Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 90-93, March 2020.

THE TRIANGLE GRAPH T_6 IS NOT SPN*

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Abstract. A real symmetric matrix A is copositive if $x'Ax \geq 0$ for every nonnegative vector x. A matrix is SPN if it is a sum of a real positive semidefinite matrix and a nonnegative matrix. Every SPN matrix is copositive, but the converse does not hold for matrices of order greater than 4. A graph G is an SPN graph if every copositive matrix whose graph is G is SPN. It is shown that the triangle graph T_6 is not SPN.

Key words. Copositive matrices, SPN matrices.

AMS subject classifications. 15B48, 15B35.

1. Introduction. A real symmetric matrix A is said to be copositive (COP) if $x'Ax \geq 0$ for every nonnegative vector x. It is said to be SPN if it can be expressed as a sum of a real positive semidefinite matrix and a nonnegative matrix. It is immediate that every SPN matrix is COP. Diananda [2] showed that every COP matrix of order less than or equal to 4 is SPN. A more comprehensive discussion of this situation is given in [5].

For matrices of order 5 and more, not every COP matrix is SPN. The question then becomes to understand which non-zero patterns allow one to deduce SPN from COP. This is done in terms of a graph G, the matrix under consideration only being allowed to have non-zero entries on the diagonal and where the adjacency matrix of G is non-zero. A graph G is said to be SPN if this additional restriction forces a COP matrix to be SPN.

In [6], Shaked-Monderer gives a comprehensive and fascinating discussion of this topic. Unfortunately the article contains an unfortunate error, see [7]. In a further paper [4], further progress is made and a number of conjectures are stated. In particular, it is conjectured that the triangle graph T_n is SPN for all $n \geq 5$. The graph T_n has vertices 1 and 2 joined by an edge and further vertices $3, 4, \ldots, n$ each joined only to the vertices 1 and 2.

The object of this article is to show that T_6 is not an SPN graph and one may conclude that T_n is also not an SPN graph for $n \geq 6$. It is known that T_5 is an SPN graph [7].

2. The example. Let

$$A = \begin{pmatrix} 1 & -\cos(\theta) & \cos(2\theta) & -\cos(\theta) & -\sin(\theta) & 1 \\ -\cos(\theta) & 1 & -\cos(\theta) & \cos(2\theta) & 1 & -\sin(\theta) \\ \cos(2\theta) & -\cos(\theta) & 1 & 0 & 0 & 0 \\ -\cos(\theta) & \cos(2\theta) & 0 & 1 & 0 & 0 \\ -\sin(\theta) & 1 & 0 & 0 & 1 & 0 \\ 1 & -\sin(\theta) & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

*Received by the editors on December 20, 2019. Accepted for publication on January 6, 2020. Handling Editor: Michael Tsatsomeros.

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A publication of the International Linear Algebra Society
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where θ is chosen in the range $0 < \theta < \frac{\pi}{6}$. Then A is carried on the graph T_6. Our objective is to show that A is COP but not SPN.

Proposition 1. *The matrix A is not SPN.*

Proof. Let $v_1 = (1, 2\cos(\theta), 1, 0, 0, 0)'$, $v_2 = (2\cos(\theta), 1, 0, 1, 0, 0)'$, $v_3 = (0, 1, \cos(\theta), 0, 0, \sin(\theta))'$ and $v_4 = (1, 0, 0, \cos(\theta), \sin(\theta), 0)'$. Then $v_j' A v_j = 0$ for $j = 1, 2, 3, 4$. Suppose that $A = P + N$ where P is positive semidefinite and N is nonnegative then $v_j' P v_j \geq 0$ and $v_j' N v_j \geq 0$ since the v_j are nonnegative vectors. It follows that $v_j' P v_j = 0$ for $j = 1, 2, 3, 4$ and then that $P v_j = 0$. For each j, let S_j be the support of v_j. Then since for $v_j' N v_j = 0$, it follows that $p_{i,k} = a_{i,k}$ for $(i, k) \in \bigcup_{j=1}^{4}(S_j \times S_j)$. Thus, we may write

$$P = \begin{pmatrix} 1 & -\cos(\theta) & \cos(2\theta) & -\cos(\theta) & -\sin(\theta) & p_{1,6} \\ -\cos(\theta) & 1 & -\cos(\theta) & \cos(2\theta) & p_{2,5} & -\sin(\theta) \\ \cos(2\theta) & -\cos(\theta) & 1 & p_{3,4} & p_{3,5} & 0 \\ -\cos(\theta) & \cos(2\theta) & p_{3,4} & 1 & 0 & p_{4,6} \\ -\sin(\theta) & p_{2,5} & p_{3,5} & 0 & 1 & p_{5,6} \\ p_{1,6} & -\sin(\theta) & 0 & p_{4,6} & p_{5,6} & 1 \end{pmatrix}.$$

Next, we study the 24 equations coming from $P v_j = 0$ for $j = 1, 2, 3, 4$. These amount to

$$\begin{aligned} 0 &= p_{1,6} - \sin(2\theta), \\ 0 &= p_{2,5} - \sin(2\theta), \\ 0 &= p_{3,4} + \cos(3\theta), \\ 0 &= p_{1,6} + \cos(\theta)p_{4,6} + \sin(\theta)p_{5,6}, \\ 0 &= p_{2,5} + \cos(\theta)p_{3,5} + \sin(\theta)p_{5,6}, \\ 0 &= \sin(2\theta)p_{1,6} + p_{4,6}, \\ 0 &= \sin(2\theta)p_{2,5} + p_{3,5}, \\ 0 &= \cos(2\theta) + \cos(\theta)p_{3,4} + \sin(\theta)p_{4,6}, \\ 0 &= \cos(2\theta) + \cos(\theta)p_{3,4} + \sin(\theta)p_{3,5} \end{aligned}$$

and for $0 < \theta < \frac{\pi}{2}$, the only solution is

$$p_{1,6} = p_{2,5} = \sin(2\theta), \quad p_{3,5} = p_{4,6} = (1 - 4\cos(\theta)^2)\sin(\theta), \quad -p_{3,4} = p_{5,6} = \cos(3\theta).$$

Thus, unless $\cos(3\theta) = 0$, $n_{3,4}$ and $n_{5,6}$ have opposite signs. This contradiction shows that the decomposition $A = P + N$ cannot exist. ∎

We need the following from [3, Lemma 3.1].

