SOLVABILITY TO SOME SYSTEMS OF MATRIX EQUATIONS
USING G-OUTER INVERSES[*]

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Abstract. Two matrix equation systems $AXA = AEA$ and $BAEAX = XAEAD$, where $A \in \mathbb{C}^{m \times n}$ and $B, D, E \in \mathbb{C}^{n \times m}$; and $AXA = AEA$, $BAEAX = B$ and $XAEAD = D$, where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times m}$, $D \in \mathbb{C}^{n \times q}$ and $E \in \mathbb{C}^{n \times m}$, are investigated and equivalent conditions for their solvability are presented. Expressions of their general solutions are established in terms of G-outer inverses of A. Specializing matrices B, D, E, these results are applied to solve various systems of matrix equations. In particular, the set of all G-outer inverses of A is described. Since the fact A is below B under the G-outer (T, S)-partial order implies that any G-outer (T, S)-inverse of B is also a G-outer (T, S)-inverse of A, an additional condition such that the converse holds is studied.

Key words. Outer inverse, G-Drazin inverse, G-outer inverse, G-outer partial order, Minus partial order.

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1. Introduction. The solvability of matrix equations as well as finding their explicit solutions have a large number of applications in physics, mechanics, control theory and many other field [2, 13]. Many matrix equations have been extended to Hilbert spaces operators [11, 14]. Generalized inverses play an important role in the investigation of matrix equations and partial orders [1, 2, 7, 8].

According to standard notation, $\mathbb{C}^{m \times n}$ is the set of $m \times n$ complex matrices. We use $rank(A)$, A^*, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to represent the rank, the conjugate transpose, the null space and the range (column space), respectively, of $A \in \mathbb{C}^{m \times n}$. The index of $A \in \mathbb{C}^{n \times n}$, denoted by $\text{ind}(A)$, is the smallest nonnegative integer k for which $rank(A^k) = rank(A^{k+1})$.

The matrix $X \in \mathbb{C}^{n \times m}$ is an outer (or inner) inverse of $A \in \mathbb{C}^{m \times n}$ if the equality $XAX = X$ ($AXA = A$) is satisfied. The set of all inner inverses of A will be denoted by $A\{1\}$. For $A, B \in \mathbb{C}^{m \times n}$, we say that A is below B under the minus partial order (denoted by $A \leq^- B$) if there exists $A^- \in A\{1\}$ such that $AA^- = BA^-$ and $A^-A = A^-B$ [6]. Also, A is below B under the space pre-order (denoted by $A \preceq^s B$) if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$ [7]. It is well known, by [7, Theorem 3.3.5], that $A \leq^- B$ implies $A \preceq^s B$.

In this paper, we focus on outer inverses of A which have fixed range and null space. Let $A \in \mathbb{C}^{m \times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Recall that X is an outer inverse of A with prescribed range T and null space S if

$$XAX = X, \quad \mathcal{R}(X) = T, \quad \mathcal{N}(X) = S.$$

It is well known that A has an outer inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$ [2]. In this case, X is unique and denoted by $A_{T,S}^{(2)}$. The notation $\mathbb{C}_{T,S}^{m \times n}$ represents the set of all $A \in \mathbb{C}^{m \times n}$

such that $A_{T,S}^{(2)}$ exists.

Some special well known kinds of outer inverses are introduced now. For $A \in \mathbb{C}^{m \times n}$, there exists the Moore-Penrose inverse of A as the unique matrix $A^\dagger = X \in \mathbb{C}^{n \times m}$ such that

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

Let $A \in \mathbb{C}^{n \times n}$ and $k = \mathrm{ind}(A)$. The Drazin inverse of A is the unique matrix $A^D = X \in \mathbb{C}^{n \times n}$ which satisfies

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA.$$

When $\mathrm{ind}(A) = 1$, $A^\# = A^D$ is called the group inverse of A.

A G-Drazin inverse of a square matrix was defined by Wang and Liu [12]. For $A \in \mathbb{C}^{n \times n}$ and $k = \mathrm{ind}(A)$, $X \in \mathbb{C}^{n \times n}$ is a G-Drazin inverse of A if

$$(1.1) \qquad\qquad AXA = A, \quad A^{k+1}X = A^k \quad \text{and} \quad XA^{k+1} = A^k.$$

This set of equations is equivalent to [3]

$$(1.2) \qquad\qquad AXA = A \quad \text{and} \quad A^k X = X A^k.$$

Observe that G-Drazin inverse is not unique in general.

