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EXTREME POINTS OF CERTAIN TRANSPORTATION POLYTOPES WITH FIXED TOTAL SUMS*

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Abstract. Transportation matrices are $m\times n$ nonnegative matrices with given row sum vector R and column sum vector S. All such matrices form the convex polytope $\mathcal{U}(R,S)$ which is called a transportation polytope and its extreme points have been classified. In this article, we consider a new class of convex polytopes $\Delta(\bar{R},\bar{S},\sigma)$ consisting of certain transportation polytopes satisfying that the sum of all elements is σ, and the row and column sum vectors are dominated componentwise by the given positive vectors $\bar{R}$ and $\bar{S}$, respectively. We characterize the extreme points of $\Delta(\bar{R},\bar{S},\sigma)$. Moreover, we give the minimal term rank and maximal permanent of $\Delta(\bar{R},\bar{S},\sigma)$.

Key words. Transportation polytopes, Extreme points, Term ranks, Permanents.

AMS subject classifications. 15B51, 52B05.

1. Introduction.

Let $R=(r_1,\ldots,r_m)$ and $S=(s_1,\ldots,s_n)$ be nonnegative vectors satisfying

$$\sigma=\sum_{i=1}^{m} r_i=\sum_{j=1}^{n} s_j. \tag{1.1}$$

The *transportation polytope* $\mathcal{U}(R,S)$ is the set of all $m\times n$ nonnegative matrices with row sum vector R and column sum vector S. The matrices in $\mathcal{U}(R,S)$ are called *transportation matrices*. Transportation polytopes model the transportation of goods from m supply locations to n demand locations. The ith supply location supplies a quantity of r_i, while the jth demand location demands a quantity of s_j. A matrix $A=[a_{i,j}]$ in $\mathcal{U}(R,S)$ models the scenario where $a_{i,j}$ is the amount of material transported from the ith supply location to the jth demand location. The ith row sum of A is denoted by $r_i(A)$, and the jth column sum of A is denoted by $s_j(A)$. Denote by $R(A)$ the row sum vector of a matrix A and by $S(A)$ the column sum vector of A, i.e.

$$R(A)=(r_1(A),\ldots,r_m(A)),S(A)=(s_1(A),\ldots,s_n(A)).$$

The polytope $\mathcal{U}(R,S)$ is nonempty if and only if (1.1) holds. Due to the applications in many optimization problems, transportation polytopes have been intensively studied by many mathematicians, e.g. [3, 17, 20, 16, 13, 15]. Given an $m\times n$ matrix A, let $\mathcal{P}(A)$ denote the $(0,1)$-matrix with 1's in the position of the nonzero entries of A and 0's elsewhere. $\mathcal{P}(A)$ is called the *pattern* of A. The *bipartite graph* $BG(A)$ of A is a graph with vertex set $\{\mathbf{r}_1,\ldots,\mathbf{r}_m\}\cup\{\mathbf{s}_1,\ldots,\mathbf{s}_n\}$, where there is an edge between $\mathbf{r}_i$ and $\mathbf{s}_j$ if and only if

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$a_{i,j} \neq 0$. A *line* of a matrix A means either a row or a column of A. Denote the set of extremal matrices of $\mathcal{U}(R,S)$ by $\mathfrak{E}(R,S)$, which was first investigated by Jurkat and Ryser in [17].

We summarize the results in the following Proposition.

PROPOSITION 1.1 ([12, 17]). *If $A \in \mathcal{U}(R,S)$, then the following conditions are equivalent:*

(i) $A \in \mathfrak{E}(R,S)$.
(ii) Every submatrix of A contains a line with at most one positive entry.
(iii) Every submatrix A' of A of size $m' \times n'$ has at most $m'+n'-1$ positive entries.
(iv) There is no matrix $B \in \mathcal{U}(R,S)$ such that $B \neq A$ and $\mathcal{P}(B) = \mathcal{P}(A)$.
(v) $BG(A)$ is a forest with no isolated vertex.

Given nonnegative vectors $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ satisfying (1.1), denote by $\mathcal{U}_{\leq}(R,S)$ the convex set of all $m \times n$ nonnegative matrices with row sum vectors dominated componentwise by R and column sum vectors dominated componentwise by S. We say that the ith row sum (resp. jth column sum) of a matrix B in $\mathcal{U}_{\leq}(R,S)$ is *unattained* if $r_i(B) < r_i$ (resp. $s_j(B) < s_j$). Equivalently, we also say the ith row (resp. jth column) vertex in $BG(B)$ is unattained. Denote the set of all extreme points in $\mathcal{U}_{\leq}(R,S)$ by $\mathfrak{E}_{\leq}(R,S)$. In [4], Brualdi gave a combinatorial classification of the extreme points of $\mathcal{U}_{\leq}(R,S)$ as the following theorem.

PROPOSITION 1.2 ([4]). *If $B \in \mathcal{U}_{\leq}(R,S)$, then the following conditions are equivalent:*

(i) $B \in \mathfrak{E}_{\leq}(R,S)$.
(ii) $BG(B)$ is a forest where at most one vertex of each tree corresponds to a row or a column of B whose sum in B is unattained.
(iii) There exists some extreme point A of $\mathcal{U}(R,S)$, such that $BG(B)$ can be obtained by deleting a set (possibly empty) of edges of a subtree from each connected components in $BG(A)$. Thus B can be obtained by replacing by zero the positive entries of A corresponding to the deleted edges.

By specializing $m = n$ and $R = S = (1, \dots, 1) \in \mathbb{R}^n$ in $\mathcal{U}(R,S)$, we obtain the convex polytope Ω_n of all $n \times n$ doubly stochastic matrices. The extreme points of Ω_n were characterized by Birkhoff [1] to be all $n \times n$ permutation matrices. Mirsky [23] investigated the convex polytope ω_n of all $n \times n$ doubly substochastic matrices which are nonnegative matrices whose row and column sums are at most 1. He proved that the extreme points of ω_n are all $n \times n$ subpermutation matrices. In [18] and [19], Katz gave the extreme points of the polytopes of symmetric doubly stochastic matrices and symmetric doubly substochastic matrices, respectively. In 1977, Cruse [7] characterized the extreme points of the polytope of centrosymmetric doubly stochastic even matrices, while the odd case was solved by Brualdi and Cao [6] in 2018. Later on, Chen, Cao and Wang [11] characterized the extreme points of the polytope of centrosymmetric doubly substochastic matrices. Moreover, Cao and Chen [9] studied the convex set ω_n^s of all $n \times n$ doubly substochastic matrices with the sum of all elements equal to s.

For a positive integer n and $0 \leq s \leq n$, denote by $\lfloor s \rfloor$ the greatest integer less than or equal to s, and denote by $\lceil s \rceil$ the smallest integer greater than or equal to s. For a vector $v = (v_1, \dots, v_n)$, denote the sum of all elements in v by $|v|$, i.e., $|v| = v_1 + \cdots + v_n$. Let

$$v_n^s = (\underbrace{1, 1, \dots, 1}_{\lfloor s \rfloor}, s - \lfloor s \rfloor, 0, \dots, 0) \in \mathbb{R}^n,$$

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which contains $\lfloor s \rfloor$ 1's and satisfies $|v_n^s| = s$. Denote by $\mathcal{RE}(v_n^s)$ the set of all rearrangements of v_n^s, i.e.,

$$\mathcal{RE}(v_n^s) = \{v \in \mathbb{R}^n :\ \exists\ \pi \in S_n, \pi(v) = (v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n)}) = v_n^s\}.$$

For $0 \le \alpha \le 1$, let $B_m(\alpha)$ be the $m \times m$ matrix in the form:

$$B_m(\alpha) = \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 1-\alpha & \alpha & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1-\alpha & \alpha \end{pmatrix}.$$

The extreme points of ω_n^s can be characterized as the following proposition.

