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UNICYCLIC GRAPHS WITH THE STRONG RECIPROCAL EIGENVALUE PROPERTY*

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Abstract. A graph G is bipartite if and only if the negative of each eigenvalue of G is also an eigenvalue of G. It is said that a graph has property (R), if G is nonsingular and the reciprocal of each of its eigenvalues is also an eigenvalue. Further, if the multiplicity of an eigenvalue equals that of its reciprocal, the graph is said to have property (SR). The trees with property (SR) have been recently characterized by Barik, Pati and Sarma. Barik, Neumann and Pati have shown that for trees the two properties are, in fact, equivalent. In this paper, the structure of a unicyclic graph with property (SR) is studied. It has been shown that such a graph is bipartite and is a corona (unless it has girth four). In the case it is not a corona, it is shown that the graph can have one of the three specified structures. Families of unicyclic graphs with property (SR) having each of these specific structures are provided.

Key words. Unicyclic graphs, Adjacency matrix, Corona, Perfect matching, Property (SR).

AMS subject classifications. 15A18, 05C50.

1. Introduction.

Let G be a simple graph on vertices $1, 2, \ldots, n$. The *adjacency matrix* of G is defined as the $n \times n$ matrix $A(G)$, with (i, j)th entry 1 if $\{i, j\}$ is an edge and 0 otherwise. Since $A(G)$ is a real symmetric matrix, all its eigenvalues are real. Throughout the *spectrum of* G is defined as

$$\sigma(G) = (\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)),$$

where $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ are the eigenvalues of $A(G)$. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G and is denoted by $\rho(G)$. If G is connected then $A(G)$ is irreducible, thus by the Perron-Frobenius theory, $\rho(G)$ is simple and is afforded by a positive eigenvector, called the *Perron vector*. A graph G is said to be *singular* (resp. *nonsingular*) if $A(G)$ is singular (resp. nonsingular).

Definition 1.1. [7] Let G_1 and G_2 be two graphs on disjoint sets of n and m vertices, respectively. The *corona* $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2.

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Note that the corona $G_1 \circ G_2$ has $n(m+1)$ vertices and $|E(G_1)|+n(|E(G_2)|+m)$ edges. Let us denote the cycle on n vertices by C_n and the complete graph on n vertices by K_n. The coronas $C_3 \circ K_2$ and $K_2 \circ C_3$ are shown in Figure 1.1.

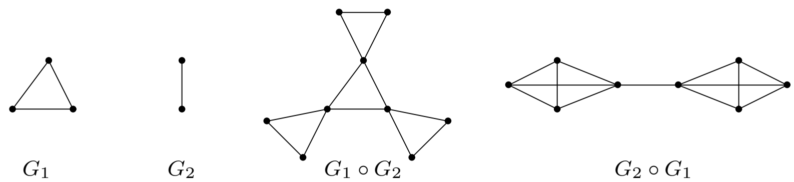

FIG. 1.1. *Corona of two graphs.*

Connected graphs in which the number of edges equals the number of vertices are called *unicyclic graphs* [7]. The unique cycle in a unicyclic graph G is denoted by Γ. If u is a vertex of the unicyclic graph G then a component T of $G-u$ not containing any vertex of Γ is called a *tree-branch* at u. We say that the tree-branch is *odd (even)* if the order of the tree-branch is odd (even). Recall that the *girth* of a unicyclic graph G is the length of Γ.

It is well known (see [4] Theorem 3.11 for example) that a graph G is bipartite if and only if the negative of each eigenvalue of G is also an eigenvalue of G. In contrast to the plus-minus pairs of eigenvalues of bipartite graphs, Barik, Pati and Sarma, [1], have introduced the notion of *graphs with property* (R). Such graphs G have the property that $\frac{1}{\lambda}$ is an eigenvalue of G whenever λ is an eigenvalue of G. When each eigenvalue λ of G and its reciprocal have the same multiplicity, then G is said to have *property* (SR). It has been proved in [1] that when $G = G_1 \circ K_1$, where G_1 is bipartite, then G has property (SR) (see Theorem 1.2). However, there are graphs with property (SR) which are not corona graphs. For example, one can easily verify that the graphs in Figure 1.2 have property (SR). Note that, in view of Lemma 2.3, H_1 cannot be a corona and that H_2 is not a corona has been argued in [1]. Note also that the graph H_1 is unicyclic and H_2 is not even bipartite.