As a generalization of the results from [3, 12], the definition of a G-Drazin inverse of a Banach space operator was presented in [10]. Notice that the definition given in [10] and the above definition of G-Drazin inverses for square matrices are equivalent in complex matrix case [10]. We give an adequate interpretation of the definition from [10]: a matrix $X \in \mathbb{C}^{n \times n}$ is a G-Drazin inverse of $A \in \mathbb{C}^{n \times n}$ if the following equations hold:

$$(1.3) \qquad\qquad AXA = A \quad \text{and} \quad A^D AX = X A^D A.$$

Recently a new generalized inverse which extends the notation of G-Drazin inverse was introduced in [9], using an outer inverse with determined range and null space. Precisely, a G-outer inverse was defined for an operator between two Banach spaces. We now give the version of this definition for complex rectangular matrices. Let $A \in \mathbb{C}_{T,S}^{m \times n}$. A matrix $X \in \mathbb{C}^{n \times m}$ is a G-outer (T,S)-inverse of A if the following equalities hold:

$$(1.4) \qquad\qquad AXA = A \quad \text{and} \quad A_{T,S}^{(2)} AX = X A A_{T,S}^{(2)}.$$

In the particular case that $A_{T,S}^{(2)} = A^D$, the G-outer (T,S)-inverse of A becomes the G-Drazin inverse of A. Notice that, by [9, Theorem 2.1], the system of equations (1.4) is equivalent to

$$(1.5) \qquad\qquad AXA = A, \quad A_{T,S}^{(2)} AX = A_{T,S}^{(2)} \quad \text{and} \quad A_{T,S}^{(2)} = X A A_{T,S}^{(2)}.$$

The G-outer inverse is not unique and we use $A\{GO, T, S\}$ to denote the set of all G-outer (T,S)-inverses of A. Obviously, $A\{GO, T, S\} \subseteq A\{1\}$.

Let $A, B \in \mathbb{C}_{T,S}^{m \times n}$. Then we say that A is below to B under the G-outer (T,S)-relation (denoted by $A \leq^{GO,T,S} B$) if there exist $C_1, C_2 \in A\{GO, T, S\}$ such that

$$AC_1 = BC_1 \quad \text{and} \quad C_2 A = C_2 B.$$

Recall that, by [9, Theorem 3.2], the G-outer (T, S)-relation is a partial order on $\mathbb{C}^{m \times n}_{T,S}$. Since $A\{GO, T, S\} \subseteq A\{1\}$, note that $A \leq^{GO,T,S} B$ yields $A \leq^{-} B$.

Let $A, E \in \mathbb{C}^{n \times n}$ and $k = \operatorname{ind}(A)$. Extending the system (1.2), the matrix equation system

$$(1.6) \qquad AXA = AEA \quad \text{and} \quad A^k EAX = XAEA^k$$

was studied in [4] and its general solution was derived in terms of a G-Drazin inverse of A. The set of all G-Drazin inverses of A was described as a consequence. Some recent results related to the G-Drazin partial order and proved by results from [4], can be found in [5].

Motivated by recent research about the system (1.6) and G-outer inverses, the main contribution of this manuscript is to present solvability of several systems of matrix equation which extend systems (1.1)-(1.6). Firstly, our goal is to investigate the following matrix equation system

$$AXA = AEA \quad \text{and} \quad BAEAX = XAEAD,$$

where $A \in \mathbb{C}^{m \times n}$ and $B, D, E \in \mathbb{C}^{n \times m}$. Obviously, for $m = n$, $k = \operatorname{ind}(A)$ and $B = D = A^{k-1}$, this system becomes (1.6), which means that we consider solvability of system which generalizes the system (1.6). The second research stream is investigation of the system

$$AXA = AEA, \quad BAEAX = B \quad \text{and} \quad XAEAD = D,$$

where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times m}$, $D \in \mathbb{C}^{n \times q}$ and $E \in \mathbb{C}^{n \times m}$. In a special case when $A \in \mathbb{C}^{m \times n}_{T,S}$, $B = D = A^{(2)}_{T,S}$ and $E = A^{\dagger}$, the previous system becomes (1.5) which means that our system is more general then (1.5). We establish the general solutions of our systems and the general solutions of the system (1.5) (or equivalently (1.4)). Also, we present some interesting applications to prove the importance of obtained expressions.

This paper is organized as follows. In Section 2, we get purely algebraic necessary and sufficient conditions for the solvability of our new systems of matrix equation. We find the general forms of their solutions. Applying these results, we investigate the solvability of matrix equation systems which generalize systems (1.1) and (1.5), and we present their general solutions. Particularizing B, D, E of our systems, we obtain more interesting applications in solving several matrix equation systems. In Section 3, we describe the set of all G-outer (T, S)-inverses of a rectangular matrix A. Recall that, by [9, Corollary 3.1], if A is below B under the G-outer (T, S)-partial order, then any G-outer (T, S)-inverse of B is also a G-outer (T, S)-inverse of A. By [5, Example 2.7], we see that the converse is not true in general. We show that the converse is valid under an additional condition. Thus, continuing previous research about G-outer (T, S)-inverses [9], we generalize several results from [4, 5].

In the end of this section, we state two auxiliary results which will be often used.

