Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

EXTREMAL PROPERTIES OF THE DISTANCE SPECTRAL RADIUS OF HYPERGRAPHS*

YANNA WANG† AND BO ZHOU‡

Abstract. The distance spectral radius of a connected hypergraph is the largest eigenvalue of its distance matrix. The unique hypertrees with minimum distance spectral radii are determined in the class of hypertrees of given diameter, in the class of hypertrees of given matching number, and in the class of non-hyperstar-like hypertrees, respectively. The unique hypergraphs with minimum and second minimum distance spectral radii are determined in the class of unicylic hypergraphs. The unique hypertree with maximum distance spectral radius is determined in the class of k-th power hypertrees of given matching number.

Key words. Distance spectral radius, Distance matrix, Hypergraph, Diameter, Matching number.

AMS subject classifications. 05C50, 05C65.

1. Introduction. A (simple) hypergraph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where every edge in $E(G)$ is a subset of $V(G)$ containing at least two vertices, see [2]. For $u, v \in V(G)$, if they are contained in some edge of G, then we say that they are adjacent, or v is a neighbor of u. For $u \in V(G)$, let $N_G(u)$ be the set of neighbors of u in G. The degree of a vertex u in G, denoted by $\delta_G(u)$, is the number of edges containing u in G. For an integer $k \geq 2$, the hypergraph G is k-uniform if every edge of G contains exactly k vertices.

For distinct vertices $v_0, \ldots, v_p$ and distinct edges $e_1, \ldots, e_p$ of G, the alternating sequence $(v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ is a path of G from v_0 to v_p of length p if $v_{i-1}, v_i \in e_i$ for $i = 1, \ldots, p$, and $e_i \cap e_j = \emptyset$ for $i, j = 1, \ldots, p$ with $j > i + 1$. For distinct vertices $v_0, \ldots, v_{p-1}$ and distinct edges $e_1, \ldots, e_p$, the alternating sequence $(v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_0)$ is a cycle of G (of length p) if $v_{i-1}, v_i \in e_i$ for $i = 1, \ldots, p$ with $v_p = v_0$, and $e_i \cap e_j = \emptyset$ for $i, j = 1, \ldots, p$ with $|i - j| > 1$ and $\{i, j\} \neq \{1, p\}$. If there is a path from u to v for any $u, v \in V(G)$, then we say that G is connected. A hypertree is a connected hypergraph with no cycles. A unicylic hypergraph is a connected hypergraph with exactly one cycle.

A path $(v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ of a hypergraph G is called a pendant path of G at v_0, if $\delta_G(v_0) \geq 2$, $\delta_G(v_i) = 2$ for $1 \leq i \leq p - 1$, $\delta_G(v) = 1$ for $v \in e_i \setminus \{v_{i-1}, v_i\}$ with $1 \leq i \leq p$, and $\delta_G(v_p) = 1$. If $p = 1$, then we call e_1 a pendant edge of G (at v_0). A hyperstar is a hypertree in which all edges are pendant edges at a common vertex. A hypertree is hyperstar-like if it consists of a single vertex, or a single edge, or some pendant paths at a vertex. A hypertree that is not hyperstar-like is said to be non-hyperstar-like.

Let G be a connected hypergraph on n vertices. For $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path connecting them in G. In particular, $d_G(u, u) = 0$. The diameter of G is $\max\{d_G(u, v) : u, v \in V(G)\}$. The distance matrix of G is defined as $D(G) = (d_G(u, v))_{u,v \in V(G)}$. The distance spectral radius of G, denoted by $\rho(G)$, is the largest eigenvalue of $D(G)$.

*Received by the editors on January 25, 2018. Accepted for publication on May 9, 2020. Handling Editor: Froilán M. Dopico. Corresponding Author: Bo Zhou.
†Basic Courses Department, Guang Dong Communication Polytechnic, Guangzhou 510650, P.R. China (yi.fuxue@163.com).
‡School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P.R. China (zhoubo@scnu.edu.cn).
Supported by the National Natural Science Foundation of China (no. 11671156).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

For a connected hypergraph G, $D(G) = D(O_G)$, where O_G is a graph with $V(O_G) = V(G)$ such that for $u, v \in V(O_G)$, $\{u, v\}$ is an edge of O_G if and only if u and v are in some edge of G. Obviously, each edge of G corresponds to a complete subgraph in O_G. We note that the distance matrix (in a metric space) was originally defined by Cayley [3] in 1841, while the distance matrix of a graph was first studied in [6].

The eigenvalues of distance matrices of graphs, arisen from a data communication problem studied by Graham and Pollack [5] in 1971, have been studied extensively, and in particular, the distance spectral radius received much attention, see the survey [1]. Sivasubramanian [15] studied properties of distance matrix of a 3-uniform hypertree. Watanabe et. al. [17] studied a q-ary extension of the classical binary addressing problem of graphs which was originally posed by Graham and Pollak [5], and found a sharp lower bound for the minimum length of addressings in terms of distance eigenvalues of uniform hypertrees. Lin and Zhou [8] and Lin et al. [10] studied the distance spectral radius of uniform hypergraphs and particularly, uniform hypertrees. Lin and Zhou [9] studied the distance spectral radius of uniform hypergraphs with cycles, and particularly, uniform unicyclic hypergraphs. Wang and Zhou [16] studied the distance spectral radius of a hypergraph that is not necessarily uniform. They proposed some graft transformations that decrease or increase the distance spectral radius of a hypergraph, determined the unique hypertrees with minimum and maximum distance spectral radius, respectively, among hypertrees on n vertices with m edges, where $1 \leq m \leq n-1$, and also determined the unique hypertrees with the first three smallest (largest, respectively) distance spectral radii among hypertrees on $n \geq 6$ vertices. Note that the hypertrees with minimum, second minimum and third minimum distance spectral radii are all hyperstar-like hypertrees.

We point out that the spectral theory of hypergraphs can be studied with matrices and tensors. In 2012, Cooper and Dutle [4] proposed the study of hypergraphs through tensors, and this new approach has been widely accepted by researchers of this area, see, e.g. [7, 13, 14]. However, to obtain eigenvalues of tensors has a high computational cost. In this regard, we see that the study of hypergraphs via matrices still has its place.

A matching of a hypergraph is a subset of edges such that any two edges have no vertex in common. The matching number of a hypergraph G, denoted by $\beta(G)$, is the maximum number of edges in a matching of G.

For $k \geq 2$ and a graph G on n vertices, the k-th power of G is defined as the k-uniform hypergraph on $n + (k-2)|E(G)|$ vertices with vertex set $V(G) \cup (\cup_{e \in E(G)} V_e)$ and edge set $\{e \cup V_e : e \in E(G)\}$, where $|V_e| = k - 2$ for $e \in E(G)$, see [7]. Obviously, the 2-nd power of G is G itself. A hypergraph is a k-th power hypertree if it is the k-th power of some tree.

In this paper, we determine the unique hypertree of given diameter with minimum distance spectral radius, the unique hypertree of given matching number with minimum distance spectral radius, the unique non-hyperstar-like hypertree with minimum distance spectral radius, the unique unicylic hypergraphs with respectively minimum and second minimum distance spectral radii, and the unique k-th power hypertree of given matching number with maximum distance spectral radius.

2. Preliminaries. Let G be a connected hypergraph. Since $D(G)$ is irreducible, by Perron-Frobenius theorem, $\rho(G)$ is simple and there is a unique unit positive eigenvector corresponding to $\rho(G)$, which is called the distance Perron vector of G, denoted by $x(G)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

Let $V(G) = \{v_1, \ldots, v_n\}$ and $x = (x_{v_1}, \ldots, x_{v_n})^T \in \mathbb{R}^n$. Then

$$x^T D(G)x = 2 \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) x_u x_v.$$

If x is unit and x has at least one nonnegative component, then by Rayleigh's principle, we have $\rho(G) \geq x^T D(G)x$ with equality if and only if $x = x(G)$.

For $x = x(G)$ and each $u \in V(G)$, we have

$$\rho(G)x_u = \sum_{v \in V(G)} d_G(u, v) x_v,$$

which is called the distance eigenequation of G at u.

The following lemma was stated in [8] for a connected uniform hypergraph. However, its proof applies to any connected hypergraph that is not necessarily uniform.

Lemma 2.1. [8] *Let G be a connected hypergraph with η being an automorphism of G and $x = x(G)$. Then $\eta(u) = v$ implies that $x_u = x_v$.*

Lemma 2.2. [16] *For $k, r \geq 2$, let G be a connected hypergraph with two pendant edges, say $e_1 = \{w_1, \ldots, w_k\}$ and $e_2 = \{v_1, \ldots, v_r\}$ at w_k and v_r, respectively. Let $x = x(G)$. Then $(\rho(G) + k)x_{w_1} - (\rho(G) + r)x_{v_1} = \rho(G)(x_{w_k} - x_{v_r})$.*

For a square nonnegative matrix M, let $\rho(M)$ be its spectral radius, i.e., the maximum modulus of its eigenvalues. We restate Corollary 2.2 in [11, p. 38]. If M and N are square nonnegative matrices, M is irreducible, $M - N$ is nonnegative, and $M - N \neq 0$, then $\rho(M) > \rho(N)$. For a connected hypergraph G, we have $\rho(G) = \rho(D(G))$ by Perron-Frobenius theorem. So we have the following lemma.

Lemma 2.3. *Let G be a connected hypergraph with $u, v \in V(G)$ such that u and v are not adjacent. Let G' be a hypergraph with $V(G') = V(G)$ such that u and v are adjacent, and two vertices in G' are adjacent if they are adjacent in G. Then $\rho(G') < \rho(G)$.*

Let G be a hypergraph with $u, v \in V(G)$ and $e_1, \ldots, e_r \in E(G)$ such that $u \notin e_i$ and $v \in e_i$ for $1 \leq i \leq r$. Let $e'_i = (e_i \setminus \{v\}) \cup \{u\}$ for $1 \leq i \leq r$. Suppose that $e'_i \notin E(G)$ for $1 \leq i \leq r$. Let G' be the hypergraph with $V(G') = V(G)$ and $E(G') = (E(G) \setminus \{e_1, \ldots, e_r\}) \cup \{e'_1, \ldots, e'_r\}$. Then we say that G' is obtained from G by moving edges $e_1, \ldots, e_r$ from v to u.