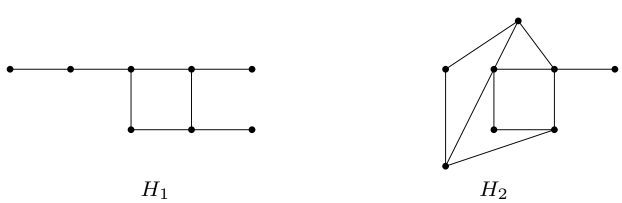

FIG. 1.2. *Graphs with property (SR) which are not coronas.*

The following result gives a class of graphs with property (SR).

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THEOREM 1.2. [1] *Let $G = G_1 \circ K_1$, where G_1 is any graph. Then λ is an eigenvalue of G if and only if $-1/\lambda$ is an eigenvalue of G. Further, if G_1 is bipartite then G has property (SR).*

In [1], it is proved that a tree T has property (SR) if and only if $T = T_1 \circ K_1$, for some tree T_1. This was strengthened in [2] where it is proved that a tree has property (R) if an only if it has property (SR).

In view of the previous example it is clear that a unicyclic graph with property (SR) may not be a corona and this motivates us to study unicyclic graphs with property (SR).

For a graph G, by $P(G;x)$ we denote the characteristic polynomial of $A(G)$. If S is a set of vertices and edges in G, by $G - S$ we mean the graph obtained by deleting all the elements of S from G. It is understood that when a vertex is deleted, all edges incident with it are deleted as well, but when an edge is deleted, the vertices incident with it are not. For a vertex v in G, $d(v)$ denotes the degree of v in G.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs on disjoint sets of m and n vertices, respectively, their union is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Let G be a graph. A *linear subgraph* L of G is a disjoint union of some edges and some cycles in G. A *k-matching* M in G is a disjoint union of k edges. If $e_1, e_2, \dots, e_k$ are the edges of a k-matching M, then we write $M = \{e_1, e_2, \dots, e_k\}$. If $2k$ is the order of G, then a k-matching of G is called a *perfect matching* of G.

Let G be a graph on n vertices. Let

$$P(G;x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \tag{1.1}$$

be the characteristic polynomial of $A(G)$. Then $a_0(G) = 1,\ a_1(G) = 0$ and $-a_2(G)$ is the number of edges in G. In general, we have (see [4] Theorem 1.3)

$$a_i = \sum_{L \in \mathcal{L}_i} (-1)^{c_1(L)} (-2)^{c_2(L)},\ i = 1, 2, \dots, n, \tag{1.2}$$

where $\mathcal{L}_i$ is the set of all linear subgraphs L of G of size i and $c_1(L)$ denotes the number of components of size 2 in L and $c_2(L)$ denotes the number of cycles in L. We note that if G has two pendant vertices with a common neighbor, then G is singular, because in that case G can not have a linear subgraph of size n. If G is bipartite, then one gets $a_i = 0$, whenever i is odd, and

$$P(G;x) = \sum_{i=0}^{[n/2]} (-1)^i \, b_{2i} \, x^{n-2i}, \tag{1.3}$$

where b_{2i} are nonnegative.

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The following results are often used to calculate the characteristic polynomials of graphs.

LEMMA 1.3. [4] *Let $e = \{u, v\}$ be an edge of G, and $\mathcal{C}(e)$ be the set of all cycles containing e. Then*

$$P(G; x) = P(G - e; x) - P(G - u - v; x) - 2 \sum_{Z \in \mathcal{C}(e)} P(G - Z; x).$$

LEMMA 1.4. [4] *Let v be a vertex in the graph G and $\mathcal{C}(v)$ be the set of all cycles containing v. Then*

$$P(G; x) = xP(G - v; x) - \sum_{u} P(G - u - v; x) - 2 \sum_{Z \in \mathcal{C}(v)} P(G - Z; x),$$

where the first summation extends over all u adjacent to v.

LEMMA 1.5. [4] *Let v be a vertex of degree 1 in the graph G and u be the vertex adjacent to v. Then*

$$P(G; x) = xP(G - v; x) - P(G - u - v; x).$$

2. Unicyclic graphs with property (SR). In this section we study the structure of unicyclic graphs with girth g and property (SR). We show that any such graph is bipartite. Further, if $g \neq 4$, then the graph is a corona graph.

