



UPPER BOUNDS ON THE ALGEBRAIC CONNECTIVITY OF GRAPHS*

ZHEN LIN[†] AND LIANYING MIAO[‡]

Abstract. The algebraic connectivity of a connected graph G is the second smallest eigenvalue of the Laplacian matrix of G . In this paper, some new upper bounds on algebraic connectivity are obtained by applying generalized interlacing to an appropriate quotient matrix.

Key words. Algebraic connectivity, Quotient matrix, Upper bound.

AMS subject classifications. 05C50, 05C82.

1. Introduction. Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, $d_G(v)$ denotes the degree of vertex v in G . Let $m(v) = \sum_{uv \in E(G)} d_G(u)/d_G(v)$ be the average of the degrees of the vertices adjacent to v . Let P_n and C_n denote the path and the cycle with n vertices, respectively. The Laplacian matrix of G , denoted by $L(G)$, is given by $L(G) = D(G) - A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the degree diagonal matrix of G , respectively. As usual, we shall index the eigenvalues of $L(G)$ in nonincreasing order, and denote them as:

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0,$$

where $\mu_{n-1}(G)$ is called algebraic connectivity by Fiedler [5], denoted by $a(G)$. It is well known that a graph is connected if and only if its algebraic connectivity is nonzero.

The algebraic connectivity of graphs is an important topic in graph theory [1, 2, 11]. In addition, the algebraic connectivity plays an important role on, among others, synchronization of coupled oscillators, network robustness, consensus problems, belief propagation, graph partitioning, and distributed filtering in sensor networks [6, 8, 10, 14, 15]. In particular, it is of considerable interest in finding the upper bound of the algebraic connectivity in the above research areas. The following classical results are obtained in a connected graph G .

In 1973, Fiedler [5] showed that

$$a(G) \leq \kappa(G) \leq \kappa'(G) \leq \delta(G), \tag{1.1}$$

and

$$a(G) \leq n - \alpha(G), \tag{1.2}$$

where $\kappa(G)$, $\kappa'(G)$, $\delta(G)$, and $\alpha(G)$ are the vertex connectivity, the edge connectivity, the minimal degree, and the independence number, respectively.

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Corresponding Author: Zhen Lin

[†]School of Mathematics and Statistics, Qinghai Normal University, Xining, 810008, Qinghai, P.R. China (lmlnzhen@163.com).

[‡]School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, Jiangsu, P.R. China (miaolianying@cumt.edu.cn).

In 2003, Fallat et al. [7] presented an upper bound in terms of edge density, that is,

$$a(G) \leq \frac{|V||E_X|}{|X||X^c|}, \quad (1.3)$$

where X is a nonempty subset of V , $X^c = V - X$ and E_X is the set of all edges with one end in X and the other end in X^c .

In 2005, Belhaiza et al. [3] gave an upper bound, that is,

$$a(G) \leq \lfloor -1 + \sqrt{1 + 2m} \rfloor, \quad (1.4)$$

where m is the number of edges in G .

In 2005, Lu et al. [12] obtained an upper bound in terms of domination number γ , that is,

$$a(G) \leq n - \gamma + \frac{n - \gamma^2}{n - \gamma}. \quad (1.5)$$

In 2007, Nikiforov [13] presented an upper bound in terms of domination number γ , that is,

$$a(G) \leq n - \gamma. \quad (1.6)$$

Let $V(G) = V_1 \cup V_2$ be a partition of G . Then $e(V_1, V_2)$ stands for the number of edges joining vertices of V_1 to vertices of V_2 . In this paper, a new upper bound on the algebraic connectivity is obtained by applying generalized interlacing inequalities to an appropriate quotient matrix as follows:

THEOREM 1.1. *Let G be a connected graph with n vertices, and let $V(G) = V_1 \cup V_2 \cup V_3$ be a partition of G with $n_i = |V_i| \leq n - 2$ for $i = 1, 2, 3$. Then*

$$a(G) \leq \min \left\{ \frac{B - \sqrt{B^2 - 4AC}}{2A} \right\}, \quad (1.7)$$

where $A = n_1 n_2 n_3$, $B = (t_1 + t_2)n_2 n_3 + (t_2 + t_3)n_1 n_2 + (t_1 + t_3)n_1 n_3$, $C = n(t_1 t_2 + t_2 t_3 + t_1 t_3)$, and $t_1 = e(V_1, V_2)$, $t_2 = e(V_1, V_3)$ and $t_3 = e(V_2, V_3)$.

