pairs of height 1. In this section, for all cases listed in Theorem 6.4, we display a basis for V and give the actions of A, A^* on this basis.

Lemma 7.1. *Suppose Theorem 6.4 (i) holds. Then*

(i) *u_0 is a basis for U_0,*

(ii) *u_i, v_i is a basis for U_i $(1\le i\le d-1)$,*

(iii) *u_d is a basis for U_d,*

(iv) *the vectors*

$$u_0,\, u_1,\, v_1,\, \dots,\, u_{d-1},\, v_{d-1},\, u_d \tag{7.108}$$

form a basis for V.

Proof. Immediate from Lemmas 2.5, 4.1 and 4.4. □

We now give the actions of R, L on the basis in Lemma 7.1.

Lemma 7.2. *Suppose Theorem 6.4 (i) holds. Then there exist scalars a_0, c_1, e_0 in $\mathbb{K}$ such that the maps R, L act on the basis (7.108) as follows.*

$$\begin{aligned}Ru_i&=u_{i+1}\quad(0\le i\le d-1),\quad Ru_d=0,\\Rv_i&=v_{i+1}\quad(1\le i\le d-2),\quad Rv_{d-1}=0,\\Lu_0&=0,\quad Lu_1=a_0u_0,\quad Lu_{i+1}=a_iu_i+b_iv_i\quad(1\le i\le d-1),\\Lv_1&=e_0u_0,\quad Lv_{i+1}=e_iu_i+c_iv_i\quad(1\le i\le d-2),\end{aligned}$$

where if d is odd, the coefficients satisfy (5.80), (5.81), (4.73), (4.74), (5.82), (5.83), (5.84) and (5.85); if d is even, the coefficients satisfy (5.93), (5.94), (4.73), (4.74), (5.95), (5.96), (5.97) and (5.98).

Proof. Immediate from Lemmas 2.11 and 7.1, and equations (4.72), (5.75), (5.76) and (5.77). □

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THEOREM 7.3. *Suppose Theorem 6.4 (i) holds. Then A, A^* act on the basis (7.108) as follows.*

$$\begin{aligned}
&Au_i = \theta_i u_i + u_{i+1} \quad (0 \le i \le d-1), \quad Au_d = \theta_d u_d,\\
&Av_i = \theta_i v_i + v_{i+1} \quad (1 \le i \le d-2), \quad Av_{d-1} = \theta_{d-1} v_{d-1},\\
&A^* u_0 = \theta^*_0 u_0, \quad A^* u_1 = a_0 u_0 + \theta^*_1 u_1,\\
&A^* u_{i+1} = a_i u_i + b_i v_i + \theta^*_{i+1} u_{i+1} \quad (1 \le i \le d-1),\\
&A^* v_1 = e_0 u_0 + \theta^*_1 v_1, \quad A^* v_{i+1} = e_i u_i + c_i v_i + \theta^*_{i+1} v_{i+1} \quad (1 \le i \le d-2).
\end{aligned}$$

Proof. Immediate from Lemma 7.2 and the definition of R, L. ∎

LEMMA 7.4. *Suppose Theorem 6.4 (ii) holds. Then*

(i) *u_0 is a basis for U_0,*

(ii) *u_i, v_i, w_i is a basis for U_i $(1 \le i \le 2)$,*

(iii) *u_3 is a basis for U_3,*

(iv) *the vectors*

$$u_0,\ u_1,\ v_1,\ w_1,\ u_2,\ v_2,\ w_2,\ u_3 \tag{7.109}$$

form a basis for V.

Proof. Similar to the proof of Lemma 7.1. ∎

We now give the actions of R, L on the basis in Lemma 7.4.

LEMMA 7.5. *Suppose Theorem 6.4 (ii) holds. Then there exist scalars a_0, e_0, f_0, s_1, t_1 in $\mathbb{K}$ such that the maps R, L act on the basis (7.109) as follows.*

$$\begin{aligned}
Ru_0 =& u_1, \quad Ru_1 = u_2, \quad Ru_2 = u_3, \quad Ru_3 = 0,\\
Rv_1 =& v_2, \quad Rv_2 = 0,\\
Rw_1 =& w_2, \quad Rw_2 = 0,\\
Lu_0 =& 0, \quad Lu_1 = a_0 u_0, \quad Lu_{i+1} = a_i u_i + b_i v_i \ (1 \leqslant i \leqslant 2),\\
Lv_1 =& e_0 u_0, \quad Lv_2 = e_1 u_1 + w_1,\\
Lw_1 =& f_0 u_0, \quad Lw_2 = f_1 u_1 + s_1 v_1 + t_1 w_1,
\end{aligned}$$

where

$$\begin{aligned}
a_1 =& 2(d-1)cc^*, \quad a_2 = a_0 - 2(2b^*c + 2bc^* + (d+3)cc^*),\\
b_1 =& 1, \quad b_2 = -1,\\
e_1 =& -e_0,\\
f_1 =& -f_0.
\end{aligned}$$

Proof. Similar to the proof of Lemma 7.2. ∎

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Theorem 7.6. *Suppose Theorem 6.4 (ii) holds. Then A, A^* act on the basis (7.109) as follows.*

$$\begin{aligned}
&Au_i = \theta_i u_i + u_{i+1} \quad (0 \le i \le 2), \quad Au_3 = \theta_3 u_3,\\
&Av_1 = \theta_1 v_1 + v_2, \quad Av_2 = \theta_2 v_2,\\
&A^*u_0 = \theta_0^* u_0, \quad A^*u_1 = a_0 u_0 + \theta_1^* u_1,\\
&A^*u_{i+1} = a_i u_i + b_i v_i + \theta_{i+1}^* u_{i+1} \quad (1 \le i \le 2),\\
&A^*v_1 = e_0 u_0 + \theta_1^* v_1, \quad A^*v_2 = e_1 u_1 + w_1 + \theta_2^* v_2,\\
&A^*\omega_1 = f_0 u_0 + \theta_1^* \omega_1, \quad A^*\omega_2 = f_1 u_1 + s_1 v_1 + t_1 \omega_1 + \theta_2^* \omega_2.
\end{aligned}$$

Proof. Similar to the proof of Theorem 7.3. ∎

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