LEMMA 2.1. *Let G be a graph on n vertices with property (SR) and $P(G; x)$ be as given in (1.1). Then $|a_i(G)| = |a_{n-i}(G)|$, for $i = 0, 1, \ldots, n$.*

Proof. Since G has property (SR), G is nonsingular. Moreover, $P(G; x)$ and $x^n P(G; \frac{1}{x})$ have the same roots. Since $P(G; x)$ is monic and the leading coefficient of $x^n P(G; \frac{1}{x})$ is $a_n = \pm 1$, it follows that $P(G; x) = \pm\, x^n P(G; \frac{1}{x})$ and the conclusion follows. ▯

LEMMA 2.2. *Let G be a unicyclic graph of order n with property (SR). Then G has a unique perfect matching. In particular $n = 2m$, for some integer m and there is an odd tree-branch of G at a vertex of the cycle.*

Proof. As G has property (SR), from (1.2), we have

$$|a_n| = 1, \text{ and } a_n(G) = \pm\Big(m_0(G) \pm 2 \times m_0(G - \Gamma)\Big), \tag{2.1}$$

where $m_0(H)$ is the number of perfect matchings of H.

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It follows from (2.1) that $m_0(G) \neq 0$, that is, G has a perfect matching. Consequently, n is even. Let $n = 2m$.

Suppose, if possible, that $m_0(G-\Gamma) > 0$, i.e. $G-\Gamma$ has a perfect matching. Thus Γ is an even cycle. As $m_0(\Gamma) = 2$, and since a matching of $G - \Gamma$ and a matching of Γ give rise to a matching of G, we have

$$2m_0(G - \Gamma) = m_0(G - \Gamma)m_0(\Gamma) \leq m_0(G).$$

By (2.1) the above inequality is strict. Thus there is a perfect matching M of G which does not contain a perfect matching of Γ. That is, in M a vertex u of Γ is matched to a vertex v of $G - \Gamma$. So there is an odd tree-branch at v. But then $G - \Gamma$ cannot have a perfect matching. This is a contradiction.

Hence $m_0(G-\Gamma) = 0$ and so $m_0(G) = 1$. The final conclusion now follows easily. ◻

Lemma 2.3. *Let G be a unicyclic graph with property (SR). If $G = G_1 \circ G_2$ then $G_2 = K_1$.*

Proof. Let G_1 be of order n_1 and G_1 be of order n_2. Suppose that $G_2 \neq K_1$. Then $n_2 \geq 2$. Notice that if G_2 has more than one isolated vertex, then G can not have a perfect matching. Therefore, G_2 must have an edge. As G is unicyclic, it follows that $n_1 = 1$ and $G_2 = K_2$ or $G_2 = K_2 + K_1$. In both the cases G does not have the property (SR). ◻

Definition 2.4. [6] Let K be any graph with a perfect matching M. An *alternating path* P (relative to M) in K is a path of odd length such that alternate edges (including the terminating ones) of P are in M.

Lemma 2.5. *Suppose G is a unicyclic graph of order $n = 2m$ with a perfect matching. Then the number of $(m-1)$-matchings of G is at least n.*

Proof. Let $M = \{e_1, e_2, \ldots, e_m\}$ be a perfect matching of G. Let $f_1, f_2, \ldots, f_m$ be the other m edges of G, which are not in M. Now,

$$\{e_1, \ldots, e_m\} - \{e_1\}, \{e_1, \ldots, e_m\} - \{e_2\}, \ldots, \{e_1, \ldots, e_m\} - \{e_m\}$$

are $(m-1)$-matchings of G. Moreover, for each f_j we have a unique alternate path $e_i f_j e_k$ of length three in G, and thereby an $(m-1)$-matching $(M - \{e_i, e_k\}) \cup \{f_j\}$ of G. ◻

Let $\mathcal{L}_G$ denote the collection of linear subgraphs of the unicyclic graph G of size $n-2$ with Γ as a component. By m_1 we denote the number of $(m-1)$-matchings of G. We note that the linear subgraphs in $\mathcal{L}_G$ have the same number of components (t say). Thus, from Equation (1.2), we have

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$$a_{n-2}(G) = (-1)^{m-1} m_1 + (-1)^t 2|\mathcal{L}_G|. \tag{2.2}$$

For the rest of the paper, we write "G is a simple corona" to mean that $G = G_1 \circ K_1$ for some graph G_1.

Lemma 2.6. *Let G be a unicyclic graph of order n with property (SR). Then the following are equivalent.*

(i) G is a simple corona,
(ii) there is no alternating path in G of length 5,
(iii) $m_1 = n$,
(iv) $\mathcal{L}_G$ is empty.

Proof. By Lemma 2.2, $n = 2m$, for some m.

(i)⇒(ii). If G is a simple corona then G has m pendant vertices and has the unique perfect matching containing all the leaves. So there is no alternating path of length 5 in G.

(ii)⇒(iii). Since G has a unique matching, arguing along the line of the previous lemma, we see that a $(m-1)$-matching is either a subset of the perfect matching or it corresponds to an alternating path. As there is no alternating path of length more than 3, it follows from the previous lemma that $m_1 = n$.