LEMMA 1.1. [2, p. 52] *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$ and $C \in \mathbb{C}^{m \times q}$. Then the equation $AYB = C$ has a solution $Y \in \mathbb{C}^{n \times p}$ if and only if $AA^{-}CB^{-}B = C$ holds for some $A^{-} \in A\{1\}$ and $B^{-} \in B\{1\}$. In this case, the general solution Y is given as $Y = A^{-}CB^{-} + Z - A^{-}AZBB^{-}$ for arbitrary $Z \in \mathbb{C}^{n \times p}$.*

LEMMA 1.2. *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$. Then $AYB = 0$ for all $Y \in \mathbb{C}^{n \times p}$ if and only if $A = 0$ or $B = 0$.*

2. Solvability to some systems of matrix equations. Firstly, we give an extension of the system (1.6) and a necessary and sufficient algebraic condition for its solvability. Using G-outer inverses, the representation of general solution is presented.

THEOREM 2.1. *Let $A \in \mathbb{C}^{m \times n}$ and $B, D, E \in \mathbb{C}^{n \times m}$. If $A^{(2)}_{\mathcal{R}(B), \mathcal{N}(D)}$ exists (or $rank(B) = rank(D) = rank(DAB)$) and $A\{GO, \mathcal{R}(B), \mathcal{N}(D)\} \neq \emptyset$, then the system*

$$(2.7) \qquad AXA = AEA \quad and \quad BAEAX = XAEAD$$

has a solution if and only if

$$(2.8) \qquad ABA(EA)^2 = (AE)^2 ADA.$$

In this case, the general solution X is given as

$$(2.9) \qquad \begin{aligned} X &= C_1 AEAC_2 + M - (I - A^- A)MAEAD(AEAD)^- \\ &\quad - (BAEA)^- BAEAM(I - AA^-) - A^- AMAA^-, \end{aligned}$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ and for arbitrary but fixed $C_1, C_2 \in A\{GO, \mathcal{R}(B), \mathcal{N}(D)\}$, $A^- \in A\{1\}$, $(AEAD)^- \in (AEAD)\{1\}$ and $(BAEA)^- \in (BAEA)\{1\}$.

Proof. In the case that X is a solution of the system (2.7), we get

$$ABA(EA)^2 = ABAE(AEA) = A(BAEAX)A = (AXA)EADA = (AE)^2 ADA,$$

i.e., (2.8) is valid.

Let (2.18) hold, $C_1, C_2 \in A\{GO, \mathcal{R}(B), \mathcal{N}(D)\}$ and $X' = C_1 AEAC_2$. By [9, Theorem 2.2], $B = C_1 AB$ and $D = DAC_2$. Now, we verify that

$$AX'A = (AC_1 A)E(AC_2 A) = AEA$$

and

$$\begin{aligned} BAEAX' &= BAE(AC_1 A)EAC_2 = C_1(ABAEAEA)C_2 = C_1 AEAEA(DAC_2) \\ &= C_1 AEAEAD = (C_1 AEAC_2)AEAD = X'AEAD, \end{aligned}$$

which imply that $X' = C_1 AEAC_2$ is a solution of the system (2.7).

Notice that a sum of $X' = C_1 AEAC_2$ and the general solution of

$$(2.10) \qquad AXA = 0 \quad and \quad BAEAX = XAEAD,$$

is the general solution of the system (2.7). Using Lemma 1.1 and $A^- \in A\{1\}$, we deduce that the equation $AXA = 0$ has a solution and its general solution is given by

$$(2.11) \qquad X = Z - A^- AZAA^-,$$

for arbitrary $Z \in \mathbb{C}^{n \times m}$. If we substitute (2.11) in $BAEAX = XAEAD$, we obtain

$$(2.12) \qquad BAEAZ(I - AA^-) = (I - A^- A)ZAEAD.$$

Multiplying the equality (2.12) on the left hand side by CA, where $C \in A\{GO, \mathcal{R}(B), \mathcal{N}(D)\}$, we show that

$$(2.13) \qquad BAEAZ(I - AA^-) = 0.$$

Let $(BAEA)^- \in (BAEA)\{1\}$. By Lemma 1.1 and $I - AA^- \in (I - AA^-)\{1\}$, the general solution of (2.13) is

$$(2.14) \qquad\qquad Z = W - (BAEA)^- BAEAW(I - AA^-)$$

for arbitrary $W \in \mathbb{C}^{n \times m}$. Applying (2.12), (2.13) and (2.14), we get

$$(2.15) \qquad\qquad (I - A^- A)WAEAD = 0.$$

For $(AEAD)^- \in (AEAD)\{1\}$, according to Lemma 1.1 and $I - A^- A \in (I - A^- A)\{1\}$, the general solution of (2.15) is represented by

$$(2.16) \qquad\qquad W = M - (I - A^- A)MAEAD(AEAD)^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$. Using (2.11), (2.14) and (2.16), we conclude that the general solution of (2.10) is expressed by

$$X = M - (I - A^- A)MAEAD(AEAD)^- - (BAEA)^- BAEAM(I - AA^-) - A^- AMAA^-.$$

Therefore, the general solution of the system (2.7) is represented by (2.9). $\qquad\qquad\qquad \square$

Notice that, [4, Theorem 2.2] can be obtained as a special case of Theorem 2.1 for $m = n$, $k = \mathrm{ind}(A)$ and $B = D = A^{k-1}$.