Proposition 1.3 ([9]). *Let $A \in \omega_n^s$. The fmollowing statements are equivalent:*

(a) $A \in \mathfrak{E}(\omega_n^s)$.
(b) There exist $R, S \in \mathcal{RE}(v_n^s)$, such that $A \in \mathfrak{E}(R, S)$.
(c) There exist $n \times n$ permutation matrices P and Q, such that

$$PAQ = I_{\lceil s \rceil - m} \oplus B_m(s + 1 - \lceil s \rceil) \oplus O_{n - \lceil s \rceil},$$

for some $1 \le m \le \lceil s \rceil$, where $I_{\lceil s \rceil - m}$ is the identity matrix of szize $\lceil s \rceil - m$ and $O_{n-\lceil s \rceil}$ is the zero matrix of size $n - \lceil s \rceil$.
(d) Each connected component of $BG(A)$ is either an isolated vertex or a path. For s not an integer, there exists one path with length $1 \le m \le \lceil s \rceil$ and $\lceil s \rceil - m$ paths with length 1 in $BG(A)$. When s is an integer, there are s paths with length 1 in $BG(A)$.

Given nonnegative vectors $\bar{R} = (\bar{r}_1, \dots, \bar{r}_m)$ and $\bar{S} = (\bar{s}_1, \dots, \bar{s}_n)$, consider the set

$$\Delta(\bar{R}, \bar{S}, \sigma) = \bigcup_{\substack{|R|=|S|=\sigma \\ R \lhd \bar{R}, S \lhd \bar{S}}} \mathcal{U}(R, S), \tag{1.2}$$

where $R \lhd \bar{R}$ means R is dominated componentwise by $\bar{R}$, and $S \lhd \bar{S}$ means S is dominated componentwise by $\bar{S}$. Here $|\bar{R}|$ is not necessarily equal to $|\bar{S}|$. As we show in the following lemma, a necessary and sufficient condition of $\Delta(\bar{R}, \bar{S}, \sigma)$ being nonempty is simply that $\sigma \le \min\{|\bar{R}|, |\bar{S}|\}$.

Lemma 1.4. *The convex polytope $\Delta(\bar{R}, \bar{S}, \sigma)$ is nonempty if and only if $0 \le \sigma \le \min\{|\bar{R}|, |\bar{S}|\}$.*

Proof. " $\Leftarrow$ " Without loss of generality, we assume that $\bar{r}_1 \le \bar{r}_2 \le \cdots \le \bar{r}_m$ and $\bar{s}_1 \le \bar{s}_2 \le \cdots \le \bar{s}_n$. For any $0 \le \sigma \le \min\{|\bar{R}|, |\bar{S}|\}$, there exist some i_0 and j_0 such that

$$\bar{r}_1 + \cdots + \bar{r}_{i_0} < \sigma \le \bar{r}_1 + \cdots + \bar{r}_{i_0} + \bar{r}_{i_0+1},$$

and

$$\bar{s}_1 + \cdots + \bar{s}_{j_0} < \sigma \le \bar{s}_1 + \cdots + \bar{s}_{j_0} + \bar{s}_{j_0+1}.$$

Thus, we can take

$$\bar{R}_0 = \left(\bar{r}_1, \dots, \bar{r}_{i_0}, \sigma - \sum_{i=1}^{i_0} \bar{r}_i, 0, \dots, 0\right),$$

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and

$$\bar{S}_0 = \left(\bar{s}_1, \ldots, \bar{s}_{j_0}, \sigma - \sum_{j=1}^{j_0} \bar{s}_i, 0, \ldots, 0\right).$$

Since $|R_0| = |S_0| = \sigma$ and $R_0 \lhd \bar{R}$, $S_0 \lhd \bar{S}$, we have $\mathcal{U}(R_0, S_0) \subseteq \Delta(\bar{R}, \bar{S}, \sigma)$. Thus $\Delta(\bar{R}, \bar{S}, \sigma)$ is nonempty.

" $\Rightarrow$ " Suppose that $\Delta(\bar{R}, \bar{S}, \sigma) \neq \emptyset$ and $A = [a_{i,j}] \in \Delta(\bar{R}, \bar{S}, \sigma)$. Since $R(A) \lhd \bar{R}, S(A) \lhd \bar{S}, \sum_{i,j=1}^n a_{i,j} = \sigma$, we have $\sigma = |R(A)| \leq |\bar{R}|$ and $\sigma = |S(A)| \leq |\bar{S}|$. Thus $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$. □

Since $\Delta(\bar{R}, \bar{S}, \sigma)$ is convex, it is natural to ask what are the extreme points of $\Delta(\bar{R}, \bar{S}, \sigma)$. We answer this question in Section 2. In Section 3, we investigate the minimal term rank $\tilde{\rho}$ of the matrices in $\Delta(\bar{R}, \bar{S}, \sigma)$. In Section 4, we study the upper bounds for the permanents of matrices in $\Delta(\bar{R}, \bar{S}, \sigma)$.

2. The extreme points of $\boldsymbol{\Delta(\bar{R}, \bar{S}, \sigma)}$. Given nonnegative vectors

$$\bar{R} = (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m), \;\; \bar{S} = (\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n),$$

and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, denote by $\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$ the set of all extreme points of $\Delta(\bar{R}, \bar{S}, \sigma)$. We then characterize the matrices in $\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$. We say that the ith row sum (resp. the jth column sum) of a matrix A in $\Delta(\bar{R}, \bar{S}, \sigma)$ is *unattained* if $0 < r_i(A) < \bar{r}_i$ (resp. $0 < s_j(A) < \bar{s}_j$). Or equivalently, the ith row sum (resp. the jth column sum) vertex in $BG(A)$ is unattained.

THEOREM 2.1. *Given nonnegative vectors $\bar{R} = (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m)$, $\bar{S} = (\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)$, and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, if $A \in \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$ and $BG(A)$ is connected, then at most one row vertex (resp. one column vertex) corresponding to a row (resp. a column) in A is unattained.*

Proof. Assume that both the row vertices $\mathbf{r}_i$ and $\mathbf{r}_j$ of the connected component corresponding to row i and row j, respectively, are unattained. There is a path connecting $\mathbf{r}_i$ and $\mathbf{r}_j$, i.e.,

$$\mathbf{r}_i \to \mathbf{s}_{j_1} \to \mathbf{r}_{i_1} \to \mathbf{s}_{j_2} \to \cdots \to \mathbf{r}_{i_{k-1}} \to \mathbf{s}_{j_k} \to \mathbf{r}_j.$$

We then add an ϵ to the entries

$$(i, j_1), (i_1, j_2), \ldots, (i_{k-1}, j_k),$$

and subtract an ϵ from the entries

$$(i_1, j_1), (i_2, j_2), \ldots, (i_{k-1}, j_{k-1}), (j, j_k).$$

The resultant matrix is denoted by A_1. We can illustrate this process by the following path

$$\mathbf{r}_i \xrightarrow{+\epsilon} \mathbf{s}_{j_1} \xrightarrow{-\epsilon} \mathbf{r}_{i_1} \xrightarrow{+\epsilon} \mathbf{s}_{j_2} \xrightarrow{-\epsilon} \cdots \xrightarrow{-\epsilon} \mathbf{r}_{i_{k-1}} \xrightarrow{+\epsilon} \mathbf{s}_{j_k} \xrightarrow{-\epsilon} \mathbf{r}_j.$$