(iii)⇒(iv). Note that by Lemma 2.1 $|a_{n-2}| = |a_2| = n$, the number of edges in G. As $m_1 = n$, it follows now from (2.2) that $\mathcal{L}_G$ is empty.

(iv)⇒(i). If $\mathcal{L}_G$ is empty, using $|a_{n-2}| = |a_2| = n$, we get that $m_1 = n$ and hence by previous lemma there is no alternating path of length more than 3. If G is not a simple corona, then there is an edge $(u, v) \in M$, the perfect matching such that $d(u), d(v) \geq 2$. Thus there is a path $[u_0, u, v, v_0]$ such that $(u_0, u), (v, v_0) \notin M$. Note that u_0 is matched to some vertex by M, and so is v_0. Then we get an alternating path of length 5, unless $(u_0, v_0) \in M$. But then G has a cycle Γ of girth 4 and G has more than one perfect matchings. This is not possible, by Lemma 2.2. Thus G is a simple corona. □

We know from Theorem 1.2 that if G is a bipartite graph which is also a simple corona, then G has property (SR). Let us ask the converse. Suppose that we have a unicyclic graph with property (SR). Is it necessarily bipartite? Is it necessarily a corona? The following lemma answers the first question affirmatively.

Theorem 2.7. *Let G be a unicyclic graph with property (SR). Then the girth g*

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of G is even. Further, if $g \not\equiv 0 \pmod 4$ then G is a simple corona.

Proof. Suppose that g is odd. Then $|\mathcal{L}_G| = 0$, and therefore by Lemma 2.6, G is a simple corona. In view of Theorem 1.2, $\lambda, -\frac{1}{\lambda}$ have the same multiplicity in $\sigma(G)$. As G has property (SR), $-\frac{1}{\lambda}, -\lambda$ have the same multiplicity in $\sigma(G)$. Thus $\lambda, -\lambda$ have the same multiplicity in $\sigma(G)$. Thus G is bipartite, contradicting the fact that G has an odd cycle. So g is even.

Let $g = 2l$. Then by (2.2), we have

$$a_{n-2}(G) = (-1)^{m-1}m_1 + (-1)^{m-l}2|\mathcal{L}_G| = (-1)^{m-1}(m_1 + (-1)^{l-1}2|\mathcal{L}_G|). \tag{2.3}$$

Since l is odd in the present case, $n = |a_{n-2}(G)| = m_1 + 2|\mathcal{L}_G|$. As $m_1 \geq n$, we must have $|\mathcal{L}_G| = 0$, and the result follows from Lemma 2.6. □

LEMMA 2.8. *Let G be a non-corona unicyclic graph with property (SR). Then G has exactly two odd tree-branches at (say) $u, v \in \Gamma$. Every other vertex on Γ is matched to a point on Γ and the distance between u and v on Γ is odd.*

Proof. From Theorem 2.7, $g \equiv 0 \pmod 4$. Moreover, by Lemma 2.6, we have $|\mathcal{L}_G| \neq 0$. By Lemma 2.2, there is at least one odd tree-branch at a vertex, say, $u \in \Gamma$. As the order of the graph and the cycle are both even, there must be another odd tree-branch, say, at $v \in \Gamma$.

Let D be a linear subgraph of G in $\mathcal{L}_G$. Clearly D misses at least one vertex from each odd tree-branch at a vertex of Γ. Since D covers $n-2$ vertices of G, it follows that the number of such odd tree-branches is at most 2. Thus G has exactly two such odd tree-branches. Thus every other vertex on Γ is matched to a point on Γ and hence the distance between u and v on Γ is odd. □

Let G be graph with a perfect matching. Then, by $\mathcal{P}_G$ we denote the set of all alternating paths of length more than 3 in G.

LEMMA 2.9. *Let G be a unicyclic graph with property (SR). Then $|\mathcal{P}_G| = 2|\mathcal{L}_G|$.*

Proof. If $g \not\equiv 0 \pmod 4$ then G is a simple corona, by Theorem 2.7. Hence $\mathcal{P}_G = \emptyset = \mathcal{L}_G$. Suppose now that $g \equiv 0 \pmod 4$. It follows from (2.3) that

$$n = |a_{n-2}(G)| = m_1 - 2|\mathcal{L}_G|. \tag{2.4}$$

Let M be the unique matching in G and $m = |M|$. Note that any $(m-1)$-matching is either a subset of M or is obtained uniquely by an alternating path. Following the argument in the proof of Lemma 2.5, we see that the number of $(m-1)$-matchings in G (recall that it is m_1) is exactly $n + |\mathcal{P}_G|$. The result follows by using (2.4). □