Lemma 2. *Let A be a real symmetric matrix with $a_{i,i} = 1$ for $i = 1, \ldots, n$. Suppose that every principal submatrix B of A, in which each off-diagonal entry is less than 1, is copositive. Then A is copositive.*

Proposition 3. *The matrix A is COP.*

Proof. By Lemma 2 and using the fact that it is sufficient to consider only maximal subsets with the stated property, we see that we need only consider the four subsets $Q_1 = \{1, 2, 3, 4\}$, $Q_2 = \{1, 3, 4, 5\}$, $Q_3 = \{2, 3, 4, 6\}$ and $Q_4 = \{3, 4, 5, 6\}$ and the corresponding principal submatrices B_j for $j = 1, 2, 3, 4$. Since these matrices are 4×4, it will suffice by the result of Diananda [2] to show that they are SPN.

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We see that $B_4 = I$ is positive semidefinite. Also

$$B_2 = \begin{pmatrix} 1 & 0 & -\cos(\theta) & -\sin(\theta) \\ 0 & 1 & 0 & 0 \\ -\cos(\theta) & 0 & 1 & 0 \\ -\sin(\theta) & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \cos(2\theta) & 0 & 0 \\ \cos(2\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is SPN since $\cos(2\theta) \geq 0$ and the first matrix in the sum is positive semidefinite. The matrix B_3 is similar to B_2. We have

$$B_1 = \begin{pmatrix} 1 & -\cos(\theta) & \cos(2\theta) & -\cos(\theta) \\ -\cos(\theta) & 1 & -\cos(\theta) & \cos(2\theta) \\ \cos(2\theta) & -\cos(\theta) & 1 & -\cos(3\theta) \\ -\cos(\theta) & \cos(2\theta) & -\cos(3\theta) & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(3\theta) \\ 0 & 0 & \cos(3\theta) & 0 \end{pmatrix},$$

where the first matrix in the sum is positive semidefinite of rank 2 and the second is nonnegative since $\cos(3\theta) \geq 0$. Thus, the B_j are SPN for $j = 1, 2, 3, 4$. ∎

3. Comments. For a real symmetric $n \times n$ matrix A, one may define

$$\operatorname{copv}(A) = \inf\{x'Ax : x \in \mathbb{R}^n_+,\ \|x\| \leq 1\},$$

where $\|x\|$ denotes the euclidean norm of the vector x. One may also define

$$\operatorname{spnv}(A) = \sup\{\lambda_{\min}(A - N) : N \text{ a nonnegative matrix}\}. \tag{3.2}$$

Here, we have denoted $\lambda_{\min}(X)$ the smallest eigenvalue of a real symmetric matrix X. Clearly, $\operatorname{spnv}(A) \leq \operatorname{copv}(A)$, A is COP if and only if $\operatorname{copv}(A) \geq 0$, and A is SPN if and only if $\operatorname{spnv}(A) \geq 0$.

It is hard to compute $\operatorname{copv}(A)$ numerically, but $\operatorname{spnv}(A)$ can be computed for real symmetric matrices A with constant diagonal (i.e., all diagonal entries equal) using semidefinite programming. To see this first observe that the definition of $\operatorname{spnv}(A)$ is unchanged if we restrict N to be a nonnegative matrix with zero diagonal in (3.2). Then we will have

$$\operatorname{spnv}(A) = \sup\{t : A - N - X - tI = 0,\ N \text{ nonnegative with zero diagonal},\ X \text{psd.}\}. \tag{3.3}$$

But in (3.3) X is forced to have a constant diagonal. Therefore,

$$\operatorname{spnv}(A) = \sup\{-x_{1,1} : X \text{ psd. with constant diagonal},\ X \leq A \text{ off diagonal}\}.$$

Thus, the computation of $\operatorname{spnv}(A)$ becomes a semidefinite programming problem over $n \times n$ real psd. matrices X with objective function $X \mapsto -x_{1,1}$, linear constraints $x_{1,1} - x_{j,j} = 0$ for $j = 2, \ldots, n$ and linear inequalities $x_{j,k} \leq a_{j,k}$ for $1 \leq j < k \leq n$.

The counterexample presented above was obtained by a computer search seeking to maximize the ratio (of negative numbers) $\operatorname{spnv}(A)/\operatorname{copv}(A)$ for matrices A with zero diagonal carried on the graph. For this, we used the well established software packages CSDP [1] and SDPA [8] to estimate $\operatorname{spnv}(A)$. Heuristic estimates were used for $\operatorname{copv}(A)$. Further comments are available at `http://www.math.mcgill.ca/drury/research/spn/`. The optimal value of θ is close to 0.366032 and in this case, the diagonal entries of A need to be replaced with 1.002873 to render the resulting matrix SPN.

Acknowledgment. I thank the referee for a very speedy referee's report and for showing me how to write the paper without reference to numerics.

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A publication of the International Linear Algebra Society
Volume 36, pp. 90-93, March 2020.

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