Similarly, by reversing the sign of ϵ, we get another matrix A_2 which can be illustrated by the following path

$$\mathbf{r}_i \xrightarrow{-\epsilon} \mathbf{s}_{j_1} \xrightarrow{+\epsilon} \mathbf{r}_{i_1} \xrightarrow{-\epsilon} \mathbf{s}_{j_2} \xrightarrow{+\epsilon} \cdots \xrightarrow{+\epsilon} \mathbf{r}_{i_{k-1}} \xrightarrow{-\epsilon} \mathbf{s}_{j_k} \xrightarrow{+\epsilon} \mathbf{r}_j.$$

It is easy to check that

$$|A_1| = |A_2| = |A| = \sigma,$$

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and

$$S(A_1) = S(A_2) = S(A).$$

The row sum vectors of A_1 and A_2 differ from $R(A)$ only on the ith and jth row sum entries, which can be shown as

$$r_i(A_1) = r_i(A) + \epsilon,\ r_i(A_2) = r_i(A) - \epsilon,$$
$$r_j(A_1) = r_j(A) - \epsilon,\ r_j(A_2) = r_j(A) + \epsilon.$$

Since both $r_i(A)$ and $r_j(A)$ in $R(A)$ are unattained, we can choose ϵ small enough to make sure that both A_1 and A_2 are in $\Delta(\bar{R}, \bar{S}, \sigma)$. Since

$$A = \frac{1}{2}A_1 + \frac{1}{2}A_2,$$

A is not an extreme point of $\Delta(\bar{R}, \bar{S}, \sigma)$. This contradicts with the given condition. The case that at most one column sum entry in $S(A)$ is unattained can be proved similarly. ∎

REMARK 2.2. When $BG(A)$ contains two or more connected components and A is extreme in $\Delta(\bar{R}, \bar{S}, \sigma)$, by the similar proof as in Theorem 2.1, we conclude that each connected component of $BG(A)$ contains at most one row vertex (resp. one column vertex) which is unattained.

THEOREM 2.3. *Given nonnegative vectors* $\bar{R} = (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m), \bar{S} = (\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)$, *and* $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, *if* $A \in \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$, *then* $BG(A)$ *contains at most one connected component with exactly one row vertex and one column vertex which are unattained.*

Proof. Assume that $BG(A)$ has two connected components, each of which contains exactly one row vertex and one column vertex unattained, implying that $BG(A)$ has two disconnected paths

$$\mathbf{r}_{i_1} \to \mathbf{s}_{j_1} \to \mathbf{r}_{i_2} \to \mathbf{s}_{j_2} \to \cdots \to \mathbf{r}_{i_p} \to \mathbf{s}_{j_p}, \tag{2.3}$$

and

$$\mathbf{r}_{i'_1} \to \mathbf{s}_{j'_1} \to \mathbf{r}_{i'_2} \to \mathbf{s}_{j'_2} \to \cdots \to \mathbf{r}_{i'_q} \to \mathbf{s}_{j'_q}, \tag{2.4}$$

which are contained in these two connected components, respectively, where $r_{i_1}(A)$ and $r_{i'_1}(A)$, $s_{j_p}(A)$ and $s_{j'_q}(A)$ are unattained. We then add an ϵ to the entries

$$(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p),$$

and subtract an ϵ from the entries

$$(i_2, j_1), (i_3, j_2), \ldots, (i_p, j_{p-1}).$$

The i_1th row sum and the j_pth column sum are increased by ϵ while other row and column sums remain unchanged. The total sum of all elements in the connected component containing the path (2.3) is increased by ϵ. We then subtract an ϵ from the entries

$$(i'_1, j'_1), (i'_2, j'_2), \ldots, (i'_q, j'_q),$$

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and add an ϵ to the entries

$$(i_2', j_1'), (i_3', j_2'), \ldots, (i_q', j_{q-1}').$$

This will decrease the i_1'th row sum and the j_q'th column sum by ϵ. The total sum of all elements in the connected component containing the path (2.4) is decreased by ϵ. Denote the resulting matrix by A^+ and note that the total sum of A^+ is still equal to σ, which remains unchanged. By reversing the sign of ϵ, we obtain another matrix A^- with the total sum equal to σ. Since $r_{i_1}(A), r_{i_1'}(A), s_{j_p}(A), s_{j_q'}(A)$ are unattained, we can choose ϵ sufficiently small such that A^+, A^- are in $\Delta(\bar{R}, \bar{S}, \sigma)$. Since

$$A = \frac{1}{2}A^+ + \frac{1}{2}A^-,$$

A is not an extreme point in $\Delta(\bar{R}, \bar{S}, \sigma)$. This contradicts with the given condition. ☐

THEOREM 2.4. *Let $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$ be nonnegative vectors, $0 \le \sigma \le \min\{|\bar{R}|, |\bar{S}|\}$, and A is a matrix in $\Delta(\bar{R}, \bar{S}, \sigma)$. A is in $\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$ if and only if $BG(A)$ is a forest and $BG(A)$ contains at most one connected component with exactly one unattained row vertex and one unattained column vertex, and all other connected components are formed by the following three types $(i), (ii), (iii)$,*

(i) All row and column sums are attained,
(ii) Exactly one row sum is not attained,
(iii) Exactly one column sum is not attained.

Proof. " $\Rightarrow$ " By the definition of $\Delta(\bar{R}, \bar{S}, \sigma)$ in (1.2), we have

$$\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma)) \subseteq \bigcup_{\substack{|R|=|S|=\sigma \\ R \lhd \bar{R}, S \lhd \bar{S}}} \mathfrak{E}(R, S),$$

where $\mathfrak{E}(R, S)$ denotes the set of all extreme points of $\mathcal{U}(R, S)$. If $A \in \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$, then there exist $R \lhd \bar{R}$ and $S \lhd \bar{S}$ with $|R| = |S| = \sigma$ such that $A \in \mathfrak{E}(R, S)$. By Proposition 1.1, $BG(A)$ is a forest. Theorem 2.1 guarantees that the connected components of $BG(A)$ are of types $(i), (ii), (iii)$, and Theorem 2.3 guarantees that $BG(A)$ contains at most one connected component with exactly one unattained row vertex and one unattained column vertex.

" $\Leftarrow$ " Let A be a matrix in $\Delta(\bar{R}, \bar{S}, \sigma)$ and $BG(A)$ satisfy the given conditions. Suppose that there exist A_1 and A_2 both in $\Delta(\bar{R}, \bar{S}, \sigma)$ such that

$$A = \lambda A_1 + (1 - \lambda) A_2, \tag{2.5}$$

for some $0 < \lambda < 1$. By Proposition 1.1, $A \in \mathfrak{E}(R(A), S(A))$ where $|R(A)| = |S(A)| = \sigma$ and $R(A) \lhd \bar{R}, S(A) \lhd \bar{S}$. Since the bipartite graphs $BG(A_1)$ and $BG(A_2)$ are subgraphs of $BG(A)$, both $BG(A_1)$ and $BG(A_2)$ are forests. Therefore, $A_1 \in \mathfrak{E}(R(A_1), S(A_1))$ and $A_2 \in \mathfrak{E}(R(A_2), S(A_2))$, where $|R(A_1)| = |S(A_1)| = |R(A_2)| = |S(A_2)| = \sigma$ and $R(A_1) \lhd \bar{R}, S(A_1) \lhd \bar{S}, R(A_2) \lhd \bar{R}, S(A_2) \lhd \bar{S}$. Moreover, if $r_i(A)$ (resp. $s_j(A)$) is attained, by (2.5) both $r_i(A_1)$ (resp. $s_j(A_1)$) and $r_i(A_2)$ (resp. $s_j(A_2)$) are attained. Since the connected component of $BG(A)$ which is of type $(i), (ii)$, or (iii) contains at most one unattained vertex, $BG(A_1)$ and $BG(A_2)$ must have exactly the same connected components as those in $BG(A)$ which are of types $(i), (ii), (iii)$. If $BG(A)$ contains at most one connected component with exactly one unattained row vertex and one unattained column vertex, then in this connected component, the only unattained row sum and the only unattained column sum are uniquely determined since the rest row and column sums are all