REMARK 1.2. Let G be the following graph in Fig. 1. Let $V(G) = V_1 \cup V_2 \cup V_3$ be a partition of G with $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3\}$ and $V_3 = \{v_4, v_5, v_6\}$. Applying (1.7), we have $a(G) \leq \frac{8}{3} \approx 2.66667$. However, applying (1.1)-(1.6), we get $a(G) \leq 3$, $a(G) \leq 4$, $a(G) \leq 3$, $a(G) \leq 3$, $a(G) \leq 6$, and $a(G) \leq 5$, respectively. In fact, $a(G) \approx 2.38196$. This example shows that our result is better than known results.

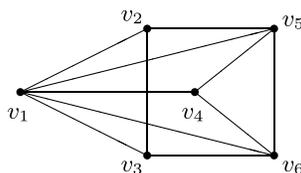


FIGURE 1. The graph G .

2. The proof of Theorem 1.1. Let M be a real symmetric partitioned matrix of order n described in the following block form:

$$\begin{pmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{pmatrix},$$

where the diagonal blocks M_{ii} are $n_i \times n_i$ matrices for any $i \in \{1, 2, \dots, t\}$ and $n = n_1 + \cdots + n_t$. For any $i, j \in \{1, 2, \dots, t\}$, b_{ij} is the average row sum of M_{ij} , that is, b_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then $\mathcal{B}(M) = (b_{ij})$ is called the quotient matrix of M .

LEMMA 2.1 ([9]). *Let M be a symmetric partitioned matrix of order n with eigenvalues $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$, and let $\mathcal{B}(M)$ be its quotient matrix with eigenvalues $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ and $n > m$. Then $\xi_i \geq \eta_i \geq \xi_{n-m+i}$ for $i = 1, 2, \dots, m$.*

The proof of Theorem 1.1. Let $\mathcal{B}(G)$ be the quotient matrix of $L(G)$ corresponding to the partition $V(G) = V_1 \cup V_2 \cup V_3$ of G . Then,

$$\mathcal{B}(G) = \begin{pmatrix} \frac{t_1+t_2}{n_1} & -\frac{t_1}{n_1} & -\frac{t_2}{n_1} \\ -\frac{t_1}{n_2} & \frac{t_1+t_3}{n_2} & -\frac{t_3}{n_2} \\ -\frac{t_2}{n_3} & -\frac{t_3}{n_3} & \frac{t_2+t_3}{n_3} \end{pmatrix}. \tag{2.1}$$

By direct computation, the characteristic polynomial of (2.1) is

$$\det(xI_n - \mathcal{B}(G)) = \frac{x}{A}(Ax^2 - Bx + C),$$

where $A = n_1n_2n_3$, $B = (t_1 + t_2)n_2n_3 + (t_2 + t_3)n_1n_2 + (t_1 + t_3)n_1n_3$, $C = n(t_1t_2 + t_2t_3 + t_1t_3)$, and $t_1 = e(V_1, V_2)$, $t_2 = e(V_1, V_3)$, $t_3 = e(V_2, V_3)$, and $n_1 + n_2 + n_3 = n$. Thus,

$$\eta_1(G) = \frac{B + \sqrt{B^2 - 4AC}}{2A},$$

and

$$\eta_2(G) = \frac{B - \sqrt{B^2 - 4AC}}{2A}.$$

As V_1, V_2 , and V_3 are arbitrary, we may take

$$\eta_2 = \min \left\{ \frac{B - \sqrt{B^2 - 4AC}}{2A} \right\}.$$

By Lemma 2.1, we have

$$a(G) \leq \min \left\{ \frac{B - \sqrt{B^2 - 4AC}}{2A} \right\}.$$

By reasoning similar to that in the proof of Theorem 1.1, we can obtain a lower bound on the largest Laplacian eigenvalue of G as follows:

THEOREM 2.2. *Let G be a connected graph with n vertices, and let $V(G) = V_1 \cup V_2 \cup V_3$ be a partition of G with $n_i = |V_i| \leq n - 2$ for $i = 1, 2, 3$. Then*

$$\mu_1(G) \geq \max \left\{ \frac{B + \sqrt{B^2 - 4AC}}{2A} \right\},$$

where $A = n_1n_2n_3$, $B = (t_1 + t_2)n_2n_3 + (t_2 + t_3)n_1n_2 + (t_1 + t_3)n_1n_3$, $C = n(t_1t_2 + t_2t_3 + t_1t_3)$, and $t_1 = e(V_1, V_2)$, $t_2 = e(V_1, V_3)$, and $t_3 = e(V_2, V_3)$.

The Laplacian spectral ratio of a connected graph G , denoted by $r_L(G)$, is defined as the quotient between the largest and second smallest Laplacian eigenvalues of G . Barahona et al. [4] showed that a graph G exhibits better synchronizability if the ratio $r_L(G)$ is as small as possible. By Theorems 1.1 and 2.2, we have

THEOREM 2.3. *Let G be a connected graph with n vertices, and let $V(G) = V_1 \cup V_2 \cup V_3$ be a partition of G with $n_i = |V_i| \leq n - 2$ for $i = 1, 2, 3$. Then*

$$r_L(G) \geq \max \left\{ \frac{B + \sqrt{B^2 - 4AC}}{B - \sqrt{B^2 - 4AC}} \right\},$$

where $A = n_1n_2n_3$, $B = (t_1 + t_2)n_2n_3 + (t_2 + t_3)n_1n_2 + (t_1 + t_3)n_1n_3$, $C = n(t_1t_2 + t_2t_3 + t_1t_3)$, and $t_1 = e(V_1, V_2)$, $t_2 = e(V_1, V_3)$, and $t_3 = e(V_2, V_3)$.

3. Corollaries. In this section, we obtain some corollaries by selecting different V_1 , V_2 , and V_3 in Theorem 1.1. Similarly, there are also corresponding results on the lower bounds of the largest Laplacian eigenvalue and the Laplacian spectral ratio. In order to avoid redundancy, we omit here.

COROLLARY 3.1. *Let G be a connected graph with n vertices. If G contains a complete bipartite induced subgraph $K_{s,t}$ with bipartition $V(K_{s,t}) = X \cup Y$, $|X| = s$, $|Y| = t$ and $s + t \leq n - 1$, then*

$$a(G) \leq \min \left\{ \frac{B - \sqrt{B^2 - 4AC}}{2A} \right\},$$

where $A = st(n - s - t)$, $B = (d_1t + d_2s)n - d_1t^2 - d_2s^2 - 2s^2t^2$, $C = (d_1d_2 - s^2t^2)n$, $d_1 = \sum_{v \in X} d_G(v)$ and $d_2 = \sum_{v \in Y} d_G(v)$.

Proof. Let $V_1 = X$ and $V_2 = Y$. Then, $|V_1| = s$, $|V_2| = t$ and $|V_3| = n - s - t$. Further, we have $e(V_1, V_2) = st$, $e(V_1, V_3) = d_1 - st$ and $e(V_2, V_3) = d_2 - st$, where $d_1 = \sum_{v \in X} d_G(v)$ and $d_2 = \sum_{v \in Y} d_G(v)$. Thus, $A = st(n - s - t)$, $B = (d_1t + d_2s)n - d_1t^2 - d_2s^2 - 2s^2t^2$ and $C = (d_1d_2 - s^2t^2)n$. By Theorem 1.1, we have the proof. \square