If we apply Theorem 2.1, we can solve more systems of matrix equation extending systems (1.3) and (1.4) which are their particular cases.

COROLLARY 2.2. *Let $A \in \mathbb{C}^{m \times n}$ and $B, D \in \mathbb{C}^{n \times m}$.*

(i) *If $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(D)}$ exists and $A\{GO, \mathcal{R}(B), \mathcal{N}(D)\} \neq \emptyset$, then the system*

$$AXA = A \quad and \quad BAX = XAD$$

has a solution if and only if

$$ABA = ADA.$$

In this case, the general solution X is given as

$$X = C + M - (I - A^- A)MAD(AD)^- - (BA)^- BAM(I - AA^-) - A^- AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ and for arbitrary but fixed $C \in A\{GO, \mathcal{R}(B), \mathcal{N}(D)\}$, $A^- \in A\{1\}$, $(AD)^- \in (AD)\{1\}$ and $(BA)^- \in (BA)\{1\}$.

(ii) *If $A^{(2)}_{\mathcal{R}(B),\mathcal{N}(B)}$ exists and $A\{GO, \mathcal{R}(B), \mathcal{N}(B)\} \neq \emptyset$, then the system*

$$AXA = A \quad and \quad BAX = XAB$$

has a solution. In this case, the general solution X is given as

$$X = C + M - (I - A^- A)MAB(AB)^- - (BA)^- BAM(I - AA^-) - A^- AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ and for arbitrary but fixed $C \in A\{GO, \mathcal{R}(B), \mathcal{N}(B)\}$, $A^- \in A\{1\}$, $(AB)^- \in (AB)\{1\}$ and $(BA)^- \in (BA)\{1\}$.

Proof. (i) For $E = A^\dagger$ in Theorem 2.1, we verify this part.

(ii) It follows by (i) when $B = D$. □

COROLLARY 2.3. *Let $A \in \mathbb{C}^{m \times n}$ be of rank r, T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, S be a subspace of $\mathbb{C}^m$ of dimension $m - s$ and $E \in \mathbb{C}^{n \times m}$. If $A_{T,S}^{(2)}$ exists and $A\{GO, T, S\} \neq \emptyset$, then the system*

$$AXA = AEA \quad and \quad A_{T,S}^{(2)}AEAX = XAEAA_{T,S}^{(2)}$$

has a solution if and only if

$$AA_{T,S}^{(2)}A(EA)^2 = (AE)^2 AA_{T,S}^{(2)}A.$$

In this case, the general solution X is given as

$$X = C_1 AEAC_2 + M - (I - A^- A)MAEAA_{T,S}^{(2)}(AEAA_{T,S}^{(2)})^-$$
$$- (A_{T,S}^{(2)}AEA)^- A_{T,S}^{(2)}AEAM(I - AA^-) - A^- AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ and for arbitrary but fixed $C_1, C_2 \in A\{GO, T, S\}$, $A^- \in A\{1\}$, $(AEAA_{T,S}^{(2)})^- \in (AEAA_{T,S}^{(2)})\{1\}$ and $(A_{T,S}^{(2)}AEA)^- \in (A_{T,S}^{(2)}AEA)\{1\}$.

Proof. If $B = D = A_{T,S}^{(2)}$ in Theorem 2.1, we prove this result. □

We now present an equivalent condition for solving a new matrix equation system which is a generalization of the system given by (1.5). Also, we obtain the general solution of this new system in terms of G-outer inverses. Beside the fact that the matrices B and D are not of the same type in Theorem 2.1 and Theorem 2.4, we observe that G-outer inverses which appear in Theorem 2.1 and Theorem 2.4 are not from the same set. Precisely, we use G-outer inverses from the set $A\{GO, \mathcal{R}(B), \mathcal{N}(D)\}$ in Theorem 2.1, but from the set $A\{GO, \mathcal{R}(D), \mathcal{N}(B)\}$ in Theorem 2.4.

THEOREM 2.4. *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times m}$, $D \in \mathbb{C}^{n \times q}$ and $E \in \mathbb{C}^{n \times m}$. If $A_{\mathcal{R}(D),\mathcal{N}(B)}^{(2)}$ exists (or $rank(D) = rank(B) = rank(BAD)$) and $A\{GO, \mathcal{R}(D), \mathcal{N}(B)\} \neq \emptyset$, then the system*

(2.17) $$AXA = AEA, \quad BAEAX = B \quad and \quad XAEAD = D$$

has a solution if and only if

(2.18) $$BA(EA)^2 = BA \quad and \quad (AE)^2 AD = AD.$$

In this case, the general solution X is given as

$$X = C_1 AEAC_2 + M - (I - A^- A)MAEAD(AEAD)^-$$
(2.19) $$- (BAEA)^- BAEAM(I - AA^-) - A^- AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ and for arbitrary but fixed $C_1, C_2 \in A\{GO, \mathcal{R}(D), \mathcal{N}(B)\}$, $A^- \in A\{1\}$, $(AEAD)^- \in (AEAD)\{1\}$ and $(BAEA)^- \in (BAEA)\{1\}$.