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attained. Therefore, we claim that $R(A_1) = R(A_2) = R(A), S(A_1) = S(A_2) = S(A)$. This implies that $A, A_1, A_2 \in \mathfrak{E}(R(A), S(A))$ and $A = A_1 = A_2$. Thus, A is extreme in $\Delta(\bar{R}, \bar{S}, \sigma)$. ◻

Corollary 2.5. *If $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$ are nonnegative vectors with $|\bar{R}| = |\bar{S}|$ and $0 \le \sigma \le |\bar{R}|$, then*

$$\{B \in \mathfrak{E}_{\le}(\bar{R}, \bar{S}) |\ |B| = \sigma\} \subseteq \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma)).$$

Proof. This follows from the fact that $\Delta(\bar{R}, \bar{S}, \sigma) = \{A \in \mathcal{U}_{\le}(\bar{R}, \bar{S}) |\ |A| = \sigma\} \subseteq \mathcal{U}_{\le}(\bar{R}, \bar{S})$ and $\{B \in \mathfrak{E}_{\le}(\bar{R}, \bar{S}) |\ |B| = \sigma\} \subseteq \Delta(\bar{R}, \bar{S}, \sigma)$. ◻

Corollary 2.6. *Let $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$ be nonnegative vectors satisfying that $|\bar{R}| = |\bar{S}|$, $0 \le \sigma < |\bar{R}|$, and B is extreme in $\mathcal{U}_{\le}(\bar{R}, \bar{S})$ with $|B| > \sigma$. If there exists one edge connecting $\boldsymbol{r}_i$ and $\boldsymbol{s}_j$ in $BG(B)$, such that either $\boldsymbol{r}_i$ or $\boldsymbol{s}_j$ is the only possible unattained vertex in its connected component, and $b_{i,j} \ge |B| - \sigma$, then the matrix obtained from B by decreasing $b_{i,j}$ by $|B| - \sigma$ is an extreme point of $\Delta(\bar{R}, \bar{S}, \sigma)$.*

Proof. From Proposition 1.2, $BG(B)$ is a forest and each connected component in $BG(B)$ contains at most one vertex unattained. By decreasing the element $b_{i,j}$, $BG(B)$ contains at most one connected component with exactly one unattained row vertex and one unattained column vertex. Thus by Theorem 2.4, the matrix obtained from B by decreasing $b_{i,j}$ by $|B| - \sigma$ is an extreme point in $\Delta(\bar{R}, \bar{S}, \sigma)$. ◻

However, $\Delta(\bar{R}, \bar{S}, \sigma)$ contains more extreme points than those in Corollary 2.6. To explicitly obtain the matrices in $\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$, we first find out all possible row sum vectors and column sum vectors for matrices in $\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$. To do that, we introduce some notations.

Denote the set $\{\bar{r}_1, \ldots, \bar{r}_m\}$ by $S_{\bar{R}}$, and the set $\{\bar{s}_1, \ldots, \bar{s}_n\}$ by $S_{\bar{S}}$. Let

$$\bar{R}_{\sum} := \left\{ \sum_{r \in W} r \,\middle|\, W \subseteq S_{\bar{R}} \right\},\ \bar{S}_{\sum} := \left\{ \sum_{s \in T} s \,\middle|\, T \subseteq S_{\bar{S}} \right\}.$$

If T or W is $\emptyset$, then $\sum_{r \in \emptyset} r := 0, \sum_{s \in \emptyset} s := 0$. Also let

$$\begin{aligned}
\mathrm{Diff}(\bar{R}_{\sum}, \bar{S}_{\sum}) &:= \left\{ r - s \middle|\ r - s \ge 0, r \in \bar{R}_{\sum}, s \in \bar{S}_{\sum} \right\}, \\
\mathrm{Diff}(\bar{S}_{\sum}, \bar{R}_{\sum}) &:= \left\{ s - r \middle|\ s - r \ge 0, r \in \bar{R}_{\sum}, s \in \bar{S}_{\sum} \right\}, \\
\mathrm{Sum}(\bar{R}_{\sum}, \bar{S}_{\sum}) &:= \left\{ r + s \middle|\ r \in \bar{R}_{\sum}, s \in \bar{S}_{\sum} \right\}, \\
\mathrm{Diff}(\sigma, \mathrm{Sum}(\bar{R}_{\sum}, \bar{S}_{\sum})) &:= \left\{ \sigma - t \middle|\ \sigma - t \ge 0, t \in \mathrm{Sum}(\bar{R}_{\sum}, \bar{S}_{\sum}) \right\}.
\end{aligned}$$

Since $0 \in \bar{R}_{\sum}$ and also $0 \in \bar{S}_{\sum}$, $\bar{R}_{\sum} \subseteq \mathrm{Diff}(\bar{R}_{\sum}, \bar{S}_{\sum})$ and $\bar{S}_{\sum} \subseteq \mathrm{Diff}(\bar{S}_{\sum}, \bar{R}_{\sum})$.

Define

$$\mathbf{R} := \left\{ r \le \sigma : r \in S_{\bar{R}} \cup \mathrm{Diff}(\bar{S}_{\sum}, \bar{R}_{\sum}) \cup \mathrm{Diff}(\sigma, \mathrm{Sum}(\bar{R}_{\sum}, \bar{S}_{\sum})) \right\},$$

and

$$\mathbf{S} := \left\{ s \le \sigma : s \in S_{\bar{S}} \cup \mathrm{Diff}(\bar{R}_{\sum}, \bar{S}_{\sum}) \cup \mathrm{Diff}(\sigma, \mathrm{Sum}(\bar{R}_{\sum}, \bar{S}_{\sum})) \right\}.$$

Lemma 2.7. *If $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$ are nonnegative vectors with $0 \le \sigma \le \min\{|\bar{R}|, |\bar{S}|\}$, and $A \in \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$, then all row sums of A are in the set $\mathbf{R}$ and all column sums of A are in the set $\mathbf{S}$.*

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Proof. By Theorem 2.4, $BG(A)$ contains at most one connected component with exactly one unattained row vertex and one unattained column vertex, and all other connected components are of three types (i), (ii) and (iii). If the connected component is of type (i) or type (iii), then the corresponding row sums are all attained which are in $S_{\bar{R}}$. If the connected component is of type (ii), then the unattained row sum is in $\mathrm{Diff}(\bar{S}_{\sum}, \bar{R}_{\sum})$ due to the compatibility of the connected component. If the connected component has exactly one unattained row vertex and one unattained column vertex, then the only unattained row sum can then be calculated by taking the difference of σ and the rest row sums, which is in $\mathrm{Diff}(\sigma, \mathrm{Sum}(\bar{S}_{\sum}, \bar{R}_{\sum}))$. Since this type of component can appear in $BG(A)$ at most once, $R(A)$ should contain at most one element which is generated in this way. The column sum case can be proved similarly. ∎

From Theorem 2.4, we have the following proposition and corollary.