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Suppose that G is a non-corona unicyclic graph with property (SR). Then by Lemma 2.8, G has exactly two odd tree-branches at vertices of Γ. For convenience, we shall always use that T_1, T_2 are the odd tree-branches of G at the vertices $w_1, w_2 \in \Gamma$, respectively, and the edges $\{w_1, v_1\}, \{w_2, v_2\}$ are edges in the unique perfect matching, where $v_i \in T_i$, respectively. We call a vertex u of T_i, $i = 1, 2$, a *distinguished vertex* if $T_i - u$ has a perfect matching. Let r_i $(i = 1, 2)$ be the number of distinguished vertices in T_i. The following relation between r_i and $\mathcal{L}_G$ is crucial for further developments.

LEMMA 2.10. *Let G be a non-corona unicyclic graph with property (SR). Then* $|\mathcal{L}_G| = r_1 r_2$.

Proof. Let D be any linear subgraph of G of size $n-2$ containing Γ. Then D will miss exactly one vertex from the trees T_1 and T_2. If these points are u_1 and u_2 respectively, then D will induce a perfect matching of $T_i - u_i$, for $i = 1, 2$. Thus u_i are distinguished points. Since the perfect matching of a forest (if it exists) is unique, each such pair (u_1, u_2) will give rise to a unique linear subgraph of G of size $n-2$ containing Γ. Hence the result follows. □

LEMMA 2.11. *Let T be a tree such that $T - v$ has a perfect matching M_v and u be another vertex in T. Suppose that $[v = v_1, \cdots, v_r = u]$ is the unique path from v to u in T. Then $T - u$ has a perfect matching M_u if and only if $r = 2k + 1$, for some k and the edges $\{v_{2i}, v_{2i+1}\} \in M_v$.*

Proof. If $[v = v_1, \cdots, v_{2k+1} = u]$ be a path such that the edges $\{v_{2i}, v_{2i+1}\} \in M_v$, then clearly,

$$M_u = M_v \cup \Big\{\{v_{2i-1}, v_{2i}\} : i = 1, \cdots, k\Big\} \setminus \Big\{\{v_{2i}, v_{2i+1}\} : i = 1, \cdots, k\Big\}$$

is a perfect matching of $T - u$.

Conversely, take a u such that $T - u$ has a perfect matching M_u. Let $[v = v_1, \cdots, v_r = u]$ is the unique path from v to u in T. Suppose if possible, that $\{v, x\} \in M_u, x \neq v_2$. In that case the component of $T - v$ which contains x is odd. Thus $T - v$ could not be a perfect matching. Thus $\{v_1, v_2\} \in M_u$ and $\{v_2, v_3\} \notin M_u$. Thus $T - u - v_1 - v_2$ has no odd components.

Obviously $\{v_1, v_2\} \notin M_v$. If $\{v_2, y\} \in M_v, y \neq v_3$ then $T - v_1 - v_2$ has an odd component containing y. But then $T - u - v_1 - v_2$ has the same odd component, a contradiction. Thus $\{v_2, v_3\} \in M_v$. Hence $T - v_3$ has a perfect matching (apply the first paragraph). Hence as in the last paragraph, $\{v_3, v_4\} \in M_u$ and $\{v_3, v_4\} \notin M_v$.

Continuing this way, we see that $\{v_{2i}, v_{2i+1}\} \in M_v$ and $\{v_{2i-1}, v_{2i}\} \in M_u$. Since $\{v_{r-1}, v_r\} \notin M_u$, we see that r is odd. □

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Theorem 2.12. *Let G be a unicyclic graph with property (SR) and girth $g \neq 4$. Then G is a simple corona.*

Proof. Suppose that G is not a simple corona. In view of Lemma 2.9 and Lemma 2.10 we have

$$|\mathcal{P}_G| = 2r_1r_2. \tag{2.5}$$

Moreover, G has exactly two odd tree-branches. Let T_i, w_i, v_i, $i = 1, 2$ be as in the discussion following Lemma 2.9. We prove that (2.5) can not hold and the result will follow.

By Lemma 2.8, both the v_1-v_2 paths are alternating and at least one of them has length more than 5. Let x_1, x_2 be the vertices on the longer paths, adjacent to w_1, w_2, respectively.