REMARK 3.2. If G is a connected graph with n vertices, we take $K_{1,1} = uv$, then

$$a(G) \leq \min \left\{ \frac{(n-1)A_1 - 2 - \sqrt{((n-1)A_1 - 2)^2 - 4n(n-2)(B_1 - 1)}}{2(n-2)} : uv \in E(G) \right\}, \quad (3.1)$$

where $A_1 = d_G(u) + d_G(v)$, $B_1 = d_G(u)d_G(v)$. It is clearly that the equality holds in (3.1) if G is a star $K_{1,n-1}$ or a complete graph K_n . Let G be the following graph in Fig. 2. Applying (3.1), we have $a(G) \leq \frac{5-\sqrt{10}}{3} \approx 0.6126$. However, applying (1.1)-(1.6), we get $a(G) \leq 1$, $a(G) \leq 2$, $a(G) \leq \frac{5}{6} \approx 0.8333$, $a(G) \leq 2$, $a(G) \leq \frac{10}{3}$ and $a(G) \leq 3$, respectively. In fact, $a(G) \approx 0.5188$. This example shows that our result is better than known results.

REMARK 3.3. If G is a connected k -regular graph with n vertices, by inequality (3.1), then

$$a(G) \leq \frac{n(k-1)}{n-2}. \quad (3.2)$$

It is easy to see that if $k < \frac{n}{2}$, then (3.2) is better than (1.1) for $a(G) \leq \delta(G)$.

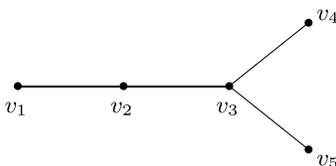


FIGURE 2. The graph G .

REMARK 3.4. Let G be a connected k -regular graph with n vertices. If G contains a complete bipartite induced subgraph $K_{s,t}$ and $s + t \leq n - 1$, then

$$a(G) \leq \frac{2kn - ks - kt - 2st - \sqrt{(2kn - ks - kt - 2st)^2 - 4(n - s - t)(k^2 - st)n}}{2(n - s - t)}.$$

REMARK 3.5. If G is a connected triangle-free graph with n vertices, then no two neighbors of a vertex v can be adjacent. Thus, there is an induced $K_{1,d_G(v)}$ in G . By Corollary 3.1, we have

$$a(G) \leq \min \left\{ \frac{B_2 - \sqrt{B_2^2 - 4A_2C_2}}{2A_2} \right\},$$

where $A_2 = n - d_G(v) - 1$, $B_2 = nd_G(v) + nm(v) - d_G^2(v) - 2d_G(v) - m(v)$, and $C_2 = nd_G(v)(m(v) - 1)$. In particular, if G is a connected k -regular triangle-free graph with n vertices, then

$$a(G) \leq \frac{k(2n - k - 3) - \sqrt{k(k^3 + 6k^2 + 9k + 4n^2 - 4n - 12kn)}}{2(n - k - 1)}, \quad (3.3)$$

and the equality holds if G is a Petersen graph or a complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$. Moreover, for a cycle C_9 , the upper bound in (3.3) is better than that of (1.1)-(1.6), respectively.

A subset S of $V(G)$ is an independent set of G if no two vertices in S are adjacent in G . A clique of G is a subset of vertices such that it induces a complete subgraph of G . Given a graph G , define $\alpha(G)$ and $\omega(G)$ (α and ω for short), the independence number and the clique number of G to be the numbers of vertices of the largest independent set and the largest clique in G , respectively. A complete split graph $S_{r,\omega}$ is a graph obtained from an independent set on r vertices and a clique on ω vertices by adding all edges between them.

COROLLARY 3.6. Let G be a connected graph with n vertices. If G contains a complete split induced subgraph $S_{r,\omega}$ with bipartition $V(S_{r,\omega}) = R \cup W$, $|R| = r$, $|W| = \omega$ and $r + \omega \leq n - 1$, then

$$a(G) \leq \frac{B - \sqrt{B^2 - 4AC}}{2A},$$

where $A = r\omega(n - r - \omega)$, $B = \omega d_1 n - \omega^2 d_1 - r^2 \omega^2 + r n d_2 - r n \omega^2 + r n \omega - r^2 d_2 - r^2 \omega$, $C = (-r^2 \omega^2 + d_1 d_2 - \omega^2 d_1 + \omega d_1)n$, $d_1 = \sum_{v \in R} d_G(v)$, and $d_2 = \sum_{v \in W} d_G(v)$.