Proposition 2.8. *Given nonnegative vectors $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$, and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, if $A \in \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$, then the number of unattained line sums in A does not exceed the number of connected components in $BG(A)$ plus one.*

Corollary 2.9. *Given nonnegative vectors $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$, and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, if $A \in \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$, then the number of unattained line sums in A is at most* $\min\{m, n\} + 1$.

Proof. Since in $BG(A)$ the number of connected components containing at least one edge is at most $\min\{m, n\}$, together with Theorem 2.4, there are at most $\min\{m, n\} + 1$ unattained vertices. ∎

Corollary 2.9 helps us to exclude those pairs $(\tilde{R}, \tilde{S})$ whose corresponding transportation polytopes do not contain any extreme points of $\Delta(\bar{R}, \bar{S}, \sigma)$, and we have

$$\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma)) \subseteq \bigcup_{\substack{|\tilde{R}|=|\tilde{S}|=\sigma \\ \tilde{R} \lhd \bar{R}, \tilde{S} \lhd \bar{S}}} \mathfrak{E}(\tilde{R}, \tilde{S}), \tag{2.6}$$

where $\tilde{R}, \tilde{S}$ satisfy the conditions in Lemma 2.7 and Corollary 2.9. When $\sigma < \min\{|\bar{R}|, |\bar{S}|\}, \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$ is a proper subset of $\bigcup \mathfrak{E}(\tilde{R}, \tilde{S})$, where $|\tilde{R}| = |\tilde{S}| = \sigma$ and $\tilde{R} \lhd \bar{R}, \tilde{S} \lhd \bar{S}$. This is because not all matrices in $\mathfrak{E}(\tilde{R}, \tilde{S})$ satisfy Theorem 2.4. In order to obtain the matrices in $\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))$, we first find out all extreme points of $\mathcal{U}(\tilde{R}, \tilde{S})$ as long as $\tilde{R}$ and $\tilde{S}$ satisfy the conditions in Lemma 2.7 and Corollary 2.9. Then remove those matrices in $\mathfrak{E}(\tilde{R}, \tilde{S})$ whose bipartite graphs do not satisfy the conditions in Theorem 2.4.

Here, we use an example to illustrate how to find the right pairs of $\tilde{R}$ and $\tilde{S}$.

Example 2.10. Given $\sigma = 4, \bar{R} = (2, 3), \bar{S} = (1, 2, 3)$, we have $S_{\bar{R}} = \{2, 3\}, S_{\bar{S}} = \{1, 2, 3\}$ and

$$\begin{aligned}
&\bar{R}_{\Sigma} = \{0, 2, 3, 5\}, \bar{S}_{\Sigma} = \{0, 1, 2, 3, 4, 5, 6\}, \\
&\mathrm{Diff}(\bar{R}_{\sum}, \bar{S}_{\sum}) = \{0, 1, 2, 3, 4, 5\}, \\
&\mathrm{Diff}(\bar{S}_{\sum}, \bar{R}_{\sum}) = \{0, 1, 2, 3, 4, 5, 6\}, \\
&\mathrm{Sum}(\bar{R}_{\sum}, \bar{S}_{\sum}) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}, \\
&\mathrm{Diff}(\sigma, \mathrm{Sum}(\bar{R}_{\sum}, \bar{S}_{\sum})) = \{0, 1, 2, 3, 4\}, \\
&\mathbf{R} = \mathbf{S} = \{0, 1, 2, 3, 4\}.
\end{aligned}$$

The row and column sum vectors of the polytopes satisfying Corollary 2.9 are as follows.

1. $\tilde{R}_1 = (1, 3),\ \tilde{S}_1 = (0, 1, 3)$;
2. $\tilde{R}_2 = (1, 3),\ \tilde{S}_2 = (0, 2, 2)$;

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3. $\tilde{R}_3 = (1,3)$, $\tilde{S}_3 = (1,0,3)$;
4. $\tilde{R}_4 = (1,3)$, $\tilde{S}_4 = (1,1,2)$;
5. $\tilde{R}_5 = (1,3)$, $\tilde{S}_5 = (1,2,1)$;
6. $\tilde{R}_6 = (2,2)$, $\tilde{S}_6 = (0,1,3)$;
7. $\tilde{R}_7 = (2,2)$, $\tilde{S}_7 = (0,2,2)$;
8. $\tilde{R}_8 = (2,2)$, $\tilde{S}_8 = (1,0,3)$;
9. $\tilde{R}_9 = (2,2)$, $\tilde{S}_9 = (1,1,2)$;
10. $\tilde{R}_{10} = (2,2)$, $\tilde{S}_{10} = (1,2,1)$.

Once we have all possible $\tilde{R}$ and $\tilde{S}$, we can also modify the Ryser's algorithm ([5, 17]) to obtain the extreme points of $\Delta(\bar{R},\bar{S},\sigma)$.

(i) Let $\tilde{R} = (\tilde{r}_1,\dots,\tilde{r}_m)$ and $\tilde{S} = (\tilde{s}_1,\dots,\tilde{s}_n)$ where $\tilde{r}_1+\cdots+\tilde{r}_m = \tilde{s}_1+\cdots+\tilde{s}_n = \sigma$. Begin with $\hat{R} = \tilde{R}$ and $\hat{S} = \tilde{S}$, and $A = [a_{i,j}]$ equal to zero matrix of size $m \times n$.

(ii) Choose $\tilde{r}_i$ and $\tilde{s}_j$ and replace $a_{i,j}$ with $\min\{\tilde{r}_i, \tilde{s}_j\}$. Let

$$\hat{R} = (\tilde{r}_1,\dots,\tilde{r}_i - \min\{\tilde{r}_i,\tilde{s}_j\},\dots,\tilde{r}_m),$$

and

$$\hat{S} = (\tilde{s}_1,\dots,\tilde{s}_j - \min\{\tilde{r}_i,\tilde{s}_j\},\dots,\tilde{s}_n).$$

If $\tilde{r}_i < \bar{r}_i$ unattained, then all the row vertices in the same connected component containing row vertex $\mathbf{r}_i$ must be attained. If $\tilde{s}_j < \bar{s}_j$ unattained, then all the column vertices in the same connected component containing column vertex $\mathbf{s}_j$ must be attained. Once there is a connected component in $BG(A)$ which contains exactly one row vertex and one column vertex which are both unattained, then all the other connected components should only contain at most one unattained vertex. Otherwise, we stop the procedure and output empty.

(iii) Repeat Step(ii) until $\hat{R}$ and $\hat{S}$ are zero vectors.