It follows from Lemma 2.11 that from any distinguished vertex (other than v_1) u_1 of T_1 there are two alternating paths to each distinguished vertex of T_2 and these paths have lengths at least 5. Similarly, from any distinguished vertex (other than v_2) u_2 of T_2 there are two alternating paths to v_1 and these paths have lengths at least 5. Thus

$$|\mathcal{P}_G| \geq 2(r_1 - 1)r_2 + 2(r_2 - 1) + 3 = 2r_1r_2 + 1,$$

where the term 3 counts the alternating path between v_1, v_2, the alternating path between v_1, x_2, and the alternating path between v_2, x_1. □

3. Non-corona unicyclic graphs with property (SR). A necessary condition for a non-corona unicyclic graph to have property (SR) is that the girth is four. One can easily see that it is not sufficient. In this section, we study the structure of a non-corona unicyclic graph with property (SR) and show that it has one of three specific structures. We supply examples to show the existence of families of graphs with property (SR) in each of these cases.

Lemma 3.1. *Let G be a non-corona unicyclic graph with property (SR). Let T_i, $i = 1, 2$ be the two odd-tree branches of G and r_i be the number of distinguished vertices in T_i, $i = 1, 2$, respectively. Then $2 \leq r_1 + r_2 \leq 3$.*

Proof. It is obvious that $2 \leq r_1 + r_2$. Note that from any distinguished vertex (other than v_1) u_1 of T_1 there are two alternating paths to each distinguished vertex of T_2, one alternating path to the vertex 3 and these paths have lengths at least 5. (See Figure 3.1).

From any distinguished vertex (other than v_2) u_2 of T_2 there are two alternating paths to v_1, one alternating path to the vertex 4 and these paths also have lengths at least 5.

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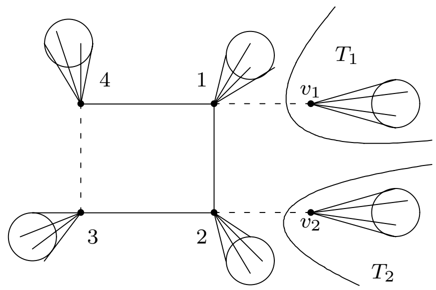

FIG. 3.1. *The graph G.*

There is one alternating path of length 5 from v_1 to v_2. Thus

$$|\mathcal{P}_G| \geq 2(r_1-1)r_2 + (r_1-1) + 2(r_2-1) + (r_2-1) + 1 = 2r_1r_2 + r_1 + r_2 - 3.$$

In view of Lemma 2.9 and Lemma 2.10, we have

$$|\mathcal{P}_G| = 2|\mathcal{L}_G| = 2r_1r_2.$$

So, $r_1 + r_2 \leq 3$. $\square$

In view of the previous lemma, we have two cases: $r_1 = r_2 = 1$ or $r_1 = 1, r_2 = 2$. Accordingly the necessary conditions on the structure of G are described by the following results.

THEOREM 3.2. *Let G be a non-corona unicyclic graph with property (SR) and T_i, r_i, $i = 1, 2$, be as discussed earlier. Suppose that $r_1 = r_2 = 1$. Then G has one of the two structures as shown in Figure 3.2, where*

(a) $F_{2a}, F_{2b}, F_{1b}, F_w$ are forests of simple corona trees; a non-pendant vertex of each tree in F_{2a}, F_{2b} is adjacent to 2; a non-pendant vertex of each tree in F_{1b} is adjacent to 1; a non-pendant vertex of each tree in F_w is adjacent to w in G.

(b) F_{1a} is a forest in which all but one tree, say T_3, are simple coronas and a non-pendant vertex of each simple corona tree is adjacent to 1 in G, and

(c) the graph induced by $1, v_1$ and vertices of T_3 has exactly one alternating path of length 5 (thus it has no alternating paths of length more than 5).

Proof. As $r_1 = r_2 = 1$, the number of alternating paths of length more than 3 has to be exactly 2. The path $[v_1, 1, 4, 3, 2, v_2]$ is an alternating path of length 5.

Notice that any even tree-branch at 3 (or 4, which is similar) will give us at least one more such path. Suppose that T is such a tree-branch at 4 and let w be the vertex adjacent to 3. Recall that w is matched to a vertex v outside the cycle, thus $v \in T$.