Proof. If X is a solution of the system (2.17), then (2.18) is satisfied:

$$BA(EA)^2 = BAE(AEA) = (BAEAX)A = BA$$

and

$$(AE)^2 AD = (AEA)EAD = A(XAEAD) = AD.$$

Suppose that (2.18) holds, $C_1, C_2 \in A\{GO, \mathcal{R}(D), \mathcal{N}(B)\}$ and $X' = C_1 AEAC_2$. Using [9, Theorem 2.2], we obtain

$$AX'A = (AC_1A)E(AC_2A) = AEA,$$
$$BAEAX' = BAE(AC_1A)EAC_2 = (BAEAEA)C_2 = BAC_2 = B$$

and

$$X'AEAD = C_1AE(AC_2A)EAD = C_1(AEAEAD) = C_1AD = D.$$

Thus, $X' = C_1 AEAC_2$ is a solution of the system (2.17).

The general solution of the system (2.17) is a sum of $X' = C_1 AEAC_2$ and the general solution of

$$(2.20) \qquad\qquad AXA = 0, \quad BAEAX = 0 \quad \text{and} \quad XAEAD = 0.$$

Let $A^- \in A\{1\}$. By Lemma 1.1, we have that the general solution of equation $AXA = 0$ is

$$(2.21) \qquad\qquad X = Z - A^- AZAA^-,$$

for arbitrary $Z \in \mathbb{C}^{n \times m}$. Substituting (2.21) in $BAEAX = 0$, we get

$$(2.22) \qquad\qquad BAEAZ(I - AA^-) = 0.$$

For $(BAEA)^- \in (BAEA)\{1\}$, according to Lemma 1.1 and $I - AA^- \in (I - AA^-)\{1\}$, the general solution of (2.22) is given by

$$(2.23) \qquad\qquad Z = W - (BAEA)^- BAEAW(I - AA^-)$$

for arbitrary $W \in \mathbb{C}^{n \times m}$. Using $XAEAD = 0$, (2.21) and (2.23), we verify that

$$(2.24) \qquad\qquad (I - A^- A)WAEAD = 0.$$

Let $(AEAD)^- \in (AEAD)\{1\}$. Applying Lemma 1.1 and $I - A^- A \in (I - A^- A)\{1\}$, the general solution of (2.24) is

$$(2.25) \qquad\qquad W = M - (I - A^- A)MAEAD(AEAD)^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$. From (2.21), (2.23) and (2.25), the general solution of the system (2.20) is given by

$$X = M - (I - A^- A)MAEAD(AEAD)^- - (BAEA)^- BAEAM(I - AA^-) - A^- AMAA^-,$$

which implies that (2.19) is the general solution of (2.17). $\qquad\qquad\qquad\qquad\qquad\qquad \square$

REMARK 2.5. Notice that, if the conditions of Theorem 2.4 and the equalities of (2.18) are satisfied, then, for $(BAEA)^- \in (BAEA)\{1\}$ and $(AEAD)^- \in (AEAD)\{1\}$,

  (i) $BAEA(BAEA)^- BA = BA$ and $AD(AEAD)^- AEAD = AD$;
  (ii) $BAEA(BAEA)^- B = B$ and $D(AEAD)^- AEAD = D$.

Indeed, for $(BAEA)^- \in (BAEA)\{1\}$,

$$BA = (BAEA)EA = BAEA(BAEA)^-(BAEAEA) = BAEA(BAEA)^- BA.$$

Multiplying the previous equality by $C \in A\{GO, \mathcal{R}(D), \mathcal{N}(B)\}$ on the right hand side, we obtain $B = BAEA(BAEA)^- B$. Let $(AEAD)^- \in (AEAD)\{1\}$. Multiplying

$$AD = AE(AEAD) = (AEAEAD)(AEAD)^- AEAD = AD(AEAD)^- AEAD$$

by C on the left hand side, we have $D = D(AEAD)^- AEAD$.

Specializing matrices B, D, E of Theorem 2.4, we obtain some interesting applications. In the particular case that $E = A^\dagger$ in Theorem 2.4, we solve the following system of three equations using a G-outer inverse.

COROLLARY 2.6. *Let* $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times m}$ *and* $D \in \mathbb{C}^{n \times q}$. *If* $A^{(2)}_{\mathcal{R}(D),\mathcal{N}(B)}$ *exists and* $A\{GO, \mathcal{R}(D), \mathcal{N}(B)\} \neq \emptyset$, *then the system*

$$AXA = A, \quad BAX = B \quad and \quad XAD = D$$

has a solution. In addition, the general solution X *is given as*

$$X = C + M - (I - A^-A)MAD(AD)^- - (BA)^-BAM(I - AA^-) - A^-AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ *and for arbitrary but fixed* $C \in A\{GO, \mathcal{R}(D), \mathcal{N}(B)\}$, $A^- \in A\{1\}$, $(AD)^- \in (AD)\{1\}$ *and* $(BA)^- \in (BA)\{1\}$.