Example 2.11. Following the results of Example 2.10, we can find all the extreme points of $\Delta(\bar{R},\bar{S},\sigma)$ where $\sigma = 4, \bar{R} = (2,3), \bar{S} = (1,2,3)$ as listed below

$$\begin{bmatrix}0&1&0\\0&0&3\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&2\end{bmatrix},\begin{bmatrix}0&1&0\\0&1&2\end{bmatrix},\begin{bmatrix}0&0&1\\0&2&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&3\end{bmatrix},\begin{bmatrix}0&0&1\\1&0&2\end{bmatrix},$$
$$\begin{bmatrix}0&1&0\\1&0&2\end{bmatrix},\begin{bmatrix}1&0&0\\0&2&1\end{bmatrix},\begin{bmatrix}0&1&0\\1&1&1\end{bmatrix},\begin{bmatrix}0&0&1\\1&2&0\end{bmatrix},\begin{bmatrix}0&1&1\\0&0&2\end{bmatrix},\begin{bmatrix}0&0&2\\0&1&1\end{bmatrix},$$
$$\begin{bmatrix}0&2&0\\0&0&2\end{bmatrix},\begin{bmatrix}0&0&2\\0&2&0\end{bmatrix},\begin{bmatrix}1&0&1\\0&0&2\end{bmatrix},\begin{bmatrix}0&0&2\\1&0&1\end{bmatrix},\begin{bmatrix}1&1&0\\0&0&2\end{bmatrix},\begin{bmatrix}0&0&2\\1&1&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&1\\0&2&0\end{bmatrix},\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix},\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix},\begin{bmatrix}0&2&0\\1&0&1\end{bmatrix}.$$

Remark 2.12. Denote $L_n = (1,1,\dots,1) \in \mathbb{R}^n$. Note that $\omega_n^s = \Delta(L_n, L_n, s)$ where $0 \le s \le n$. By Lemma 2.7, $\mathbf{R} = \mathbf{S} = \{0, 1, s - \lfloor s \rfloor\}$. Therefore, Proposition 1.3 easily follows from Theorem 2.4.

3. The minimal term rank. Let $A = [a_{i,j}]$ be an $m \times n$ nonnegative matrix. The *term rank* of A is the maximal number $\rho(A)$ of positive elements of A with no two of them on a line. The fundamental

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minimax theorem of König-Egerváry asserts that $\rho(A)$ equals the minimal number of lines in A that contain all positive elements in A [5]. In this section, we investigate the minimal term rank $\tilde{\rho}$ of the matrices in the class $\Delta(\bar{R}, \bar{S}, \sigma)$. Given set S, denote by $\tilde{\rho}(S)$ the minimal term rank of the matrices in S. We first show that the minimal term rank can only be achieved on the extreme points of $\Delta(\bar{R}, \bar{S}, \sigma)$.

Lemma 3.1. *Let S be a bounded convex polytope of $m \times n$ matrices, and $\mathfrak{E}(S)$ be the set of all extreme points of S which is nonempty. Then*

$$\tilde{\rho}(S) = \tilde{\rho}(\mathfrak{E}(S)).$$

Proof. Since $\mathfrak{E}(S) \subseteq S$, $\tilde{\rho}(S) \leq \tilde{\rho}(\mathfrak{E}(S))$. On the other hand, for any $A \in S$, there exist $A_1, \ldots, A_k \in \mathfrak{E}(S)$ such that

$$A = \lambda_1 A_1 + \cdots + \lambda_k A_k,$$

where $0 \leq \lambda_1, \ldots, \lambda_k \leq 1$ and $\lambda_1 + \cdots + \lambda_k = 1$. Thus, a cover of A is also a cover of A_i for $1 \leq i \leq k$. By the minimax theorem this implies that $\rho(A) \geq \rho(A_i)$ for $1 \leq i \leq k$, and $\tilde{\rho}(S) \geq \tilde{\rho}(\mathfrak{E}(S))$. Therefore, the lemma holds. ∎

Theorem 3.2. *Let $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m)$ and $\bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$ be nonnegative vectors, and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$. We have*

$$\tilde{\rho}(\Delta(\bar{R}, \bar{S}, \sigma)) = \min\{\tilde{\rho}(\mathfrak{E}(\tilde{R}, \tilde{S})) : |\tilde{R}| = |\tilde{S}| = \sigma,\ \tilde{R} \lhd \bar{R}, \tilde{S} \lhd \bar{S}\},$$

where $\tilde{R}, \tilde{S}$ satisfy the conditions in Lemma 2.7 and Corollary 2.9.

Proof. Recall that (2.6) says

$$\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma)) \subseteq \bigcup_{\substack{|\tilde{R}|=|\tilde{S}|=\sigma \\ \tilde{R} \lhd \bar{R}, \tilde{S} \lhd \bar{S}}} \mathfrak{E}(\tilde{R}, \tilde{S}) \subseteq \Delta(\bar{R}, \bar{S}, \sigma)),$$

where $\tilde{R}, \tilde{S}$ satisfy the conditions in Lemma 2.7 and Corollary 2.9. Therefore,

$$\tilde{\rho}(\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma))) \geq \min\{\tilde{\rho}(\mathfrak{E}(\tilde{R}, \tilde{S})) : |\tilde{R}| = |\tilde{S}| = \sigma,\ \tilde{R} \lhd \bar{R}, \tilde{S} \lhd \bar{S}\} \geq \tilde{\rho}(\Delta(\bar{R}, \bar{S}, \sigma)).$$

By Lemma 3.1, $\tilde{\rho}(\Delta(\bar{R}, \bar{S}, \sigma)) = \tilde{\rho}(\mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma)))$. Therefore the theorem holds. ∎

For any nonnegative real vector $R = (a_1, a_2, \ldots, a_l)$, denote the rearrangement of the elements in R in ascending order by

$$a'_1 \leq a'_2 \leq \cdots \leq a'_l,$$

and in descending order by

$$a^*_1 \geq a^*_2 \geq \cdots \geq a^*_l.$$

We denote

$$R' = (a'_1, a'_2, \ldots, a'_l), R^* = (a^*_1, a^*_2, \ldots, a^*_l).$$

The following theorem comes from Jurkat and Ryser [17].

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Theorem 3.3 ([17]). *Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be nonnegative vectors satisfying $|R| = |S| = \sigma$. Then*

$$\tilde{\rho}(\mathcal{U}(R, S)) = \rho(E_{R', S^*}),$$

where $E_{R',S^*} \in \mathfrak{E}(R', S^*)$ is constructed inductively by choosing the position $(1, 1)$ in the submatrix under consideration. Moreover,

$$\rho(E_{R',S^*}) = \min\{e + f\},$$

where the minimum is taken over all pairs $e \geq 0, f \geq 0$ satisfying

$$r_1^* + r_2^* + \cdots + r_e^* + s_1^* + s_2^* + \cdots + s_f^* \geq \sigma.$$

Given $\bar{R}, \bar{S}$ nonnegative vectors and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, there exist i_σ and j_σ such that

$$\bar{r}_1^* + \bar{r}_2^* + \cdots + \bar{r}_{i_\sigma - 1}^* < \sigma \leq \bar{r}_1^* + \bar{r}_2^* + \cdots + \bar{r}_{i_\sigma}^*,$$

and

$$\bar{s}_1^* + \bar{s}_2^* + \cdots + \bar{s}_{j_\sigma - 1}^* < \sigma \leq \bar{s}_1^* + \bar{s}_2^* + \cdots + \bar{s}_{j_\sigma}^*.$$

Let

$$R_\rho^* := \left(\bar{r}_1^*, \ldots, \bar{r}_{i_\sigma - 1}^*, \sigma - \sum_{s=1}^{i_\sigma - 1} \bar{r}_s^*, 0, \ldots, 0\right),$$
$$S_\rho^* := \left(\bar{s}_1^*, \ldots, \bar{s}_{j_\sigma - 1}^*, \sigma - \sum_{t=1}^{j_\sigma - 1} \bar{s}_t^*, 0, \ldots, 0\right).$$

Since $0 < \sigma - \sum_{s=1}^{i_\sigma - 1} \bar{r}_s^* \leq \bar{r}_{i_\sigma}^*, 0 < \sigma - \sum_{t=1}^{j_\sigma - 1} \bar{s}_t^* \leq \bar{s}_{j_\sigma}^*$, the only possible unattained row sum in R_ρ^* is the i_σth row sum, and the only possible unattained column sum in S_ρ^* is the j_σth column sum. By Theorem 2.4

$$\mathfrak{E}(R_\rho^*, S_\rho^*) \subseteq \mathfrak{E}(\Delta(\bar{R}, \bar{S}, \sigma)),$$

which implies that

$$\tilde{\rho}(\mathcal{U}(R_\rho^*, S_\rho^*)) \geq \tilde{\rho}(\Delta(\bar{R}, \bar{S}, \sigma)). \tag{3.7}$$

On the other hand, for any $\tilde{R}, \tilde{S}$ satisfying $\tilde{R} \lhd \bar{R}, \tilde{S} \lhd \bar{S}$, and $|\tilde{R}| = |\tilde{S}| = \sigma$, by Theorem 3.3 we have

$$\tilde{\rho}(\mathcal{U}(\tilde{R}, \tilde{S})) \geq \tilde{\rho}(\mathcal{U}(R_\rho^*, S_\rho^*)). \tag{3.8}$$

(3.7) and (3.8) together give us the following theorem.