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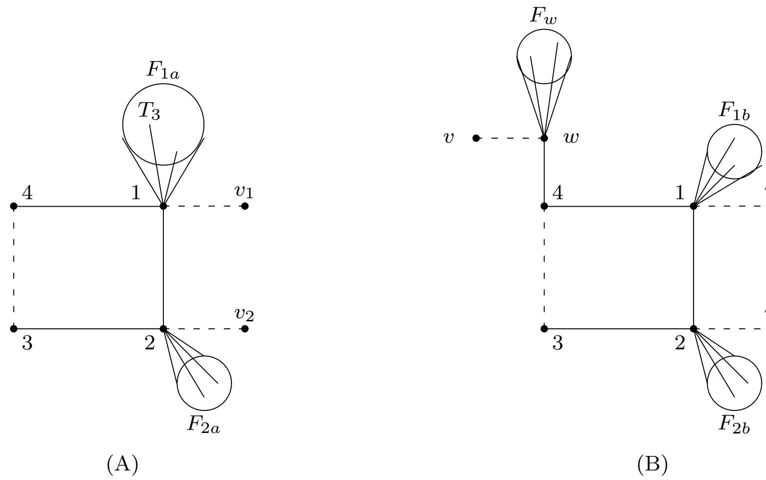

FIG. 3.2. *Case of* $r_1 = r_2 = 1$.

So the path $[v_2, 2, 3, 4, w, v]$ is an alternating path of length 5. If $d(v) \geq 2$, then we will have an alternating path of length at least 7. Therefore $d(v) = 1$. It follows that the other tree-branches at w are even and have perfect matchings and cannot have an alternating path of length at least 5. Thus they are simple corona trees. It follows that only a non-pendant vertex of each simple corona tree can be adjacent to w in G. Further in this case, a similar argument shows that the forests F_{1b}, F_{2b} consist of simple corona trees only and only a non-pendant vertex of each simple corona tree in F_{1b}, F_{2b} is adjacent to 2, 3, respectively. Since we already have the required number of alternating paths of length 5, we see that there is no tree-branch at vertex 4. The rest of the proof is now routine. ☐

THEOREM 3.3. *Let G be a non-corona unicyclic graph with property (SR) and T_i, r_i, $i = 1, 2$, be as discussed earlier. Suppose that $r_1 = 1, r_2 = 2$. Then the structure of G is as shown in Figure 3.3, where F_1 and F_5 are forests of simple corona trees; a non-pendant vertex of each tree in F_1 (resp. F_5) is adjacent to the vertex 1 (resp. 5) in G.*

Proof. Similar to the proof of Theorem 3.2. ☐

We note that each of the graphs with structures described above can be obtained from certain simple corona graphs by deleting two specific pendant vertices with distance 3. The following two theorems give necessary and sufficient conditions for unicyclic graphs of structures as in Figure 3.2(B) and Figure 3.3 to have property (SR).

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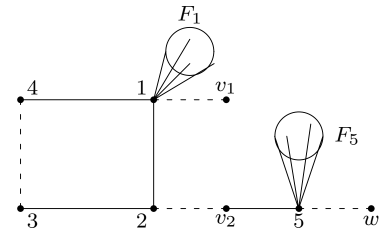

FIG. 3.3. *Case of* $r_1 = 1, r_2 = 2$.

THEOREM 3.4. *Let G be a unicyclic graph having structure as in Figure 3.2(B). Let T and T_4 be the components of $G - 2 - 4$ containing the vertices 1 and w respectively. Then G has property (SR) if and only if*

$$P(T;x)P(T_4 - w - w';x) = P(T - 1 - v_1;x)P(T_4;x). \tag{3.1}$$

In particular, if T and T_4 in G are isomorphic with 1 and w as corresponding vertices, then G has property (SR).

Proof. Let e be the edge with end vertices 2 and 3. Using Lemma 1.3 and Lemma 1.4, we get

$$\begin{aligned}
P(G;x) &= P(G-e;x) - P(G-3-2;x) - 2P(G-\Gamma;x)\\
&= P(G-e;x) - \Big[xP(G-3-2-4;x) - P(G-3-2-4-w;x)\\
&\quad -P(G-3-2-4-1;x)\Big] - 2P(G-\Gamma;x)\\
&= P(G-e;x) - x^2P(G-3-2-4-v_2;x) + P(G-3-2-4-w;x)\\
&\quad -P(G-\Gamma;x)\\
&= P(G-e;x) - x^2P(G-3-2-4-v_2;x)\\
&\quad +x^2P(T;x)P(T_4-w-w';x)P(F_{2b};x) - x^2P(T-1-v_1;x)P(T_4;x)P(F_{2b};x).
\end{aligned}$$