Lemma 2.4. [16] *Let G be a hypergraph with connected induced subhypergraphs G_0, H_1 and H_2 such that there are two adjacent vertices w_1 and w_2 in G_0 with $N_{G_0}(w_1) \setminus \{w_2\} = N_{G_0}(w_2) \setminus \{w_1\}$, $V(H_i) \cap V(G_0) = \{w_i\}$ for $i = 1, 2$, $V(H_1) \cap V(H_2) = \emptyset$, $V(G) = V(G_0) \cup V(H_1) \cup V(H_2)$, and $E(G) = E(G_0) \cup E(H_1) \cup E(H_2)$. Suppose that $|V(H_i)| \geq 2$ for $i = 1, 2$. Let G' be the hypergraph obtained from G by moving all edges containing w_2 except the edges in $E(G_0)$ from w_2 to w_1. Then $\rho(G) > \rho(G')$.*

Let G be a hypergraph with $e_1, e_2 \in E(G)$ and $u_1, \ldots, u_s \in V(G)$ such that $u_1, \ldots, u_s \notin e_1$ and $u_1, \ldots, u_s \in e_2$, where $|e_2| - s \geq 2$. Let $e'_1 = e_1 \cup \{u_1, \ldots, u_s\}$ and $e'_2 = e_2 \setminus \{u_1, \ldots, u_s\}$. Suppose that $e'_1, e'_2 \notin E(G)$. Let G' be the hypergraph with $V(G') = V(G)$ and $E(G') = (E(G) \setminus \{e_1, e_2\}) \cup \{e'_1, e'_2\}$. Then we say that G' is obtained from G by moving vertices $u_1, \ldots, u_s$ from e_2 to e_1.

For a connected hypergraph G with $V_1 \subseteq V(G)$, let $\sigma_G(V_1)$ be the sum of the entries of the distance Perron vector of G corresponding to the vertices in V_1. Furthermore, if all the vertices of V_1 induce a connected subhypergraph H of G, then we write $\sigma_G(H)$ instead of $\sigma_G(V_1)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

For $e \in E(G)$, let $G - e$ be the subhypergraph of G obtained by deleting e.

3. Distance spectral radius of hypertrees with given diameter. A hypertree is a loose path if there is a path containing all its vertices. For $2 \leq d \leq n-1$, let T_n^d be a loose path of the form $(v_0, e_1, v_1, e_2, \ldots, v_{d-1}, e_d, v_d)$, where $\left|e_{\lceil \frac{d}{2} \rceil}\right| = n - d + 1$ and $|e_i| = 2$ for $i \neq \lceil \frac{d}{2} \rceil$. Let T_n^1 be the hypertree on n vertices consisting of a single edge.

LEMMA 3.1. *Suppose that d is even with $2 \leq d < n-1$. Let $x = x(T_n^d)$ and $u \in e_{\frac{d}{2}} \setminus \left\{v_{\frac{d}{2}-1}, v_{\frac{d}{2}}\right\}$. Then*

(i) $x_{v_{d-i}} > x_{v_i}$ *for* $i = 0, \ldots, \frac{d}{2} - 1$;

(ii) $\left(\left\lceil \frac{\left|e_{\frac{d}{2}}\right|}{2} \right\rceil - 1\right) x_u + \sum_{i=0}^{\frac{d}{2}-1} x_{v_i} > \sum_{i=\frac{d}{2}+1}^{d} x_{v_i}$.

Proof. Let $T = T_n^d$. Since $d < n-1$, we have $\left|e_{\frac{d}{2}}\right| \geq 3$. By Lemma 2.1, $x_w = x_u$ for $w \in e_{\frac{d}{2}} \setminus \left\{v_{\frac{d}{2}-1}, v_{\frac{d}{2}}\right\}$.

Suppose first that $d = 2$. From the distance eigenequations of T at v_2 and v_0, we have

$$\rho(T)\left(x_{v_2} - x_{v_0}\right) = (|e_1| - 2)x_u + 2x_{v_0} - 2x_{v_2},$$

i.e.,

$$(|e_1| - 2)x_u = (\rho(T) + 2)(x_{v_2} - x_{v_0}).$$

Obviously, $(|e_1| - 2)x_u > 0$. Thus, $x_{v_2} > x_{v_0}$, proving (i). Furthermore, we have

$$(|e_1| - 2)x_u > 2(x_{v_2} - x_{v_0}),$$

and thus, $\left(\left\lceil \frac{|e_1|}{2} \right\rceil - 1\right) x_u + x_{v_0} > x_{v_2}$, proving (ii).

Now suppose that $d \geq 4$. By Lemma 2.2, we have

$$\rho(T)(x_{v_{d-1}} - x_{v_1}) = (\rho(T) + 2)(x_{v_d} - x_{v_0}).$$

For $2 \leq i \leq \frac{d}{2} - 1$ with $d \geq 6$, from the distance eigenequations of T at $v_{d-(i-1)}$, v_{i-1}, v_{d-i}, and v_i, we have

$$\rho(T)(x_{v_{d-(i-1)}} - x_{v_{i-1}}) - \rho(T)(x_{v_{d-i}} - x_{v_i}) = 2\sum_{j=0}^{i-1}(x_{v_j} - x_{v_{d-j}}),$$

i.e.,

$$\rho(T)(x_{v_{d-i}} - x_{v_i}) = (\rho(T) + 2)(x_{v_{d-(i-1)}} - x_{v_{i-1}}) + 2\sum_{j=0}^{i-2}(x_{v_{d-j}} - x_{v_j}).$$

Thus, it follows that $x_{v_{d-i}} - x_{v_i}$ for $0 \leq i \leq \frac{d}{2} - 1$ are all positive, or zero, or negative.

From the distance eigenequations of T at $v_{\frac{d}{2}+1}$ and $v_{\frac{d}{2}-1}$, we have

$$\rho(T)\left(x_{v_{\frac{d}{2}+1}} - x_{v_{\frac{d}{2}-1}}\right) = \left(\left|e_{\frac{d}{2}}\right| - 2\right) x_u + 2\sum_{i=0}^{\frac{d}{2}-1} x_{v_i} - 2\sum_{i=\frac{d}{2}+1}^{d} x_{v_i},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

i.e.,

$$\left(\left|e_{\frac{d}{2}}\right|-2\right)x_u=(\rho(T)+2)\left(x_{v_{\frac{d}{2}+1}}-x_{v_{\frac{d}{2}-1}}\right)+2\sum_{i=0}^{\frac{d}{2}-2}(x_{v_{d-i}}-x_{v_i}).$$

Obviously, $\left(\left|e_{\frac{d}{2}}\right|-2\right)x_u>0$. Thus, $x_{v_{d-i}}>x_{v_i}$ for $0\le i\le\frac{d}{2}-1$. This proves (i).

Note that

$$\frac{1}{2}\rho(T)\left(x_{v_{\frac{d}{2}+1}}-x_{v_{\frac{d}{2}-1}}\right)=\frac{\left|e_{\frac{d}{2}}\right|-2}{2}x_u+\sum_{i=0}^{\frac{d}{2}-1}x_{v_i}-\sum_{i=\frac{d}{2}+1}^{d}x_{v_i}.$$

By (i), we have $x_{v_{\frac{d}{2}+1}}-x_{v_{\frac{d}{2}-1}}>0$. Now (ii) follows immediately. ∎

THEOREM 3.2. *Let T be a hypertree on $n\ge 6$ vertices with diameter d, where $1\le d\le n-1$. Then $\rho(T)\ge\rho(T_n^d)$ with equality if and only if $T\cong T_n^d$.*

Proof. It is trivial if $d=1$.

Suppose that $d\ge 2$. Let T be a hypertree on n vertices with diameter d that minimizes the distance spectral radius.

Let $P=(v_0,e_1,v_1,\ldots,v_{d-1},e_d,v_d)$ be a path of length d in T.

Suppose that there exists a vertex v_i with $1\le i\le d-1$ of degree at least three. Let $\delta_T(v_i)=t\ge 3$. Then T consists of t subhypertrees $T_1,\ldots,T_t$ such that $|V(T_i)|\ge 2$ for $1\le i\le t$ and $T_1,\ldots,T_t$ have exactly one vertex v_i in common. We may assume that $e_i\in E(T_1)$ and $e_{i+1}\in E(T_2)$. Let T' be the hypertree obtained from T by deleting edge e_i and all edges in $E(T_3)$, and adding an edge $e_i\cup V(T_3)$. Obviously, the diameter of T' is d. By Lemma 2.3, $\rho(T')<\rho(T)$, a contradiction. Thus, $\delta_T(v_i)=2$ for $i=1,\ldots,d-1$.

Suppose that there exists an edge e_i with $2\le i\le d-1$ of size at least three, whose deletion yields at least three nontrivial components, i.e., there exists a vertex w in $e_i\setminus\{v_{i-1},v_i\}$ with $\delta_T(w)\ge 2$. Let T_w be the component in $T-e_i$ containing w. Let T'' be the hypertree obtained from T by deleting edge e_i and all edges in $E(T_w)$, and adding an edge $e_i\cup V(T_w)$. Obviously, T'' also has diameter d. By Lemma 2.3, $\rho(T'')<\rho(T)$, a contradiction. Thus, $T-e_i$ has exactly two nontrivial components for $2\le i\le d-1$. It follows that $T=P$. If $d=n-1$, then $T\cong T_n^d$. Suppose that $d<n-1$. Let $x=x(T_n^d)$.

Case 1. d is even.