Proof. Let $V_1 = R$ and $V_2 = W$. Then, $|V_1| = r$, $|V_2| = \omega$ and $|V_3| = n - r - \omega$. Further, we have $e(V_1, V_2) = r\omega$, $e(V_1, V_3) = d_1 - r\omega$, and $e(V_2, V_3) = d_2 - \omega(r + \omega - 1)$, where $d_1 = \sum_{v \in R} d_G(v)$ and $d_2 = \sum_{v \in W} d_G(v)$. Thus, $A = r\omega(n - r - \omega)$, $B = \omega d_1 n - \omega^2 d_1 - r^2 \omega^2 + r n d_2 - r n \omega^2 + r n \omega - r^2 d_2 - r^2 \omega$, and $C = (-r^2 \omega^2 + d_1 d_2 - \omega^2 d_1 + \omega d_1)n$. By Theorem 1.1, we have the proof. \square

REMARK 3.7. Let G be a connected k -regular graph with n vertices. If G contains a complete split induced subgraph $S_{r,\omega}$ and $r + \omega \leq n - 1$, then

$$a(G) \leq \frac{B_1 - \sqrt{B_1^2 - 4A_1C_1}}{2A_1},$$

where $A_1 = n - r - \omega$, $B_1 = 2kn - kr - k\omega - r\omega - n\omega + n - r$, and $C_1 = (k^2 - k\omega - r\omega + k)n$. In particular, if $\omega = 2$, then the graph $S_{r,2}$ is called book graph, that is, the graph $S_{r,2}$ consists of r triangles sharing an edge. Let G be a connected k -regular graph with n vertices. If G contains a book induced subgraph $S_{r,2}$ and $r \leq n - 3$, then

$$a(G) \leq \frac{B_2 - \sqrt{B_2^2 - 4A_2C_2}}{2A_2},$$

where $A_2 = n - r - 2$, $B_2 = 2kn - kr - 2k - n - 3r$, and $C_2 = (k^2 - k - 2r)n$. In particular, if G is a connected k -regular graph with n vertices and at least one triangle, then

$$a(G) \leq \frac{(k-2)n}{n-3},$$

and the equality holds if G is a triangular prism or a complete graph K_n .

COROLLARY 3.8. Let G be a connected graph with n vertices and independence number $\alpha = |S| \geq 2$, and let $V(G) = V_1 \cup V_2 \cup V_3$ be a partition of G . If $V_1 \cup V_2 = V(S)$ and $n_i = |V_i| \leq n - 2$ for $i = 1, 2$, then

$$a(G) \leq \frac{b - \sqrt{b^2 - 4n(n-\alpha)c}}{2(n-\alpha)},$$

where $b = (d_1^* - d_2^*)n_1 + (d_1^* + d_2^*)n - d_1^*\alpha$, $c = d_1^*d_2^*$, $d_1^* = \sum_{v \in V_1} d_G(v)/n_1$, and $d_2^* = \sum_{v \in V_2} d_G(v)/n_2$.

Proof. By hypothesis, we have $V_3 = V(G) - V_1 - V_2$ and $|V_3| = n - \alpha$. Further, we have $e(V_1, V_2) = 0$, $e(V_1, V_3) = \sum_{v \in V_1} d_G(v)$, and $e(V_2, V_3) = \sum_{v \in V_2} d_G(v)$. Thus, $A = n_1n_2(n - \alpha)$, $B = n_1n_2[(d_1^* - d_2^*)n_1 + (d_1^* + d_2^*)n - d_1^*\alpha]$, and $C = nn_1n_2d_1^*d_2^*$. By Theorem 1.1, we have the proof. \square