Theorem 3.4. *Let $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m)$ and $\bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$ be nonnegative vectors, and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$. Then*

$$\tilde{\rho}(\Delta(\bar{R}, \bar{S}, \sigma)) = \tilde{\rho}(\mathfrak{E}(R_\rho^*, S_\rho^*)).$$

Corollary 3.5. *For $0 \leq s \leq n$, we have*

$$\tilde{\rho}(\omega_n^s) = \tilde{\rho}(\Delta(L_n, L_n, s)) = \tilde{\rho}(\mathfrak{E}(v_n^s, v_n^s)) = \lceil s \rceil.$$

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4. The maximal permanent. In this section, we consider $\Delta(\bar{R}, \bar{S}, \sigma)$ where $\bar{R}$ and $\bar{S}$ are both n dimensional nonnegative real vectors. For an $n \times n$ matrix A, the *permanent* of A is defined by

$$per(A) = \sum_{\pi \in S_n} a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)},$$

where S_n is the symmetric group of size n. The following theorem is given by Jurkat and Ryser [17].

THEOREM 4.1 ([17]). *Let A be a matrix in $\mathcal{U}(R, S)$, then*

$$per(A) \leq \prod_{i=1}^{n} \min\{r'_i, s'_i\}, \tag{4.9}$$

and equality is attained in (4.9) by a matrix A in $\mathfrak{E}(R', S')$ with the main diagonal

$$a_{i,i} = \min\{r'_i, s'_i\}, \text{ for all } 1 \leq i \leq n.$$

We then have the following lemma.

LEMMA 4.2. *Given nonnegative vectors $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_n), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$, and $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, let A be a matrix in $\Delta(\bar{R}, \bar{S}, \sigma)$. Then*

$$per(A) \leq \prod_{i=1}^{n} \min\{\bar{r}'_i, \bar{s}'_i\}. \tag{4.10}$$

Moreover, when $\sigma \geq \sum_{i=1}^{n} \min\{\bar{r}'_i, \bar{s}'_i\}$, there exist matrices with the main diagonal $a_{i,i} = \min\{\bar{r}'_i, \bar{s}'_i\}$ for all $1 \leq i \leq n$ which make (4.10) hold as an equality.

Proof. Recall that

$$\Delta(\bar{R}, \bar{S}, \sigma) = \bigcup_{\substack{|R|=|S|=\sigma \\ R \lhd \bar{R}, S \lhd \bar{S}}} \mathcal{U}(R, S).$$

For any R, S satisfying $R \lhd \bar{R}$, and $S \lhd \bar{S}, |R| = |S| = \sigma$, we have

$$r'_1 \leq \bar{r}'_1, r'_2 \leq \bar{r}'_2, \ldots, r'_n \leq \bar{r}'_n, \quad s'_1 \leq \bar{s}'_1, s'_2 \leq \bar{s}'_2, \ldots, s'_n \leq \bar{s}'_n.$$

Therefore, $\mathcal{U}(R', S') \subseteq \Delta(\bar{R}', \bar{S}', \sigma)$ and for all $1 \leq i \leq n$ we have $\min\{r'_i, s'_i\} \leq \min\{\bar{r}'_i, \bar{s}'_i\}$. This implies that

$$\prod_{i=1}^{n} \min\{r'_i, s'_i\} \leq \prod_{i=1}^{n} \min\{\bar{r}'_i, \bar{s}'_i\}.$$

Moreover, by Theorem 4.1, for any $A \in \mathcal{U}(R, S)$,

$$per(A) \leq \prod_{i=1}^{n} \min\{r'_i, s'_i\}.$$

Thus for all $A \in \Delta(\bar{R}, \bar{S}, \sigma)$,

$$per(A) \leq \prod_{i=1}^{n} \min\{\bar{r}'_i, \bar{s}'_i\}.$$

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When $\sum_{i=1}^{n}\min\{\bar{r}'_i,\bar{s}'_i\} \le \sigma \le \min\{|\bar{R}|,|\bar{S}|\}$, there exist $R=(r_1,\dots,r_m), S=(s_1,\dots,s_n)$ satisfying that $|R|=|S|=\sigma$ and for all $1\le i\le n$,

$$\min\{\bar{r}'_i,\bar{s}'_i\} \le r'_i \le \bar{r}'_i,\ \min\{\bar{r}'_i,\bar{s}'_i\} \le s'_i \le \bar{s}'_i.$$

In this case there exists a matrix $A\in\mathcal{U}(R,S)$ such that there exists exactly one diagonal of A formed by $\min\{\bar{r}'_1,\bar{s}'_1\},\dots,\min\{\bar{r}'_n,\bar{s}'_n\}$. Thus, the equality in (4.10) can be achieved. ∎

When $m=n, \bar{R}=\bar{S}=(1,\dots,1)\in\mathbb{R}^n$, and $0\le\sigma\le n$, $\Delta(\bar{R},\bar{S},\sigma)$ is the set of all doubly substochastic matrices with total sum equal to σ. By Lemma 4.2, for all $A\in\Delta(\bar{R},\bar{S},\sigma), per(A)\le 1$, where $per(A)=1$ if and only if $\sigma=n$ and A is a permutation matrix ([10, 21]).

Lemma 4.3. *Let $X=(x_1,\dots,x_n), \bar{X}=(\bar{x}_1,\dots,\bar{x}_n)$ be nonnegative vectors satisfying that $0\le x_i\le\bar{x}_i$ for all $1\le i\le n$, $\bar{x}_1\le\dots\le\bar{x}_n$ and $|X|=x_1+\dots+x_n=\sigma$ which is fixed. Let $1\le i_\sigma\le n$ satisfying that*

$$\bar{x}_{i_\sigma} < \frac{\sigma-\sum_{i=1}^{i_\sigma-1}\bar{x}_i}{n-i_\sigma+1},\ \bar{x}_{i_\sigma+1} \ge \frac{\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i}{n-i_\sigma}.$$

Then

$$\prod_{i=1}^{n} x_i \le \prod_{i=1}^{i_\sigma}\bar{x}_i\left(\frac{\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i}{n-i_\sigma}\right)^{n-i_\sigma}. \tag{4.11}$$