By Lemma 3.1 (i), we have

$$\sum_{i=0}^{\frac{d}{2}-1}x_{v_i}<\sum_{i=\frac{d}{2}}^{d}x_{v_i}. \tag{3.1}$$

Suppose there exist some k with $1\le k\le\frac{d}{2}$ and some ℓ with $\frac{d}{2}+1\le\ell\le d$ such that $|e_k|\ge 3$ and $|e_\ell|\ge 3$. We may assume that $\sum_{i=1}^{\frac{d}{2}}|e_i|\ge\sum_{i=\frac{d}{2}+1}^{d}|e_i|$. Let T^* be the hypertree obtained from T by moving all vertices in $e_i\setminus\{v_{i-1},v_i\}$ for each $i\ne\frac{d}{2}$ from e_i to $e_{\frac{d}{2}}$. Obviously, $T^*\cong T_n^d$. Let $u\in e_k\setminus\{v_{k-1},v_k\}$. By Lemmas 2.1 and 3.1 (ii),

$$\sum_{i=1}^{\frac{d}{2}}(|e_i|-2)x_u+\sum_{i=0}^{\frac{d}{2}-1}x_{v_i}>\sum_{i=\frac{d}{2}+1}^{d}x_{v_i}. \tag{3.2}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

As we pass from T to T^*, for $i = 1, \dots, \frac{d}{2}-1$ (with $d \geq 4$), the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_{\frac{d}{2}}, \dots, v_d\right\}$ is decreased by $\frac{d}{2} - i$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_0, \dots, v_{\frac{d}{2}-1}\right\}$ is increased by at most $\frac{d}{2}-i$; for $i = \frac{d}{2}+1, \dots, d$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_0, \dots, v_{\frac{d}{2}-1}\right\}$ is decreased by $i - \frac{d}{2}$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\cup_{i=1}^{\frac{d}{2}}(e_i \setminus \{v_{i-1}, v_i\})$ is decreased by at least $i - \frac{d}{2}$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_{\frac{d}{2}+1}, \dots, v_d\right\}$ is increased by at most $i - \frac{d}{2}$, and the distances between all other vertex pairs are decreased or remain unchanged. Let $F = \sum_{i=1}^{\frac{d}{2}-1} \sigma_{T^*}(e_i \setminus \{v_{i-1}, v_i\}) \left(\frac{d}{2} - i\right) \left(\sum_{i=\frac{d}{2}}^{d} x_{v_i} - \sum_{i=0}^{\frac{d}{2}-1} x_{v_i}\right)$. Then $F = 0$ if $d = 2$, and from (3.1), $F > 0$ if $d \geq 4$. Thus, from (3.2), we have

$$\begin{aligned}
\frac{1}{2}(\rho(T) - \rho(T^*)) &\geq \frac{1}{2} x^\top (D(T) - D(T^*))x \\
&\geq F + \sum_{i=\frac{d}{2}+1}^{d} \sigma_{T^*}(e_i \setminus \{v_{i-1}, v_i\}) \left(i - \frac{d}{2}\right) \left(\sum_{i=1}^{\frac{d}{2}} (|e_i| - 2)x_u + \sum_{i=0}^{\frac{d}{2}-1} x_{v_i} - \sum_{i=\frac{d}{2}+1}^{d} x_{v_i}\right) \\
&\geq \sum_{i=\frac{d}{2}+1}^{d} \sigma_{T^*}(e_i \setminus \{v_{i-1}, v_i\}) \left(i - \frac{d}{2}\right) \left(\sum_{i=1}^{\frac{d}{2}} (|e_i| - 2)x_u + \sum_{i=0}^{\frac{d}{2}-1} x_{v_i} - \sum_{i=\frac{d}{2}+1}^{d} x_{v_i}\right) \\
&> 0,
\end{aligned}$$

and so $\rho(T^*) < \rho(T)$, a contradiction. Therefore, we have either $|e_i| = 2$ for $i = 1, \dots, \frac{d}{2}$ or $|e_i| = 2$ for $i = \frac{d}{2} + 1, \dots, d$. If $d = 2$, then $T \cong T_n^d$. Suppose that $d \geq 4$. Suppose that $T \ncong T_n^d$. Then we may assume that $|e_i| = 2$ for $i = \frac{d}{2} + 1, \dots, d$, but $|e_i| \geq 3$ for some $i = 1, \dots, \frac{d}{2} - 1$.

Let $\widehat{T}$ be the hypertree obtained from T by moving all vertices in $e_i \setminus \{v_{i-1}, v_i\}$ for each $i = 1, \dots, \frac{d}{2} - 1$ from e_i to $e_{\frac{d}{2}}$. Obviously, $\widehat{T} \cong T_n^d$.

As we pass from T to $\widehat{T}$, for $i = 1, \dots, \frac{d}{2} - 1$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_{\frac{d}{2}}, \dots, v_d\right\}$ is decreased by $\frac{d}{2} - i$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_0, \dots, v_{\frac{d}{2}-1}\right\}$ is increased by at most $\frac{d}{2} - i$, and the distance between any other vertex pair is decreased or remains unchanged. Then, from (3.1), we have

$$\begin{aligned}
\frac{1}{2}(\rho(T) - \rho(\widehat{T})) &\geq \frac{1}{2} x^\top (D(T) - D(\widehat{T}))x \\
&\geq \sum_{i=1}^{\frac{d}{2}-1} \sigma_{\widehat{T}}(e_i \setminus \{v_{i-1}, v_i\}) \left(\frac{d}{2} - i\right) \left(\sum_{i=\frac{d}{2}}^{d} x_{v_i} - \sum_{i=0}^{\frac{d}{2}-1} x_{v_i}\right) \\
&> 0,
\end{aligned}$$

and thus, $\rho(\widehat{T}) < \rho(T)$, a contradiction. Therefore, $T \cong T_n^d$.

Case 2. d is odd.

Let $A = \sum_{i=0}^{\frac{d-1}{2}} x_{v_i}$ and $B = \sum_{i=\frac{d+1}{2}}^{d} x_{v_i}$. By Lemma 2.1, $x_{v_i} = x_{v_{d-i}}$ for $i = 0, \dots, \frac{d-1}{2}$, and thus, $A = B$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

Suppose that there exists some i with $1 \le i \le d$ and $i \neq \frac{d+1}{2}$, such that $|e_i| \ge 3$. Let T^* be the hypertree obtained from T by moving all vertices in $e_i \setminus \{v_{i-1}, v_i\}$ for each $i \neq \frac{d+1}{2}$ from e_i to $e_{\frac{d+1}{2}}$. Obviously, $T^* \cong T_n^d$.

As we pass from T to T^*, for $i = 1, \ldots, \frac{d-1}{2}$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_{\frac{d+1}{2}}, \ldots, v_d\right\}$ is decreased by $\frac{d+1}{2} - i$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_0, \ldots, v_{\frac{d-3}{2}}\right\}$ is increased by at most $\frac{d+1}{2} - i$, for $i = \frac{d+3}{2}, \ldots, d$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_0, \ldots, v_{\frac{d-1}{2}}\right\}$ is decreased by $i - \frac{d+1}{2}$, the distance between a vertex of $e_i \setminus \{v_{i-1}, v_i\}$ and a vertex of $\left\{v_{\frac{d+3}{2}}, \ldots, v_d\right\}$ is increased by at most $i - \frac{d+1}{2}$, and the distances between all other vertex pairs are decreased or remain unchanged. Then

$$\begin{aligned}
\frac{1}{2}(\rho(T) - \rho(T^*)) &\ge \frac{1}{2} x^\top (D(T) - D(T^*)) x \\
&\ge \sum_{i=1}^{\frac{d-1}{2}} \sigma_{T^*}(e_i \setminus \{v_{i-1}, v_i\}) \left(\frac{d+1}{2} - i\right) \left(B - A + x_{v_{\frac{d-1}{2}}}\right) \\
&\quad + \sum_{i=\frac{d+3}{2}}^{d} \sigma_{T^*}(e_i \setminus \{v_{i-1}, v_i\}) \left(i - \frac{d+1}{2}\right) \left(A - B + x_{v_{\frac{d+1}{2}}}\right) \\
&= \sum_{i=1}^{\frac{d-1}{2}} \sigma_{T^*}(e_i \setminus \{v_{i-1}, v_i\}) \left(\frac{d+1}{2} - i\right) x_{v_{\frac{d-1}{2}}} \\
&\quad + \sum_{i=\frac{d+3}{2}}^{d} \sigma_{T^*}(e_i \setminus \{v_{i-1}, v_i\}) \left(i - \frac{d+1}{2}\right) x_{v_{\frac{d+1}{2}}} \\
&> 0,
\end{aligned}$$

and thus, $\rho(T^*) < \rho(T)$, a contradiction. Therefore, $|e_i| = 2$ for $1 \le i \le d$ with $i \neq \frac{d+1}{2}$. It follows that $T \cong T_n^d$. ∎

4. Distance spectral radius of non-hyperstar-like hypertrees. For $n \ge 6$, let H_n be a hypertree on n vertices obtained from T_{n-3}^1 with edge $e = \{w_1, \ldots, w_{n-3}\}$ by attaching a pendant edge $\{u_i, w_i\}$ to w_i for each $i = 1, 2, 3$. Let H'_n be the hypertree obtained from H_n by deleting edges e and $\{u_2, w_2\}$, and adding edges $e \setminus \{w_2\}$, $\{u_2, w_1\}$ and $\{w_2, w_3\}$.

LEMMA 4.1. *Let H_n and H'_n be defined above. Then $\rho(H_n) < \rho(H'_n)$.*

Proof. Let $x = x(H_n)$. By Lemma 2.1, $x_{w_1} = x_{w_2} = x_{w_3}$ and $x_{u_1} = x_{u_2} = x_{u_3}$. For $v \in V(H_n) \setminus \{w_1, u_1\}$, $2d_{H_n}(v, w_1) - d_{H_n}(v, u_1) \ge 0$. From the distance eigenequations of H_n at w_1 and u_1, we have $\rho(H_n)(2x_{w_1} - x_{u_1}) \ge 2x_{u_1} - x_{w_1}$, which implies that $(\rho(H_n) + 1)(2x_{w_1} - x_{u_1}) \ge x_{u_1} + x_{w_1} > 0$. Thus, $2x_{w_1} > x_{u_1}$.