COROLLARY 3.9. Let G be a connected graph with n vertices and clique number ω , and let $V(G) = V_1 \cup V_2 \cup V_3$ be a partition of G . If $V_1 \cup V_2 = V(K_\omega)$ and $n_i = |V_i| \leq n - 2$ for $i = 1, 2$, then

$$a(G) \leq \frac{b - \sqrt{b^2 - 4n(n-\omega)c}}{2(n-\omega)},$$

where $b = (d_1^* - d_2^*)n_1 + (d_1^* + d_2^* + 2)n - (n + d_1^* + 1)\omega$, $c = (d_1^* - d_2^*)n_1 - (d_1^* + 1)\omega + d_1^*d_2^* + d_1^* + d_2^* + 1$, $d_1^* = \sum_{v \in V_1} d_G(v)/n_1$, and $d_2^* = \sum_{v \in V_2} d_G(v)/n_2$.

Proof. By hypothesis, we have $V_3 = V(G) - V_1 - V_2$ and $|V_3| = n - \omega$. Further, we have $e(V_1, V_2) = n_1n_2$, $e(V_1, V_3) = \sum_{v \in V_1} d_G(v) - r(\omega - 1)$, and $e(V_2, V_3) = \sum_{v \in V_2} d_G(v) - s(\omega - 1)$. Thus, $A = n_1n_2(n - \omega)$, $B = n_1n_2[(d_1^* - d_2^*)n_1 + (d_1^* + d_2^* + 2)n - (n + d_1^* + 1)\omega]$, and $C = nn_1n_2[(d_1^* - d_2^*)n_1 - (d_1^* + 1)\omega + d_1^*d_2^* + d_1^* + d_2^* + 1]$. By Theorem 1.1, we have the proof. \square

Recall that the diameter of a connected graph G is the maximum distance between any two vertices of G , denoted by $d(G)$ (d for short). The girth of G , denoted by $g(G)$ (g for short), is the length of a shortest cycle in G , with the girth of an acyclic graph being infinite.

COROLLARY 3.10. Let G be a connected graph with n vertices and diameter d , and let $V(G) = V_1 \cup V_2 \cup V_3 = \{v_1, v_2, \dots, v_n\}$ be a partition of G and $V_1 \cup V_2 = V(P_{d+1}) = \{v_1, v_2, \dots, v_{d+1}\}$, then

(i) If d is an odd integer and $V_1 = \{v_1, v_3, \dots, v_d\}$, then

$$a(G) \leq \frac{b_o - \sqrt{b_o^2 - 16nc_o}}{2(d+1)(n-d-1)},$$

where $b_o = (2n-d-1)(d_1+d_2) - 2d^2 - 2d$, $c_o = (n-d-1)(d_1d_2 - d^2)$, $d_1 = \sum_{v \in V_1} d_G(v)$, and $d_2 = \sum_{v \in V_2} d_G(v)$.

(ii) If d is an even integer and $V_1 = \{v_1, v_3, \dots, v_{d+1}\}$, then

$$a(G) \leq \frac{b_e - \sqrt{b_e^2 - 16nc_e}}{2d(d+2)(n-d-1)},$$

where $b_e = d(2n-d)(d_1+d_2) + 4(n-d-1)d_2 - 2d^3 - 4d^2$, $c_e = d(d+2)(n-d-1)(d_1d_2 - d^2)$, $d_1 = \sum_{v \in V_1} d_G(v)$, and $d_2 = \sum_{v \in V_2} d_G(v)$.

Proof.

(i) If d is an odd integer and $V_1 = \{v_1, v_3, \dots, v_d\}$, then we have $|V_1| = |V_2| = \frac{d+1}{2}$. Thus, $V_3 = V(G) - V_1 - V_2$ and $|V_3| = n-d-1$. Further, $e(V_1, V_2) = d$, $e(V_1, V_3) = \sum_{v \in V_1} d_G(v) - d$, $e(V_2, V_3) = \sum_{v \in V_2} d_G(v) - d$. Therefore, we have $A = \frac{(n-d-1)(d+1)^2}{4}$, $B = \frac{1}{4}(d+1)[(2n-d-1)(d_1+d_2) - 2d^2 - 2d]$ and $C = n(d_1d_2 - d^2)$. By Theorem 1.1, we have the proof.