Proof. Recall that, by [9, Theorem 2.4], $C_1AC_2 \in A\{GO, T, S\}$, for $C_1, C_2 \in A\{GO, T, S\}$. Using $E = A^\dagger$ in Theorem 2.4, we obtain this consequence. ☐

Choosing $B = D = A^{(2)}_{T,S}$ in Theorem 2.4, we present the system which generalizes (1.5) and get that its solvability is equivalent with some algebraic conditions. We also describe the general form of its solutions.

COROLLARY 2.7. *Let* $A \in \mathbb{C}^{m \times n}$ *be of rank* r, T *be a subspace of* $\mathbb{C}^n$ *of dimension* $s \leq r$, S *be a subspace of* $\mathbb{C}^m$ *of dimension* $m - s$ *and* $E \in \mathbb{C}^{n \times m}$. *If* $A^{(2)}_{T,S}$ *exists and* $A\{GO, T, S\} \neq \emptyset$, *then the system*

$$AXA = AEA, \quad A^{(2)}_{T,S}AEAX = A^{(2)}_{T,S} \quad and \quad XAEAA^{(2)}_{T,S} = A^{(2)}_{T,S}$$

has a solution if and only if

$$A^{(2)}_{T,S}A(EA)^2 = A^{(2)}_{T,S}A \quad and \quad (AE)^2AA^{(2)}_{T,S} = AA^{(2)}_{T,S}.$$

In this case, the general solution X *is given as*

$$X = C_1AEAC_2 + M - (I - A^-A)MAEAA^{(2)}_{T,S}(AEAA^{(2)}_{T,S})^-$$
$$- (A^{(2)}_{T,S}AEA)^-A^{(2)}_{T,S}AEAM(I - AA^-) - A^-AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ *and for arbitrary but fixed* $C_1, C_2 \in A\{GO, T, S\}$, $A^- \in A\{1\}$, $(AEAD)^- \in (AEAD)\{1\}$ *and* $(BAEA)^- \in (BAEA)\{1\}$.

Proof. If we assume that $B = D = A^{(2)}_{T,S}$ in Theorem 2.4, we get this result. ☐

Setting $E = A^{(2)}_{T,S}$ in Corollary 2.7, we show the next consequence.

COROLLARY 2.8. *Let* $A \in \mathbb{C}^{m \times n}$ *be of rank* r, T *be a subspace of* $\mathbb{C}^n$ *of dimension* $s \leq r$ *and* S *be a subspace of* $\mathbb{C}^m$ *of dimension* $m - s$. *If* $A^{(2)}_{T,S}$ *exists and* $A\{GO, T, S\} \neq \emptyset$, *then the system*

$$AXA = AA^{(2)}_{T,S}A, \quad A^{(2)}_{T,S}AX = A^{(2)}_{T,S} \quad and \quad XAA^{(2)}_{T,S} = A^{(2)}_{T,S}$$

has a solution. In addition, the general solution X *is given as*

$$X = A^{(2)}_{T,S} + M - (I - A^-A)MAA^{(2)}_{T,S} - A^{(2)}_{T,S}AM(I - AA^-) - A^-AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ *and for arbitrary but fixed* $A^- \in A\{1\}$.

Proof. Notice that, for $C_1, C_2 \in A\{GO, T, S\}$, $(C_1 A A_{T,S}^{(2)}) A C_2 = A_{T,S}^{(2)} A C_2 = A_{T,S}^{(2)}$. Applying $E = A_{T,S}^{(2)}$ in Corollary 2.7 and according to $(AA_{T,S}^{(2)})^- \in (AA_{T,S}^{(2)})\{1\}$ and $(A_{T,S}^{(2)}A)^- \in (A_{T,S}^{(2)}A)\{1\}$, we prove this result.   □

If $A_{T,S}^{(2)} = A^D$ in Corollary 2.8, we verify the next corollary.

COROLLARY 2.9. *Let $A \in \mathbb{C}^{n \times n}$ and $k = \text{ind}(A)$. Then the system*

$$AXA = AA^D A, \quad A^D AX = A^D \quad and \quad XAA^D = A^D$$

has a solution. In addition, the general solution X is given as

$$X = A^D + M - (I - A^- A)MAA^D - A^D AM(I - AA^-) - A^- AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ and for arbitrary but fixed $A^- \in A\{1\}$.

We can easily check that the system appeared in Corollary 2.9 has a solution if and only if the system $AXA = AA^D A$ and $A^k X = X A^k$ considered in [4, Corollary 3.5], has a solution, and these two systems have the same general solution forms.

Applying Theorem 2.4 for $A \in \mathbb{C}^{n \times n}$, $k = \text{ind}(A)$ and $B = D = A^k$, we consider solvability of an extension of system (1.1).