As we pass from H_n to H'_n, the distance between u_2 and w_2 is increased by 2, the distance between u_2 and a vertex of $\{u_1, w_1\}$ is decreased by 1, the distance between w_2 and a vertex of $\{u_1, w_1\}$ is increased by

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

1, and the distances between all other vertex pairs are increased or remain unchanged. Then

$$\begin{aligned}\frac{1}{2}(\rho(H_n')-\rho(H_n)) &\geq \frac{1}{2}x^\top(D(H_n')-D(H_n))x \\ &\geq 2x_{u_2}x_{w_2}-x_{u_2}x_{u_1}-x_{u_2}x_{w_1}+x_{w_2}x_{u_1}+x_{w_2}x_{w_1} \\ &= x_{u_2}(2x_{w_1}-x_{u_1})+x_{w_1}^2 \\ &> 0,\end{aligned}$$

and thus, $\rho(H_n) < \rho(H_n')$. ◻

THEOREM 4.2. *Let T be a non-hyperstar-like hypertree on $n \geq 6$ vertices. Then $\rho(T) \geq \rho(H_n)$ with equality if and only if $T \cong H_n$.*

Proof. Let T be a non-hyperstar-like hypertree on n vertices that minimizes the distance spectral radius.

Let d be the diameter of T. Obviously, $d \geq 3$. Let $P = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ be a path of length d in T.

Suppose that $d \geq 4$. Let T' be the hypertree obtained from T by moving all edges containing v_2 except e_2 from v_2 to v_1. Let T'' be the hypertree obtained from T by moving all edges containing v_{d-2} except e_{d-1} from v_{d-2} to v_{d-1}. Since T is non-hyperstar-like, one of T' and T'', say T', is non-hyperstar-like. By Lemma 2.4, $\rho(T') < \rho(T)$, a contradiction. Thus, $d = 3$. Therefore, T is a hypertree obtainable from T_k^1 with edge $e = \{w_1, \ldots, w_k\}$ by attaching t_i pendant edges to w_i for $1 \leq i \leq k$, where $t_1 \geq t_2 \geq \cdots \geq t_k \geq 0$ and $t_2 \geq 1$.

Suppose that $t_2 \geq 2$.

Suppose that $t_3 \geq 1$. Let T''' be the hypertree obtained from T by moving all edges containing w_3 except e from w_3 to w_1. Obviously, T''' is non-hyperstar-like. By Lemma 2.4, $\rho(T''') < \rho(T)$, a contradiction. Thus, $t_3 = \cdots = t_k = 0$.

Suppose that $t_1 \geq 3$. Let $e_1, \ldots, e_{t_1}$ be t_1 pendant edges at w_1. Let T^* be the hypertree obtained from T by deleting edges $e_1, \ldots, e_{t_1-1}$ and adding a pendant edge $\cup_{i=1}^{t_1-1} e_i$ at w_1. Obviously, T^* is non-hyperstar-like. By Lemma 2.3, we have $\rho(T^*) < \rho(T)$, a contradiction. Thus, $t_1 = 2$. It follows that $t_1 = t_2 = 2$ and $t_3 = \cdots = t_k = 0$.

If $n = 6$, then $T \cong H_n'$.

Suppose that $n \geq 7$. Let e_1^1 and e_1^2 be two pendant edges at w_1. Let e_2^1 and e_2^2 be two pendant edges at w_2. For $i = 1, 2$, choose $u_1^i \in e_1^i \setminus \{w_1\}$ and $u_2^i \in e_2^i \setminus \{w_2\}$.

Suppose that $|e_j^i| \geq 3$ for some $j, i \in \{1, 2\}$, say $|e_1^1| \geq 3$. Let $z_1 \in e_1^1 \setminus \{w_1, u_1^1\}$. Let $\widehat{T}$ be the hypertree obtained from T by moving all vertices in $e_j^i \setminus \{w_j, u_j^i\}$ from e_j^i to e for each $j = 1, 2$ and $i = 1, 2$. Obviously, $\widehat{T} \cong H_n'$.

Let $x = x(\widehat{T})$. By Lemma 2.1, $x_{w_1} = x_{w_2}$, $x_{u_1^1} = x_{u_1^2} = x_{u_2^1} = x_{u_2^2}$, and $x_v = x_{z_1}$ for $v \in V(\widehat{T}) \setminus \{w_1, w_2, u_1^1, u_1^2, u_2^1, u_2^2\}$.

As we pass from T to $\widehat{T}$, for $i, j = 1, 2$, the distance between a vertex of $e_j^i \setminus \{w_j, u_j^i\}$ and u_j^i is increased by 1, the distance between a vertex of $e_j^i \setminus \{w_j, u_j^i\}$ and $e \setminus \{w_j\} \cup \{u_\ell^1, u_\ell^2\}$ with $\ell = \{1, 2\} \setminus \{j\}$ is decreased

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

by 1, and the distances between all other vertex pairs are decreased or remain unchanged. Then

$$\begin{aligned}\frac{1}{2}(\rho(T)-\rho(\widehat{T})) &\geq \frac{1}{2}x^\top(D(T)-D(\widehat{T}))x\\ &\geq \sum_{j=1}^{2}\sum_{i=1}^{2}\sigma_{\widehat{T}}(e_j^i\setminus\{w_j,u_j^i\})\left(\sum_{s=1}^{k}x_{w_s}-x_{w_j}+x_{u_\ell^1}+x_{u_\ell^2}-x_{u_j^i}\right)\\ &=\sum_{j=1}^{2}\sum_{i=1}^{2}(|e_j^i|-2)x_{z_1}\left(\sum_{s=1}^{k}x_{w_s}-x_{w_j}+x_{u_\ell^1}+x_{u_\ell^2}-x_{u_j^i}\right)\\ &>\sum_{j=1}^{2}\sum_{i=1}^{2}(|e_j^i|-2)x_{z_1}\left(\sum_{s=1}^{k}x_{w_s}-x_{w_j}\right)\\ &>0,\end{aligned}$$

and thus, $\rho(\widehat{T})<\rho(T)$, a contradiction. Thus, $T\cong H_n'$.

By Lemma 4.1, $\rho(H_n)<\rho(T)$, a contradiction. It follows that $t_2=1$. Since T is non-hyperstar-like, we have $t_3=1$.

Suppose that $k\geq 4$ and $t_4=1$. Let $\widetilde{T}$ be the hypertree obtained from T by moving all edges containing w_4 except e from w_4 to w_1. Obviously, $\widetilde{T}$ is non-hyperstar-like. By Lemma 2.4, $\rho(\widetilde{T})<\rho(T)$, a contradiction. Thus, $t_4=\cdots=t_k=0$.

Suppose that $t_1\geq 2$. Let $e_1',\ldots,e_{t_1}'$ be t_1 pendant edges at w_1. Let T_1 be the hypertree obtained from T by deleting edges $e_1',\ldots,e_{t_1}'$ and adding a pendant edge $\cup_{i=1}^{t_1}e_i'$ at w_1. Obviously, T_1 is non-hyperstar-like. By Lemma 2.3, we have $\rho(T_1)<\rho(T)$, a contradiction. Thus, $t_1=1$. It follows that $t_1=t_2=t_3=1$ and $t_4=\cdots=t_k=0$ for $k\geq 4$. For $i=1,2,3$, let e_i'' be the pendant edge at w_i in T, and choose $u_i\in e_i''\setminus\{w_i\}$. We may assume that $|e_1''|\geq|e_2''|\geq|e_3''|\geq 2$.

Suppose that $|e_1''|\geq 3$. Let $z_2\in e_1''\setminus\{w_1,u_1\}$. Let T_2 be the hypertree obtained from T by moving all vertices in $e_i''\setminus\{w_i,u_i\}$ from e_i'' to e for each $i=1,2,3$. Obviously, $T_2\cong H_n$. Let $x=x(T_2)$. By Lemma 2.1, $x_{w_1}=x_{w_2}=x_{w_3}$, $x_{u_1}=x_{u_2}=x_{u_3}$, and $x_v=x_{z_2}$ for $v\in V(T_2)\setminus(\cup_{i=1}^{3}\{w_i,u_i\})$.

As we pass from T to T_2, for $i=1,2,3$, the distance between a vertex of $e_i''\setminus\{w_i,u_i\}$ and u_i is increased by 1, the distance between a vertex of $e_i''\setminus\{w_i,u_i\}$ and $e\setminus\{w_i\}\cup\{u_s,u_t\}$ with $\{s,t\}=\{1,2,3\}\setminus\{i\}$ is decreased by 1, and the distances between all other vertex pairs are decreased or remain unchanged. Then

$$\begin{aligned}\frac{1}{2}(\rho(T)-\rho(T_2)) &\geq \frac{1}{2}x^\top(D(T)-D(T_2))x\\ &\geq \sum_{i=1}^{3}\sigma_{T_2}(e_i''\setminus\{w_i,u_i\})\left(\sum_{j=1}^{k}x_{w_j}-x_{w_i}+x_{u_s}+x_{u_t}-x_{u_i}\right)\\ &>\sum_{i=1}^{3}(|e_i''|-2)x_{z_2}\left(\sum_{j=1}^{k}x_{w_j}-x_{w_i}\right)\\ &>0,\end{aligned}$$

and thus, $\rho(T_2)<\rho(T)$, a contradiction. It follows that $|e_1|=|e_2|=|e_3|=2$. Thus, $T\cong H_n$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

5. Distance spectral radius of unicylic hypergraphs. In this section, we determine the unique unicylic hypergraphs with minimum and second minimum distance spectral radius, respectively.

A unicyclic hypergraph is a loose cycle if there is a cycle containing all its vertices. For $n \geq 3$, let U_n^1 be the loose cycle of length two on n vertices such that the sizes of the edges are 2 and n.

Let K_3 be a triangle on 3 vertices. If G is a unicylic hypergraph on 3 vertices, then $G \cong U_3^1$ or K_3, and obviously, $\rho(U_3^1) = \rho(K_3)$.