Proof. If $x_1=0$, then $\prod_{i=1}^{n}x_i=0$ and (4.11) holds. If $x_1=\bar{x}_1>0$, then cancel x_1 from both sides of (4.11) and consider $X'=(x_2,\dots,x_n), \bar{X}'=(\bar{x}_2,\dots,\bar{x}_n), |X'|=x_2+\dots+x_n=\sigma-x_1$. Thus without loss of generality, we suppose $0<x_1<\bar{x}_1$ and first prove the following inequality

$$\prod_{i=1}^{i_\sigma}\bar{x}_i\left(\frac{\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i}{n-i_\sigma}\right)^{n-i_\sigma} > x_1\prod_{i=2}^{i_\sigma}\bar{x}_i\left(\frac{\sigma-x_1-\sum_{i=2}^{i_\sigma}\bar{x}_i}{n-i_\sigma}\right)^{n-i_\sigma}, \tag{4.12}$$

which is equivalent to

$$\frac{\bar{x}_1}{x_1} > \left(\frac{\sigma-x_1-\sum_{i=2}^{i_\sigma}\bar{x}_i}{\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i}\right)^{n-i_\sigma}. \tag{4.13}$$

To show (4.13), we consider the function $f(x)=\ln x+(n-i_\sigma)\ln(\sigma-\sum_{i=2}^{i_\sigma}\bar{x}_i-x)$, where $x\in(0,\bar{x}_1]$. Since

$$f'(x)=\frac{1}{x}-\frac{n-i_\sigma}{\sigma-\sum_{i=2}^{i_\sigma}\bar{x}_i-x},$$

and

$$f''(x)=-\frac{1}{x^2}-\frac{n-i_\sigma}{(\sigma-\sum_{i=2}^{i_\sigma}\bar{x}_i-x)^2}<0,$$

$f'(x)$ is decreasing on $(0,\bar{x}_1]$. Thus for $x\in(0,\bar{x}_1)$,

$$\begin{aligned} f'(x) > f'(\bar{x}_1) &= \frac{1}{\bar{x}_1}-\frac{n-i_\sigma}{\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i} \\ &= \frac{\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i-(n-i_\sigma)\bar{x}_1}{\bar{x}_1(\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i)} \ge \frac{(n-i_\sigma)(\bar{x}_{i_\sigma}-\bar{x}_1)}{\bar{x}_1(\sigma-\sum_{i=1}^{i_\sigma}\bar{x}_i)} \ge 0. \end{aligned}$$

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Therefore, $f(x)$ is increasing on $(0, \bar{x}_1]$, which implies that $f(x_1) < f(\bar{x}_1)$ if $x_1 < \bar{x}_1$, i.e.

$$\ln \bar{x}_1 + (n - i_\sigma) \ln \left(\sigma - \sum_{i=2}^{i_\sigma} \bar{x}_i - \bar{x}_1 \right) \geq \ln x_1 + (n - i_\sigma) \ln \left(\sigma - \sum_{i=2}^{i_\sigma} \bar{x}_i - x_1 \right).$$

From the above inequality, we can easily obtain (4.13). Thus (4.12) holds and we have

$$\begin{aligned} \prod_{i=1}^{i_\sigma} \bar{x}_i \left(\frac{\sigma - \sum_{i=1}^{i_\sigma} \bar{x}_i}{n - i_\sigma} \right)^{n-i_\sigma} &> x_1 \prod_{i=2}^{i_\sigma} \bar{x}_i \left(\frac{\sigma - x_1 - \sum_{i=2}^{i_\sigma} \bar{x}_i}{n - i_\sigma} \right)^{n-i_\sigma} \\ &> x_1 x_2 \prod_{i=3}^{i_\sigma} \bar{x}_i \left(\frac{\sigma - x_1 - x_2 - \sum_{i=3}^{i_\sigma} \bar{x}_i}{n - i_\sigma} \right)^{n-i_\sigma} \\ &> \cdots > \prod_{i=1}^{i_\sigma} x_i \left(\frac{\sigma - \sum_{i=1}^{i_\sigma} x_i}{n - i_\sigma} \right)^{n-i_\sigma}. \end{aligned}$$

By the arithmetic-geometric inequality,

$$\left(\frac{\sigma - \sum_{i=1}^{i_\sigma} x_i}{n - i_\sigma} \right)^{n-i_\sigma} \geq x_{i_\sigma+1} \cdots x_n.$$

Thus, the lemma holds. □

For any $A \in \Delta(\bar{R}, \bar{S}, \sigma)$, by Lemma 4.2

$$per(A) \leq \max \left\{ \prod_{i=1}^{n} \min\{r'_i, s'_i\} : |R| = |S| = \sigma, R \lhd \bar{R}, S \lhd \bar{S} \right\}.$$

For all $1 \leq i \leq n, \min\{r'_i, s'_i\} \leq \bar{t}_i$ where $\bar{t}_i = \min\{\bar{r}'_i, \bar{s}'_i\}$. Note that $\sum_{i=1}^{n} \min\{r'_i, s'_i\} \leq \sigma$. Lemma 4.2 and Lemma 4.3 imply the following theorem.

THEOREM 4.4. *Given nonnegative vectors* $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_n), \bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$, $0 \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, *let* $\bar{t}_i = \min\{\bar{r}'_i, \bar{s}'_i\}$ *and* A *be a matrix in* $\Delta(\bar{R}, \bar{S}, \sigma)$.

1. *If* $0 \leq \sigma \leq \sum_{i=1}^{n} \bar{t}_i, 0 \leq i_\sigma \leq n$ *such that for* $i_\sigma \neq 0$,

$$\bar{t}_{i_\sigma} < \frac{\sigma - \sum_{i=1}^{i_\sigma - 1} \bar{t}_i}{n - i_\sigma + 1}, \ \bar{t}_{i_\sigma+1} \geq \frac{\sigma - \sum_{i=1}^{i_\sigma} \bar{t}_i}{n - i_\sigma},$$

and set $i_\sigma = 0$ *when* $\bar{t}_1 \geq \frac{\sigma}{n}$, *then*

$$per(A) \leq \begin{cases} \prod_{i=1}^{i_\sigma} \bar{t}_i \left(\frac{\sigma - \sum_{i=1}^{i_\sigma} \bar{t}_i}{n - i_\sigma} \right)^{n-i_\sigma} & \text{if} \quad i_\sigma \neq 0 \\ \left(\frac{\sigma}{n} \right)^n & \text{if} \quad i_\sigma = 0 \end{cases}.$$

The equality holds when $A = [a_{i,j}]$ *is a diagonal matrix such that if* $i_\sigma \neq 0$, *then*

$$a_{i,i} = \begin{cases} \bar{t}_i & \text{if} \quad 1 \leq i \leq i_\sigma \\ \frac{\sigma - \sum_{i=1}^{i_\sigma} \bar{t}_i}{n - i_\sigma} & \text{if} \quad i_\sigma + 1 \leq i \leq n \end{cases},$$

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and if $i_\sigma = 0$, *then*

$$a_{i,i} = \frac{\sigma}{n} \quad \text{for} \quad 1 \leq i \leq n.$$

2. *If* $\sum_{i=1}^{n} \bar{t}_i \leq \sigma \leq \min\{|\bar{R}|, |\bar{S}|\}$, *then*

$$per(A) \leq \prod_{i=1}^{n} \bar{t}_i.$$

The equality holds when A *is in* $\mathfrak{E}(R, S)$ *where* $R \lhd \bar{R}, S \lhd \bar{S}, |R| = |S| = \sigma$, *and there exist two rearrangements* $i_1, \ldots, i_n; j_1, \ldots, j_n$ *of* $1, \ldots, n$ *such that*

$$a_{i_k, j_k} = \min\{\bar{r}'_i, \bar{s}'_i\} = \bar{t}_i, \ (k, i = 1, \ldots, n).$$

Corollary 4.5. *For* $0 \leq s \leq n$ *and* $A \in \omega_n^s$,

$$per(A) \leq \left(\frac{s}{n}\right)^n.$$

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