(ii) If d is an even integer and $V_1 = \{v_1, v_3, \dots, v_{d+1}\}$, then we have $|V_1| = \frac{d+2}{2}$ and $|V_2| = \frac{d}{2}$. Thus, $V_3 = V(G) - V_1 - V_2$ and $|V_3| = n-d-1$. Further, $e(V_1, V_2) = d$, $e(V_1, V_3) = \sum_{v \in V_1} d_G(v) - d$, $e(V_2, V_3) = \sum_{v \in V_2} d_G(v) - d$. Therefore, we have $A = \frac{d(d+2)(n-d-1)}{4}$, $B = \frac{1}{4}[d(2n-d)(d_1+d_2) + 4(n-d-1)d_2 - 2d^3 - 4d^2]$ and $C = n(d_1d_2 - d^2)$. By Theorem 1.1, we have the proof. \square

COROLLARY 3.11. Let G be a connected graph with n vertices and girth g , and let $V(G) = V_1 \cup V_2 \cup V_3 = \{v_1, v_2, \dots, v_n\}$ be a partition of G and $V_1 \cup V_2 = V(C_g) = \{v_1, v_2, \dots, v_g\}$.

(i) If g is an even integer and $V_1 = \{v_1, v_3, \dots, v_{g-1}\}$, then

$$a(G) \leq \frac{b_e - \sqrt{b_e^2 - 16nc_e}}{2g(n-g)},$$

where $b_e = (2n-g)(d_1+d_2) - 2g^2$, $c_e = (n-g)(d_1d_2 - g^2)$, $d_1 = \sum_{v \in V_1} d_G(v)$ and $d_2 = \sum_{v \in V_2} d_G(v)$.

(ii) If g is an odd integer and $V_1 = \{v_1, v_3, \dots, v_g\}$, then

$$a(G) \leq \frac{b_o - \sqrt{b_o^2 - 16nc_o}}{2(g^2-1)(n-g)},$$

where $b_o = (2gn - g^2 - 1)(d_1+d_2-2) - 2(n-g)(d_1-d_2) - 2(g-1)(g^2+1) + 4(n-1)$, $c_o = (g^2-1)(n-g)[d_1d_2 - 2d_2 - (g-1)^2]$, $d_1 = \sum_{v \in V_1} d_G(v)$, and $d_2 = \sum_{v \in V_2} d_G(v)$.

Proof.

(i) If g is an even integer and $V_1 = \{v_1, v_3, \dots, v_{g-1}\}$, then we have $|V_1| = |V_2| = \frac{g}{2}$. Thus, $V_3 = V(G) - V_1 - V_2$ and $|V_3| = n-g$. Further, $e(V_1, V_2) = g$, $e(V_1, V_3) = \sum_{v \in V_1} d_G(v) - g$, $e(V_2, V_3) = \sum_{v \in V_2} d_G(v) - g$. Therefore, we have $A = \frac{(n-g)g^2}{4}$, $B = \frac{g}{4}[(2n-g)(d_1+d_2) - 2g^2]$ and $C = n(d_1d_2 - g^2)$. By Theorem 1.1, we have the proof.

- (ii) If g is an odd integer and $V_1 = \{v_1, v_3, \dots, v_g\}$, then we have $|V_1| = \frac{g+1}{2}$ and $|V_2| = \frac{g-1}{2}$. Thus, $V_3 = V(G) - V_1 - V_2$ and $|V_3| = n - g$. Further, $e(V_1, V_2) = g - 1$, $e(V_1, V_3) = \sum_{v \in V_1} d_G(v) - g - 1$, $e(V_2, V_3) = \sum_{v \in V_2} d_G(v) - g + 1$. Therefore, we have $A = \frac{(g^2-1)(n-g)}{4}$, $B = \frac{1}{4}[(2gn - g^2 - 1)(d_1 + d_2 - 2) - 2(n - g)(d_1 - d_2) - 2(g - 1)(g^2 + 1) + 4(n - 1)]$, and $C = n[d_1d_2 - 2d_2 - (g - 1)^2]$. By Theorem 1.1, we have the proof. \square

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