THEOREM 5.1. *Let G be a unicylic hypergraph on $n \geq 4$ vertices. Then $\rho(G) \geq n-1$ with equality if and only if $G \cong U_n^1$.*

Proof. Let g be the length of the unique cycle C of G. Let d be the diameter of G. If $g \geq 3$, since $n \geq 4$, then $d \geq 2$. If $g = 2$ and $G \ncong U_n^1$, then there is a vertex outside C or the sizes of both edges of C are at least 3, implying that $d \geq 2$. Therefore, we have either $G \cong U_n^1$ or $d \geq 2$. By Corollary 2.2 in [11, p. 38], U_n^1 is the unique unicylic hypergraph on $n \geq 4$ vertices with minimum distance spectral radius, which is $n-1$. □

LEMMA 5.2. *Let G be a hypergraph consisting of three connected subhypergraphs G_0, G_1, G_2 such that G_0 is a cycle of length two, where $E(G_0) = \{e_1, e_2\}$ with $e_1 \cap e_2 = \{u, v\}$, $V(G_1) \cap V(G_2) = \emptyset$, $V(G_1) \cap V(G_0) = \{u\}$, $V(G_2) \cap V(G_0) = \{v\}$, and $E(G) = E(G_0) \cup E(G_1) \cup E(G_2)$. Let $|e_i| = n_i$ for $i = 1, 2$. If $n_1 - 2 \geq n_2 \geq 2$, let $w_1 \in e_1 \setminus \{u, v\}$ and G' be the hypergraph obtained from G by moving vertex w_1 from e_1 to e_2. Then $\rho(G) < \rho(G')$.*

Proof. Let $x = x(G)$. Let $w_2 \in e_2 \setminus \{u, v\}$ if $n_2 \geq 3$. By Lemma 2.1, $x_z = x_{w_1}$ if $z \in e_1 \setminus \{u, v\}$, and $x_z = x_{w_2}$ if $z \in e_2 \setminus \{u, v\}$.

Let $V_1 = V(G) \setminus ((e_1 \cup e_2) \setminus \{u, v\})$. Note that for $z \in V_1$, $d_G(w_1, z) = d_G(w_2, z)$. From the distance eigenequations of G at w_1 and w_2, we have

$$\rho(G)x_{w_1} = (n_1 - 3)x_{w_1} + 2(n_2 - 2)x_{w_2} + \sum_{z \in V_1} d_G(w_1, z)x_z,$$

$$\rho(G)x_{w_2} = 2(n_1 - 2)x_{w_1} + (n_2 - 3)x_{w_2} + \sum_{z \in V_1} d_G(w_2, z)x_z.$$

Then

$$\rho(G)(x_{w_1} - x_{w_2}) = (n_2 - 1)x_{w_2} - (n_1 - 1)x_{w_1},$$

i.e.,

$$(\rho(G) + n_1 - 1)x_{w_1} = (\rho(G) + n_2 - 1)x_{w_2}. \tag{5.3}$$

By interlacing theorem, we have $\rho(G) \geq \rho(T_{n_2}^1) = n_2 - 1$. So, from (5.3), we have

$$\frac{x_{w_2}}{x_{w_1}} = 1 + \frac{n_1 - n_2}{\rho(G) + n_2 - 1} \geq 1 + \frac{2}{\rho(G) + n_2 - 1} \geq 1 + \frac{1}{\rho(G)} > \frac{\rho(G) + 2}{\rho(G) + 1},$$

and thus,

$$(\rho(G) + 1)x_{w_2} - (\rho(G) + 2)x_{w_1} > 0. \tag{5.4}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

As we pass from G to G', the distance between w_1 and a vertex of $e_1 \setminus \{u, v, w_1\}$ is increased by 1, the distance between w_1 and a vertex of $e_2 \setminus \{u, v\}$ is decreased by 1, and the distance between any other vertex pair remains unchanged. Thus,

$$\frac{1}{2}(\rho(G') - \rho(G)) \geq \frac{1}{2}x^\top(D(G') - D(G))x = x_{w_1}\left((n_1 - 3)\, x_{w_1} - (n_2 - 2)x_{w_2}\right).$$

This, together with (5.3) and (5.4), implies that

$$\frac{1}{2}(\rho(G') - \rho(G)) \geq x_{w_1}((\rho(G) + 1)x_{w_2} - (\rho(G) + 2)x_{w_1}) > 0,$$

and thus, $\rho(G) < \rho(G')$. □

For $n \geq 4$, let U_n^2 be the loose cycle of length two on n vertices such that the sizes of the edges are 3 and $n-1$, and let U_n^3 be the loose cycle of length three on n vertices such that the size of the edges are 2, 2 and $n-1$.

THEOREM 5.3. *Let G be a unicylic hypergraph on $n \geq 4$ vertices, where $G \ncong U_n^1$. Then $\rho(G) \geq \rho^*$ with equality if and only if $G \cong U_n^2$ or U_n^3, where ρ^* is the the largest root of the equation $\rho^3 + (3-n)\rho^2 + (12 - 5n)\rho + 4 - 2n = 0$.*

Proof. Let G be a unicylic hypergraph on n vertices with $G \ncong U_n^1$ that minimizes the distance spectral radius.

Let C be the unique cycle in G with $E(C) = \{e_1, \ldots, e_g\}$ such that $v_{i-1}, v_i \in e_i$ for $i = 1, \ldots, g$ with $v_g = v_0$, where $g \geq 2$. For $i = 0, \ldots, g-1$, let T_i be the component in $G - E(C)$ containing v_i. If $g = 2$, then we assume that $|e_1| \geq |e_2|$.

Suppose that there exists a vertex v_i with $0 \leq i \leq g-1$ of degree at least three, i.e., $|V(T_i)| \geq 2$. Let $e_0 = e_g$ if $i = 0$. Let G' be the unicyclic hypergraph obtained from G by deleting edge e_i and all edges in $E(T_i)$, and adding an edge $e_i \cup V(T_i)$ if $g \geq 3$, and let G' be the unicyclic hypergraph obtained from G by deleting edge e_0 and all edges in $E(T_i)$, and adding an edge $e_0 \cup V(T_i)$ if $g = 2$. Obviously, $G' \ncong U_n^1$. By Lemma 2.3, $\rho(G') < \rho(G)$, a contradiction. Thus, $\delta_T(v_i) = 2$ for $i = 0, \ldots, g-1$.

Suppose that there exists an edge with $1 \leq i \leq g$ of size at least three, whose deletion gives at least two nontrivial components, i.e., there exists a vertex w in $e_i \setminus \{v_{i-1}, v_i\}$ with $\delta_G(w) \geq 2$. Let T_w be the component in $G - e_i$ containing w. Suppose first that $g \geq 3$ or $g = 2$ and $|e_2| \geq 3$. Let G'' be the unicyclic hypergraph obtained from G by deleting edge e_i and all edges in $E(T_w)$, and adding an edge $e_i \cup V(T_w)$. Obviously, $G'' \ncong U_n^1$. By Lemma 2.3, $\rho(G'') < \rho(G)$, a contradiction. So assume that $g = 2$ and $|e_2| = 2$. Then $i = 1$. Let G'' be the unicyclic hypergraph obtained from G by moving the edges containing w except e_1 from w to v_0. Then $\delta_{G''}(v_0) > 2$, but as $D(G'')$ is permutationally similar to $D(G)$, we have $\rho(G'') = \rho(G)$, a contradiction. Therefore, $G - e_i$ has exactly one nontrivial component for $1 \leq i \leq g$. That is, $G = C$.

Suppose that $g \geq 4$. Let G^* be the unicyclic hypergraph obtained from G by deleting edges e_1 and e_2, and adding an edge $e_1 \cup e_2$. Obviously, $G^* \ncong U_n^1$. By Lemma 2.3, $\rho(G^*) < \rho(G)$, a contradiction. Thus, $g = 2$ or $g = 3$.

Case 1. $g = 2$.

Since $G \ncong U_n^1$, then $|e_2| \geq 3$. If $n = 4, 5$, then $G \cong U_n^2$. Suppose that $n \geq 6$ and $|e_2| \geq 4$. Let $w \in e_2 \setminus \{v_1, v_0\}$. Let H be the unicyclic hypergraph obtained from G by moving vertex w from e_2 to e_1. By Lemma 5.2, $\rho(H) < \rho(G)$, a contradiction. Thus, $|e_2| = 3$. Therefore, $G \cong U_n^2$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

Case 2. $g = 3$.

We may assume that $|e_3| \geq |e_2| \geq |e_1| \geq 2$. Suppose that $|e_2| \geq 3$. Let H be the unicyclic hypergraph obtained from G by moving all vertices in $e_i \setminus \{v_{i-1}, v_i\}$ from e_i to e_3 for $i = 1, 2$. Obviously, $H \cong U_n^3$. Let $e'_1 = \{v_0, v_1\}$, $e'_2 = \{v_1, v_2\}$ and $e'_3 = V(G) \setminus \{v_1\}$ be the edges of H.

Let $x = x(H)$. By Lemma 2.1, $x_{v_0} = x_{v_2}$, and x_w is a constant for any $w \in e'_3 \setminus \{v_2, v_0\}$. From the distance eigenequations of H at $w \in e'_3 \setminus \{v_0, v_2\}$, v_0 and v_1, we have

$$\begin{aligned}
\rho(H)x_w &= 2x_{v_0} + 2x_{v_1} + (n-4)x_w,\\
\rho(H)x_{v_0} &= x_{v_0} + x_{v_1} + (n-3)x_w,\\
\rho(H)x_{v_1} &= 2x_{v_0} + 2(n-3)x_w.
\end{aligned}$$

Then

$$\rho(H)(x_w + x_{v_0} - x_{v_1}) = -x_w + 3x_{v_1} + x_{v_0} > -x_w + 3x_{v_1},$$

which implies $(\rho(H) + 1)(x_w + x_{v_0} - x_{v_1}) > x_{v_0} + 2x_{v_1} > 0$. Thus, for any $w \in e_3 \setminus \{v_2, v_0\}$, we have $x_w + x_{v_0} - x_{v_1} > 0$.