COROLLARY 2.10. *Let $A \in \mathbb{C}^{n \times n}$, $k = \text{ind}(A)$ and $E \in \mathbb{C}^{n \times n}$. Then the system*

$$AXA = AEA, \quad A^{k+1} EAX = A^k \quad and \quad XAEA^{k+1} = A^k$$

has a solution if and only if

$$A^{k+1}(EA)^2 = A^{k+1} \quad and \quad (AE)^2 A^{k+1} = A^{k+1}.$$

In this case, the general solution X is given as

$$\begin{aligned} X = {}&C_1 AEAC_2 + M - (I - A^- A)MAEA^{k+1}(AEA^{k+1})^- \\ &- (A^{k+1}EA)^- A^{k+1} EAM(I - AA^-) - A^- AMAA^-, \end{aligned}$$

for arbitrary $M \in \mathbb{C}^{n \times m}$ and for arbitrary but fixed $C_1, C_2 \in A\{GD\}$, $A^- \in A\{1\}$, $(AEA^{k+1})^- \in (AEA^{k+1})\{1\}$ and $(A^{k+1}EA)^- \in (A^{k+1}EA)\{1\}$.

3. General representations for G-outer inverses. In this section, we firstly describe the set of all G-outer (T, S)-inverses of A using one particular G-outer (T, S)-inverse of A. Precisely, we present two general representations for G-outer inverses in terms of only one parameter M and in terms of two parameters U and V.

THEOREM 3.1. *Let $A \in \mathbb{C}^{m \times n}$ be of rank r, T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. If $A_{T,S}^{(2)}$ exists and $A\{GO, T, S\} \ne \emptyset$, then*

$$\begin{aligned} A\{GO, T, S\} = {}&\{C + (I - A^- A)M(I - AA_{T,S}^{(2)}) + (I - A_{T,S}^{(2)}A)M(I - AA^-) \\ &- (I - A^- A)M(I - AA^-), \ M \in \mathbb{C}^{n \times m} \text{ is arbitrary}\} \end{aligned}$$
(3.26)

$$\begin{aligned} = {}&\{C + (I - A^- A)V(I - AA_{T,S}^{(2)}) + (I - A_{T,S}^{(2)}A)U(I - AA^-), \\ &U, V \in \mathbb{C}^{n \times m} \text{ are arbitrary}\} \end{aligned}$$
(3.27)

for arbitrary but fixed $C \in A\{GO, T, S\}$ and $A^- \in A\{1\}$.

Proof. For $E = A^\dagger$ in Corollary 2.7, we observe that the general solution of the system (1.5) is given by

$$C + M - (I - A^- A)MAA_{T,S}^{(2)} - A_{T,S}^{(2)}AM(I - AA^-) - A^- AMAA^-,$$

for arbitrary $M \in \mathbb{C}^{n \times m}$, $C \in A\{GO, T, S\}$ and $A^- \in A\{1\}$. By some elementary calculations, we obtain (3.26).

If we set $M = U(I - AA^-) + (I - A^- A)V$ in (3.26), for arbitrary $U, V \in \mathbb{C}^{n \times m}$, by direct calculations, we get (3.27). For $U = M$ and $V = (I - A^- A)M - M(I - AA^-)$ in (3.27), for arbitrary $M \in \mathbb{C}^{n \times m}$, it follows that (3.26) holds.   □

Remark that, if $m = n$ and $A_{T,S}^{(2)} = A^D$ in Theorem 3.1, we obtain [4, Theorem 3.2]. Thus, Theorem 3.1 generalizes [4, Theorem 3.2].

We now consider the range spaces of A and B in the case that any G-outer (T, S)-inverse of B is a G-outer (T, S)-inverse of A.

THEOREM 3.2. *Let $A, B \in \mathbb{C}^{m \times n}$ be of rank r, T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Suppose that $A_{T,S}^{(2)}$ and $B_{T,S}^{(2)}$ exist. If $\emptyset \neq B\{GO, T, S\} \subseteq A\{GO, T, S\}$, then*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ *or* $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$;
(ii) $\mathcal{R}(AA_{T,S}^{(2)}) \subseteq \mathcal{R}(BB_{T,S}^{(2)})$ *and* $\mathcal{R}((A_{T,S}^{(2)}A)^*) \subseteq \mathcal{R}((B_{T,S}^{(2)}B)^*)$.

Proof. Applying Theorem 3.1, the set of all G-outer inverses of B is given by

$$(3.28) \qquad X = D + (I - B^- B)V(I - BB_{T,S}^{(2)}) + (I - B_{T,S}^{(2)}B)U(I - BB^-),$$

for arbitrary $U, V \in \mathbb{C}^{n \times m}$, $D \in B\{GO, T, S\}$ and $B^- \in B\{1\}$.