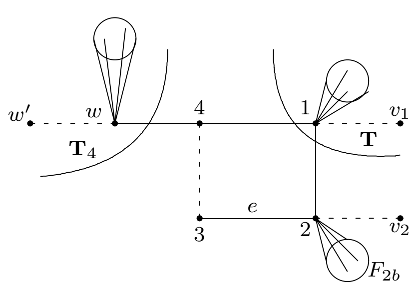

FIG. 3.4. *The trees T and T_4 in G.*

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The components of each of the graphs in the parentheses of the last expression are simple corona trees, and therefore have property (SR). Let $n = 2m$. Then for $P(G-e;x)$, the coefficient a_n is $(-1)^m$, and we have

$$P(G-e;x) = (-1)^m x^n\, P\left(G-e;\frac{1}{x}\right).$$

Similarly, we have

$$P(G-3-2-4-v_2;x) = (-1)^{m-2}x^{n-4}\, P\left(G-3-2-4-v_2;\frac{1}{x}\right),$$

$$P(T;x)P(T_4-w-w';x)P(F_{2b};x) = (-1)^{m-3}x^{n-6}\, P\left(T;\frac{1}{x}\right) P\left(T_4-w-w';\frac{1}{x}\right) P\left(F_{2b};\frac{1}{x}\right),$$

$$P(T-1-v_1;x)P(T_4;x)P(F_{2b};x) = (-1)^{m-3}x^{n-6}\, P\left(T-1-v_1;\frac{1}{x}\right) P\left(T_4;\frac{1}{x}\right) P\left(F_{2b};\frac{1}{x}\right).$$

This gives

$$\begin{aligned}(-1)^m x^n\, P\left(G;\frac{1}{x}\right) &= P(G-e;x) - x^2P(G-3-2-4-v_2;x)\\ &\quad -x^4P(T;x)P(T_4-w-w';x)P(F_{2b};x)\\ &\quad +x^4P(T-1-v_1;x)P(T_4;x)P(F_{2b};x).\end{aligned}$$

Thus G has property (SR) if and only if

$$P(G;x) = (-1)^m x^n\, P\left(G;\frac{1}{x}\right),$$

which is the case if and only if

$$\begin{aligned}&x^2P(T;x)P(T_4-w-w';x)P(F_{2b};x) - x^2P(T-1-v_1;x)P(T_4;x)P(F_{2b};x)\\ &= -x^4P(T;x)P(T_4-w-w';x)P(F_{2b};x) + x^4P(T-1-v_1;x)P(T_4;x)P(F_{2b};x).\end{aligned}$$

That is G has property (SR) if and only if

$$(x^4+x^2)P(T;x)P(T_4-w-w';x)P(F_{2b};x) = (x^4+x^2)P(T-1-v_1;x)P(T_4;x)P(F_{2b};x).$$

This completes the proof of the first assertion. The second assertion now follows. □

Theorem 3.5. *Let G be a unicyclic graph having structure as in Figure 3.3. Let T_1 and T_5 be the components (trees) of the graph $G-2-4-v_2$ containing vertices 1 and 5 respectively. Then G has property (SR) if and only if*

$$P(T_1;x)P(T_5-5-w;x) = P(T_1-1-v_1;x)P(T_5;x).$$

In particular, if T_1 and T_5 in G are isomorphic with 1 and 5 as corresponding vertices, then G has property (SR).

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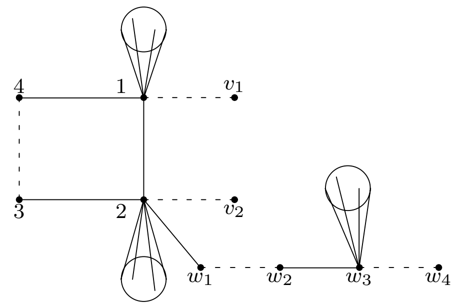

FIG. 3.5. *A subclass of graphs of Figure 3.2(A).*

Proof. Similar to the proof of Theorem 3.4. □

THEOREM 3.6. *Let G be unicyclic graph of the form as shown in figure 3.5. Let T and T' be the components of $G-2$ containing the vertices 1 and w_1 respectively. Then G has property (SR) if and only if*

$$P(T;x)P(T'-w_1-w_2;x) = P(T-3-4;x)P(T';x).$$

In particular, if T and T' are isomorphic with 1 and w_3 as corresponding vertices, then G satisfies property SR.

The proof is omitted as it is similar to that of earlier theorems.

To summarize, we have seen that a unicyclic graph G with girth $g \neq 4$ has property (SR) if and only if it is a simple corona. In case $g = 4$, G has property (SR) if it is either a simple corona or has one of the forms as described in Figures 3.2(A), 3.2(B) and 3.3. Theorems 3.4, 3.5 and 3.6 supply classes of non-corona unicyclic graphs with property (SR) which are in the forms 3.2(B), 3.3 and 3.2(A), respectively.

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