As we pass from G to H, the distance between a vertex of $e_1 \setminus \{v_0, v_1\}$ and v_1 is increased by 1, the distance between a vertex of $e_1 \setminus \{v_0, v_1\}$ and $e_3 \setminus \{v_0\}$ is decreased by 1, the distance between a vertex of $e_2 \setminus \{v_1, v_2\}$ and v_1 is increased by 1, the distance between a vertex of $e_2 \setminus \{v_1, v_2\}$ and $e_3 \setminus \{v_2\}$ is decreased by 1, and the distances between all other vertex pairs are decreased or remain unchanged. Then, for any $w \in e_3 \setminus \{v_2, v_0\}$, we have

$$\begin{aligned}
\frac{1}{2}(\rho(G) - \rho(H)) &\geq \frac{1}{2}x^\top (D(G) - D(H))x\\
&\geq \sigma_H(e_1 \setminus \{v_0, v_1\})\,(\sigma_H(e_3 \setminus \{v_0\}) - x_{v_1})\\
&\quad + \sigma_H(e_2 \setminus \{v_1, v_2\})\,(\sigma_H(e_3 \setminus \{v_2\}) - x_{v_1})\\
&\geq \sigma_H(e_1 \setminus \{v_0, v_1\})\,(x_w + x_{v_2} - x_{v_1})\\
&\quad + \sigma_H(e_2 \setminus \{v_1, v_2\})\,(x_w + x_{v_0} - x_{v_1})\\
&= \sigma_H(e'_3 \setminus e_3)\,(x_w + x_{v_0} - x_{v_1})\\
&> 0,
\end{aligned}$$

and thus, $\rho(G) > \rho(H)$, a contradiction. It follows that $|e_2| = 2$. Therefore, $G \cong U_n^3$.

By combining Cases 1 and 2, we conclude that $G \cong U_n^2$ or U_n^3. With proper labelling of the vertices, U_n^2 and U_n^3 have the same distance matrix, and thus, $\rho(U_n^2) = \rho(U_n^3)$. From the distance eigenequations of U_n^3 used above in Case 2, it is easily seen that $\rho(U_n^3)$ is the largest root of the equation $\rho^3 + (3-n)\rho^2 + (12 - 5n)\rho + 4 - 2n = 0$. ◻

6. Distance spectral radius of hypertrees with given matching number. For $2 \leq \beta \leq \lfloor \frac{n}{2} \rfloor$, let $F_{n,\beta}$ be the hypertree obtained from a hypertree on $n - \beta$ vertices consisting of a single edge e by attaching a pendant edge of size two to each of β chosen vertices of e, respectively. Obviously, $\beta(F_{n,\beta}) = \beta$.

THEOREM 6.1. *Let T be a hypertree on n vertices with matching number β, where $2 \leq \beta \leq \lfloor \frac{n}{2} \rfloor$. Then $\rho(T) \geq \rho(F_{n,\beta})$ with equality if and only if $T \cong F_{n,\beta}$.*

Proof. Let T be a hypertree on n vertices with matching number β that minimizes the distance spectral radius.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

Let $M=\{e_1,\dots,e_\beta\}$ be a maximum matching in T. Suppose that there is an edge in M, say e_1, which has two vertices of degree at least two, say w_1 and w_2. Let T' be the hypertree obtained from T by moving all edges containing w_2 except e_1 from w_2 to w_1. Obviously, M is also a maximum matching of T'. By Theorem 2.4, $\rho(T)>\rho(T')$, a contradiction. Thus, $e_1,\dots,e_\beta$ are all pendant edges.

For $i=1,\dots,\beta$, let v_i be the unique vertex in e_i of degree at least two. Let $V_1=V(T)\setminus\cup_{i=1}^{\beta}e_i\setminus\{v_i\}$. Obviously, $|V_1|\geq\beta\geq 2$. Suppose that there are two vertices in V_1, say z_1 and z_2, which are not adjacent. Let T'' be the hypertree obtained from T by deleting all edges in T except the edges in M, and adding an edge V_1. Obviously, $\beta(T'')=\beta$. By Lemma 2.3, $\rho(T)>\rho(T'')$, a contradiction. Thus, any two vertices in V_1 are adjacent, which implies that V_1 is an edge.

We may assume that $|e_1|\geq\cdots\geq|e_\beta|\geq 2$. Suppose that $|e_1|\geq 3$. For $i=1,\dots,\beta$, choose a vertex v_i' of degree one in e_i. Let $w\in e_1\setminus\{v_1,v_1'\}$. Let T^* be the hypertree obtained from T by moving all the vertices in $e_i\setminus\{v_i,v_i'\}$ from e_i to e for each $i=1,\dots,\beta$. Obviously, $T^*\cong F_{n,\beta}$. Let $x=x(T^*)$. By Lemma 2.1, $x_{v_1}=\cdots=x_{v_\beta}$, $x_{v_1'}=\cdots=x_{v_\beta'}$ and $x_v=x_w$ if $v\in V(T)\setminus\cup_{i=1}^{\beta}\{v_i,v_i'\}$.

As we pass from T to T^*, for $i=1,\dots,\beta$, the distance between a vertex of $e_i\setminus\{v_i,v_i'\}$ and v_i' is increased by 1, the distance between a vertex of $e_i\setminus\{v_i,v_i'\}$ and $\{v_j,v_j'\}$ with $j\in\{1,\dots,\beta\}\setminus\{i\}$ is decreased by 1, and the distances between all other vertex pairs are decreased or remain unchanged. Thus,

$$\begin{aligned}\frac{1}{2}(\rho(T)-\rho(T^*)) &\geq \frac{1}{2}x^\top(D(T)-D(T^*))x\\ &\geq \sum_{i=1}^{\beta}\sigma_{T^*}(e_i\setminus\{v_i,v_i'\})\left(\sum_{\substack{1\leq j\leq\beta\\ j\neq i}}(x_{v_j'}+x_{v_j})-x_{v_i'}\right)\\ &> \sum_{i=1}^{\beta}\sigma_{T^*}(e_i\setminus\{v_i,v_i'\})\sum_{\substack{1\leq j\leq\beta\\ j\neq i}}x_{v_j}\\ &>0,\end{aligned}$$

and therefore, $\rho(T)>\rho(T^*)$, a contradiction. It follows that $|e_1|=\cdots=|e_\beta|=2$, i.e., $T\cong F_{n,\beta}$. □

7. Distance spectral radius of power hypertrees with given matching number. Nath and Paul [12] determined the unique tree with maximum distance spectral radius among trees on n vertices with matching number β, where $1\leq\beta\leq\lfloor\frac{n}{2}\rfloor$. In this section, we determine the unique hypertree with maximum distance spectral radius among k-th power hypertrees with m edges and matching number β, where $1\leq\beta\leq\lfloor\frac{m+1}{2}\rfloor$.

LEMMA 7.1. [16] *For $t\geq 3$, let G be a hypergraph consisting of t connected subhypergraphs $G_1,\dots,G_t$ such that $|V(G_i)|\geq 2$ for $1\leq i\leq t$ and $V(G_i)\cap V(G_j)=\{u\}$ for $1\leq i<j\leq t$. Suppose that $\emptyset\neq I\subseteq\{3,\dots,t\}$. Let $v\in V(G_2)\setminus\{u\}$ and G' be the hypergraph obtained from G by moving all the edges containing u in G_i for all $i\in I$ from u to v. If $\sigma_G(G_1)\geq\sigma_G(G_2)$, then $\rho(G)<\rho(G')$.*

For positive integers p, q and d, let $T_{2d}(p,q)$ be the k-uniform hypertree obtained from a k-uniform loose path

$$(u_d,e_d,u_{d-1},\dots,u_2,e_2,u_1,e_1,u_0(v_0),e_1',v_1,e_2',v_2,\dots,v_{d-1},e_d',v_d)$$

by attaching $p-1$ pendant edges to u_{d-1} and $q-1$ pendant edges to v_{d-1}. In particular, $T_2(p,q)$ is a

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

k-uniform hyperstar with $p+q$ edges. Let $T_{2d}(0,1)$ be a k-uniform loose path on $2d-1$ edges.

Lemma 7.2. *For $d \geq 2$ and $2 \leq p \leq q$, $\rho(T_{2d}(p,q)) > \rho(T_{2d}(p-1,q+1))$.*

Proof. Let $H = T_{2d}(p,q)$. Let H' be the hypergraph obtained from H by moving the pendant edge e_d from u_{d-1} to v_{d-1}. Obviously, $H' \cong T_{2d}(p-1,q+1)$.

Let $x = x(H')$. By Lemma 2.1, the entry of x corresponding to each vertex of degree one in the $p-1$ pendant edges containing u_{d-1} is the same, which we denote by μ, the entry of x corresponding to each vertex of degree one in the $q+1$ pendant edges containing v_{d-1} is the same, all equal to x_{u_d}, the entry of x corresponding to each vertex of $e_i\backslash\{u_{i-1},u_i\}$ for $i=1,\ldots,d-1$ is the same, which we denote by a_i, the entry of x corresponding to each vertex of $e'_i\backslash\{v_{i-1},v_i\}$ is the same, which we denote by b_i.