(i) Set $U = 0$ in (3.28). From $X, D \in B\{GO, T, S\} \subseteq A\{GO, T, S\}$, we have that $A = AXA = ADA$ implying $A(I - B^- B)V(I - BB_{T,S}^{(2)})A = 0$, for arbitrary $V \in \mathbb{C}^{n \times m}$. Applying Lemma 1.2, note that $A(I - B^- B) = 0$ or $(I - BB_{T,S}^{(2)})A = 0$ which give $\mathcal{R}(A^*) \subseteq \mathcal{N}((I - B^- B)^*) = \mathcal{R}(B^*(B^-)^*) = \mathcal{R}(B^*)$ or $\mathcal{R}(A) \subseteq \mathcal{N}(I - BB_{T,S}^{(2)}) = \mathcal{R}(BB_{T,S}^{(2)}) = \mathcal{R}(B)$.

(ii) By $X, D \in B\{GO, T, S\} \subseteq A\{GO, T, S\}$, we deduce that $A_{T,S}^{(2)} = XAA_{T,S}^{(2)} = DAA_{T,S}^{(2)}$. For $U = 0$ in (3.28), we obtain $(I - B^- B)V(I - BB_{T,S}^{(2)})AA_{T,S}^{(2)} = 0$, for arbitrary $V \in \mathbb{C}^{n \times m}$. Using Lemma 1.2, we conclude that $(I - BB_{T,S}^{(2)})AA_{T,S}^{(2)} = 0$, i.e., $\mathcal{R}(AA_{T,S}^{(2)}) \subseteq \mathcal{N}(I - BB_{T,S}^{(2)}) = \mathcal{R}(BB_{T,S}^{(2)})$.

In a similar way, by $A_{T,S}^{(2)} = A_{T,S}^{(2)}AX = A_{T,S}^{(2)}AD$, for $V = 0$ in (3.28), we get $A_{T,S}^{(2)}A(I - B_{T,S}^{(2)}B)U(I - BB^-) = 0$. Hence, $A_{T,S}^{(2)}A(I - B_{T,S}^{(2)}B) = 0$, which yields $\mathcal{R}((A_{T,S}^{(2)}A)^*) \subseteq \mathcal{N}((I - B_{T,S}^{(2)}B)^*) = \mathcal{R}((B_{T,S}^{(2)}B)^*)$. □

For $A_{T,S}^{(2)} = A^D$ and $B_{T,S}^{(2)} = B^D$ in Theorem 3.2, we get [4, Theorem 3.8] as a consequence.

Observe that $A \leq^{GO,T,S} B$ gives $B\{GO, T, S\} \subseteq A\{GO, T, S\}$ by [9, Corollary 3.1]. It is interesting to study when the converse implication holds. Applying an additional condition, we prove that the converse is satisfied.

THEOREM 3.3. *Let $A, B \in \mathbb{C}^{m \times n}$ be of rank r, T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Suppose that $A_{T,S}^{(2)}$ and $B_{T,S}^{(2)}$ exist and $A\{GO, T, S\} \neq \emptyset \neq B\{GO, T, S\}$. Then the following statements are equivalent:*

(i) $A \leq^{GO,T,S} B$;

(ii) $B\{GO, T, S\} \subseteq A\{GO, T, S\}$ *and* $A \preceq^s B$.

Proof. (i) $\Rightarrow$ (ii): It follows by [9, Corollary 3.1] and the implications $A \leq^{GO,T,S} B \Rightarrow A \leq^- B \Rightarrow A \preceq^s B$.

(ii) $\Rightarrow$ (i): Let $C \in A\{GO, T, S\}$ and $D \in B\{GO, T, S\}$. The hypothesis $B\{GO, T, S\} \subseteq A\{GO, T, S\}$ implies that $D \in A\{GO, T, S\}$. If $C_1 = CAD$ and $C_2 = DAC$, by [9, Theorem 2.4], we have that $C_1, C_2 \in A\{GO, T, S\}$. The assumption $A \preceq^s B$ gives $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$. Thus, there exist $Z \in \mathbb{C}^{n \times n}$ and $W \in \mathbb{C}^{m \times m}$ such that $A = BZ = WB$. Now, we get

$$C_1 A = C(ADA) = CA = CWB = C(WB)DB = (CAD)B = C_1 B$$

and

$$AC_2 = (ADA)C = AC = BZC = BD(BZ)C = B(DAC) = BC_2,$$

that is, $A \leq^{GO,T,S} B$.   □

In the special case that $A_{T,S}^{(2)} = A^D$ and $B_{T,S}^{(2)} = B^D$, Theorem 3.3 recovers [5, Theorem 2.4].

REMARK 3.4. Notice that the results of this paper are also true for Hilbert space operators. This fact can be proved using the same algebraic techniques as in this paper, and it is not need to use Hilbert space operator techniques. In particular, we can assume that A, B, D, E are bounded linear operators between corresponding Hilbert spaces in Theorem 2.4 such that operators A, $AEAD$ and $BAEA$ have closed range. Then (2.17) is the system of operator equations and we prove an equivalent condition for its solvability and its general solution form in the same manner as in the proof of Theorem 2.4.

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