For $i=1,\ldots,d-1$, from the distance eigenequations of H' at u_i and v_i, we have

$$\begin{aligned}\rho(H')x_{u_i} =& \sum_{j=0}^{d-1}(i+j)x_{v_j} + \sum_{j=1}^{d-1}(k-2)(i+j)b_j + (k-1)(q+1)(d+i)x_{u_d}\\ &+\sum_{j=0}^{d-1}|i-j|x_{u_j} + \sum_{j=1}^{i}(k-2)(i-j+1)a_j\\ &+\sum_{j=i+1}^{d-1}(k-2)(j-i)a_j + (k-1)(p-1)(d-i)\mu,\end{aligned}$$

$$\begin{aligned}\rho(H')x_{v_i} =& \sum_{j=0}^{d-1}|i-j|x_{v_j} + \sum_{j=1}^{i}(k-2)(i-j+1)b_j + \sum_{j=i+1}^{d-1}(k-2)(j-i)b_j\\ &+(k-1)(q+1)(d-i)x_{u_d} + \sum_{j=0}^{d-1}(i+j)x_{u_j}\\ &+\sum_{j=1}^{d-1}(k-2)(i+j)a_j + (k-1)(p-1)(d+i)\mu,\end{aligned}$$

and for $k \geq 3$ and $i=1,\ldots,d-1$, from the distance eigenequations of H' at a vertex in $e_i\backslash\{u_{i-1},u_i\}$ and a vertex in $e'_i\backslash\{v_{i-1},v_i\}$, respectively, we have

$$\begin{aligned}\rho(H')a_i =& \sum_{j=0}^{d-1}(i+j)x_{v_j} + \sum_{j=1}^{d-1}(k-2)(i+j)b_j + (k-1)(q+1)(d+i)x_{u_d}\\ &+\sum_{j=0}^{i-1}(i-j)x_{u_j} + \sum_{j=i}^{d-1}(j-i+1)x_{u_j} + \sum_{j=1}^{d-1}(k-2)(|i-j|+1)a_j\\ &+(k-1)(p-1)(d-i+1)\mu - a_i,\end{aligned}$$

$$\begin{aligned}\rho(H')b_i =& \sum_{j=0}^{d-1}(i+j)x_{u_j} + \sum_{j=1}^{d-1}(k-2)(i+j)a_j + (k-1)(p-1)(d+i)\mu\\ &+\sum_{j=0}^{i-1}(i-j)x_{v_j} + \sum_{j=i}^{d-1}(j-i+1)x_{v_j} + \sum_{j=1}^{d-1}(k-2)(|i-j|+1)b_j\\ &+(k-1)(q+1)(d-i+1)x_{u_d} - b_i.\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

Then

$$\begin{aligned}\rho(H')\,(x_{u_i}-x_{v_i}) &= 2i\sum_{j=i}^{d-1}(x_{v_j}-x_{u_j})+2i\sum_{j=i+1}^{d-1}(k-2)(b_j-a_j)\\ &\quad+2i(k-1)(q+1)x_{u_d}-2i(k-1)(p-1)\mu\\ &\quad+\sum_{j=0}^{i-1}2j(x_{v_j}-x_{u_j})+\sum_{j=1}^{i}(k-2)(2j-1)(b_j-a_j),\end{aligned}$$

$$\begin{aligned}(\rho(H')+1)\,(a_i-b_i) &= (2i-1)\sum_{j=i}^{d-1}(x_{v_j}-x_{u_j})+(2i-1)\sum_{j=i+1}^{d-1}(k-2)(b_j-a_j)\\ &\quad+(2i-1)(k-1)(q+1)x_{u_d}-(2i-1)(k-1)(p-1)\mu\\ &\quad+\sum_{j=0}^{i-1}2j(x_{v_j}-x_{u_j})+\sum_{j=1}^{i}(k-2)(2j-1)(b_j-a_j).\end{aligned}$$

Let $A=\sum_{j=1}^{d-1}((k-2)a_j+x_{u_j})+(k-1)(p-1)\mu$ and $B=\sum_{j=1}^{d-1}((k-2)b_j+x_{v_j})+(k-1)(q+1)x_{u_d}$. Now we prove $A<B$. Suppose this is not true. Next we prove that $a_i\le b_i$ and $x_{u_i}\le x_{v_i}$ by induction on i for $1\le i\le d-1$. For $i=1$,

$$\begin{aligned}(\rho(H')+1)\,(a_1-b_1) &= \sum_{j=1}^{d-1}(x_{v_j}-x_{u_j})+\sum_{j=2}^{d-1}(k-2)(b_j-a_j)+(k-1)(q+1)x_{u_d}\\ &\quad-(k-1)(p-1)\mu+(k-2)(b_1-a_1)\\ &= B-A\\ &\le 0,\end{aligned}$$

we have $a_1\le b_1$, and then

$$\begin{aligned}\rho(H')\,(x_{u_1}-x_{v_1}) &= 2\sum_{j=1}^{d-1}(x_{v_j}-x_{u_j})+2\sum_{j=2}^{d-1}(k-2)(b_j-a_j)\\ &\quad+2(k-1)(q+1)x_{u_d}-2(k-1)(p-1)\mu+(k-2)(b_1-a_1)\\ &= 2(B-A)+(k-2)(a_1-b_1)\le 0,\end{aligned}$$

implying that $x_{u_1}\le x_{v_1}$. Now suppose that $i\ge 2$, $a_j\le b_j$ and $x_{u_j}\le x_{v_j}$ for $1\le j\le i-1$. Then

$$\begin{aligned}(\rho(H')+1)\,(a_i-b_i)-\rho(H')\left(x_{u_{i-1}}-x_{v_{i-1}}\right) &= \sum_{j=i}^{d-1}\left((k-2)\,b_j+x_{v_j}-(k-2)\,a_j-x_{u_j}\right)\\ &\quad+(k-1)(q+1)x_{u_d}-(k-1)(p-1)\mu\\ &= (B-A)-\sum_{j=1}^{i-1}\left((k-2)\,(b_j-a_j)+(x_{v_j}-x_{u_j})\right)\\ &\le 0.\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

Thus, $(\rho(H')+1)(a_i-b_i) \le \rho(H')\left(x_{u_{i-1}}-x_{v_{i-1}}\right) \le 0$, from which we have $a_i \le b_i$. Note that

$$\begin{aligned}\rho(H')(x_{u_i}-x_{v_i})-(\rho(H')+1)(a_i-b_i) &= \sum_{j=i}^{d-1}\left(x_{v_j}-x_{u_j}\right)+\sum_{j=i+1}^{d-1}(k-2)(b_j-a_j)\\ &\quad +(k-1)(q+1)x_{u_d}-(k-1)(p-1)\mu\\ &=(B-A)-\sum_{j=1}^{i-1}\left(x_{v_j}-x_{u_j}\right)-\sum_{j=1}^{i}(k-2)(b_j-a_j)\\ &\le 0.\end{aligned}$$

Thus, $\rho(H')(x_{u_i}-x_{v_i}) \le (\rho(H')+1)(a_i-b_i) \le 0$, implying that $x_{u_i} \le x_{v_i}$. It follows that $a_i \le b_i$ and $x_{u_i} \le x_{v_i}$ for $1 \le i \le d-1$. Thus, $\sum_{j=1}^{d-1}((k-2)a_j+x_{u_j}) \le \sum_{j=1}^{d-1}((k-2)b_j+x_{v_j})$.

By Lemma 2.2, $(\rho(H')+k)(\mu-x_{u_d})=\rho(H')(x_{u_{d-1}}-x_{v_{d-1}}) \le 0$, and thus, $\mu \le x_{u_d}$. This is impossible, because it would imply that $A<B$. Therefore, $A<B$.

As above, we have $a_i>b_i$ and $x_{u_i}>x_{v_i}$ for $1 \le i \le d-1$, and since

$$\rho(H')(x_{u_i}-x_{v_i}) > (\rho(H')+1)(a_i-b_i) > \rho(H')(x_{u_{i-1}}-x_{v_{i-1}}) > 0,$$

we have $x_{u_i}-x_{v_i}>a_i-b_i$ for $1 \le i \le d-1$ and $x_{u_i}-x_{v_i}>x_{u_{i-1}}-x_{v_{i-1}}$ for $2 \le i \le d-1$. By Lemma 2.2, $\mu-x_{u_d}=\frac{\rho(H')}{\rho(H')+k}(x_{u_{d-1}}-x_{v_{d-1}})<x_{u_{d-1}}-x_{v_{d-1}}$.

As we pass from H to H', the distance between a vertex of $e_d\backslash\{u_{d-1}\}$ and a vertex of degree one in the remaining $p-1$ pendant edges at u_{d-1} is increased by $2d-2$ $(p \ge 2)$, the distance between a vertex of $e_d\backslash\{u_{d-1}\}$ and a vertex of degree one in the q pendant edges at v_{d-1} is decreased by $2d-2$, for $0 \le i \le d-1$,the distance between a vertex of $e_d\backslash\{u_{d-1}\}$ and u_i is increased by $2i$, the distance between a vertex of $e_d\backslash\{u_{d-1}\}$ and v_i is decreased by $2i$, the distance between a vertex of $e_d\backslash\{u_{d-1}\}$ and $e_i\backslash\{u_{i-1},u_i\}$ is increased by $2i-1$, the distance between a vertex of $e_d\backslash\{u_{d-1}\}$ and $e_i'\backslash\{v_{i-1},v_i\}$ is decreased by $2i-1$, and the distances between all other vertex pairs remain unchanged. Thus,

$$\frac{1}{2}(\rho(H)-\rho(H')) \ge \frac{1}{2}x^\top(D(H)-D(H'))x=(k-1)x_{u_d}W, \tag{7.5}$$

where

$$W=(k-1)(2d-2)(qx_{u_d}-(p-1)\mu)+\sum_{i=1}^{d-1}2i(x_{v_i}-x_{u_i})+(k-2)\sum_{i=1}^{d-1}(2i-1)(b_i-a_i).$$

Let

$$F=\sum_{i=1}^{d-1}(2i+(k-2)(2i-1))+2(p-1)(k-1)(d-1).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

By the distance eigenequations of H' at u_{d-1} and v_{d-1}, we have

$$\begin{aligned}\rho(H')(x_{u_{d-1}}-x_{v_{d-1}}) &= (k-1)(2d-2)\left((q+1)\,x_{u_d}-(p-1)\,\mu\right)\\ &\quad+\sum_{i=1}^{d-1}2i(x_{v_i}-x_{u_i})+(k-2)\sum_{i=1}^{d-1}(2i-1)(b_i-a_i)\\ &= W+(k-1)(2d-2)x_{u_d}\\ &= 2W+\sum_{i=1}^{d-1}2i(x_{u_i}-x_{v_i})+(k-2)\sum_{i=1}^{d-1}(2i-1)(a_i-b_i)\\ &\quad+(k-1)(2d-2)\left((p-1)\mu-(q-1)x_{u_d}\right)\\ &< 2W+\sum_{i=1}^{d-1}2i(x_{u_i}-x_{v_i})+(k-2)\sum_{i=1}^{d-1}(2i-1)(x_{u_i}-x_{v_i})\\ &\quad+(k-1)(2d-2)(p-1)\left(\mu-x_{u_d}\right)\\ &< 2W+F(x_{u_{d-1}}-x_{v_{d-1}}).\end{aligned}$$

That is,

$$2W > (\rho(H')-F)(x_{u_{d-1}}-x_{v_{d-1}}). \tag{7.6}$$

For any $w\in V(H')$, there is a subhypergraph H^* of H' (obtained from H' by removing $q-p+2$ pendant edges at v_{d-1} and the resulting isolated vertices) such that $w\in V(H^*)$ and $H^*\cong T_{2d}(p-1,p-1)$. Then

$$\sum_{z\in V(H')} d_{H'}(w,z) > \sum_{z\in V(H^*)} d_{H^*}(w,z).$$

It is easy to see that

$$\begin{aligned}\sum_{z\in V(H^*)} d_{H^*}(w,z) &\geq \sum_{z\in V(H^*)} d_{H^*}(u_0,z)\\ &= \sum_{i=1}^{d-1}\Bigg(d_{H^*}(u_0,u_i)+d_{H^*}(u_0,v_i)\\ &\quad+\sum_{z\in e_i\setminus\{u_{i-1},u_i\}} d_{H^*}(u_0,z)+\sum_{z\in e'_i\setminus\{v_{i-1},v_i\}} d_{H^*}(u_0,z)\Bigg)\\ &\quad+(p-1)\left(\sum_{z\in e_d\setminus\{u_{d-1}\}} d_{H^*}(u_0,z)+\sum_{z\in e'_d\setminus\{v_{d-1}\}} d_{H^*}(u_0,z)\right)\\ &= \sum_{i=1}^{d-1}(2i+2(k-2)i)+2(p-1)(k-1)d\\ &> F.\end{aligned}$$

Thus, $\sum_{z\in V(H')} d_{H'}(w,z) > F$ for any $w\in V(H')$. Since $\rho(H')$ is bounded below by the minimum row sum of $D(H')$, we have $\rho(H') > F$. Recall that $x_{u_{d-1}} > x_{v_{d-1}}$. So, by (7.6), we have $W>0$. Now, from (7.5), we have $\rho(H) > \rho(H')$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

For positive integers p, q and d, let $T_{2d+1}(p,q)$ be the k-uniform hypertree obtained from a k-uniform loose path

$$(u_d, e_d, u_{d-1}, \ldots, u_2, e_2, u_1, e_1, u_0, e_0, v_0, e_1', v_1, e_2', v_2, \ldots, v_{d-1}, e_d', v_d)$$

by attaching $p-1$ pendant edges to u_{d-1} and $q-1$ pendant edges to v_{d-1}.

Theorem 7.3. *Let T be a k-th power hypertree with m edges and matching number β, where $1 \le \beta \le \lfloor \frac{m+1}{2} \rfloor$. Then $\rho(T) \le \rho(T_{2\beta}(\lfloor \frac{m-2\beta+2}{2} \rfloor, \lceil \frac{m-2\beta+2}{2} \rceil))$ with equality if and only if $T \cong T_{2\beta}(\lfloor \frac{m-2\beta+2}{2} \rfloor, \lceil \frac{m-2\beta+2}{2} \rceil)$.*

Proof. It is trivial when $\beta = 1$.

Suppose that $\beta \ge 2$. Let T be a k-th hypertree with m edges and matching number β that maximizes the distance spectral radius.

If $\beta = \frac{m+1}{2}$, then recalling that $T_{2\beta}(0,1)$ is the unique hypertree with maximum distance spectral radius among k-uniform hypertrees with m edges [8] and noting that $\beta(T_{2\beta}(0,1)) = \frac{m+1}{2}$, we have $T \cong T_{2\beta}(0,1)$, as desired.

Suppose that $2 \le \beta \le \lfloor \frac{m}{2} \rfloor$. Let M be a maximum matching in T.

If there is no vertex of degree at least three in T, then since T is a k-th power hypertree, we have $T \cong T_m(1,1)$ with $\beta = \frac{m}{2}$, as desired.

Suppose that there exists a vertex u in T of degree at least three. Let $\delta_T(u) = t \ge 3$, and $E_T(u) = \{e_1, \ldots, e_t\}$. Then T consists of t subhypertrees $T_1, \ldots, T_t$ such that $|E(T_i)| \ge 1$ and $e_i \in E(T_i)$ for $1 \le i \le t$, $\cup_{i=1}^t E(T_i) = E(T)$, and $T_1, \ldots, T_t$ have exactly one vertex u in common. Suppose that $|E(T_1)|, |E(T_2)| \ge 2$. We consider three cases.

Case 1. $e_3 \notin M$.

We may assume that $\sigma_T(T_1) \ge \sigma_T(T_2)$. Let w be a vertex of degree at least two contained in some pendant edge in T_2. Let T' be the hypertree obtained from T by moving edge e_3 from u to w. Then $\beta(T') = \beta$, and by Lemma 7.1, $\rho(T') > \rho(T)$, a contradiction.

Case 2. $e_3 \in M$ and $|E(T_3)| \ge 2$.

Since $e_3 \in M$, we have $e_1 \notin M$. We may assume that $\sigma_T(T_2) \ge \sigma_T(T_3)$. Let z be a vertex of degree at least two contained in some pendant edge in T_3. Let T'' be the hypertree obtained from T by moving edge e_1 from u to z. Then $\beta(T'') = \beta$, and by Lemma 7.1, $\rho(T'') > \rho(T)$, a contradiction.

Case 3. $e_3 \in M$ and $|E(T_3)| = 1$.

Let v be a vertex of degree one in e_3. Let T^* be the hypertree obtained from T by moving edge e_1 from u to v. Then $\beta(T^*) = \beta$, and by Lemma 2.4, $\rho(T^*) > \rho(T)$, a contradiction.

By combining Cases 1–3, we conclude that, among the t subhypertrees $T_1, \ldots, T_t$ of T containing u, only one has at least two edges, as T is not a k-uniform hyperstar. Since u is arbitrary and T is a k-th power hypertree, it follows that $T \cong T_\ell(p,q)$ for some positive integers p, q and ℓ with $q \ge p \ge 1$, $q \ge 2$, $3 \le \ell \le 2\beta$, and $p + q + \ell = m + 2$. Note that $\beta = \lceil \frac{\ell}{2} \rceil$. If ℓ is odd, then by Lemma 2.4, $\rho(T) = \rho(T_\ell(p,q)) < \rho(T_{\ell+1}(p, q-1))$, a contradiction. Thus, ℓ is even and $\ell = 2\beta$. If $q - p \ge 2$, then by Lemma 7.2, we have $\rho(T_{2\beta}(p,q)) < \rho(T_{2\beta}(p+1, q-1))$, a contradiction. It follows that $q - p = 0, 1$, i.e., $T \cong T_{2\beta}(\lfloor \frac{m-2\beta+2}{2} \rfloor, \lceil \frac{m-2\beta+2}{2} \rceil)$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 411-429, July 2020.

Acknowledgements. The authors thank the editors and the referees for their kind suggestions and comments on the paper.

REFERENCES

[1] M. Aouchiche and P. Hansen. Distance spectra of graphs: A survey. *Linear Algebra and its Applications*, 458:301–386, 2014.

[2] C. Berge. *Hypergraphs: Combinatorics of Finite Sets*. North-Holland, Amsterdam, 1989.

[3] A. Cayley. A theorem in the geometry of position. *Cambridge Journal of Mathematics*, 2:267–271, 1841.

[4] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra and its Applications*, 436:3268–3292, 2012.

[5] R.L. Graham and H.O. Pollak. On the addressing problem for loop switching. *The Bell System Technical Journal*, 50:2495–2519, 1971.

[6] S.L. Hakimi and S.S. Yau. Distance matrix of a graph and its realizability. *Quarterly of Applied Mathematics*, 22:305–317, 1965.

[7] S. Hu, L. Qi, and J. Shao. Cored hypergraphs, power hypergraphs and their Laplacian H-eigenvalues. *Linear Algebra and its Applications*, 439:2980–2998, 2013.

[8] H. Lin and B. Zhou. Distance spectral radius of uniform hypergraphs. *Linear Algebra and its Applications*, 506:564–578, 2016.

[9] H. Lin and B. Zhou. On distance spectral radius of uniform hypergraphs with cycles. *Discrete Applied Mathematics*, 239:125–143, 2018.

[10] H. Lin, B. Zhou, and Y. Li. On distance spectral radius of uniform hypergraphs. *Linear and Multilinear Algebra*, 66:497–513, 2018.

[11] H. Minc. *Nonnegative Matrices*. John Wiley & Sons, New York, 1988.

[12] M. Nath and S. Paul. On the distance spectral radius of trees. *Linear and Multilinear Algebra*, 61:847–855, 2013.

[13] V. Nikiforov. *Analytic methods for uniform hypergraphs*, *Linear Algebra and its Applications*, 457:455–535, 2014.

[14] K.J. Pearson and T. Zhang. On spectral hypergraph theory of the adjacency tensor. *Graphs and Combinatorics*, 30:1233–1248, 2014.

[15] S. Sivasubramanian. q-analogs of distance matrices of 3-hypertrees. *Linear Algebra and its Applications*, 431:1234–1248, 2009.

[16] Y. Wang and B. Zhou. On distance spectral radius of hypergraphs. *Linear and Multilinear Algebra*, 66:2232–2246, 2018.

[17] S. Watanabe, K. Ishi, and M. Sawa. A q-analogue of the addressing problem of graphs by Graham and Pollak. *SIAM Journal on Discrete Mathematics*, 26:527–536, 2012.