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ADDITIVE MAPS ON RANK K BIVECTORS*

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Abstract. Let $\mathcal{U}$ and $\mathcal{V}$ be linear spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, such that $\dim \mathcal{U} = n \geqslant 2$ and $|\mathbb{F}| \geqslant 3$. Let $\bigwedge^2 \mathcal{U}$ be the second exterior power of $\mathcal{U}$. Fixing an even integer k satisfying $\frac{n-1}{2} \leqslant k \leqslant n$, it is shown that a map $\psi : \bigwedge^2 \mathcal{U} \to \bigwedge^2 \mathcal{V}$ satisfies

$$\psi(u + v) = \psi(u) + \psi(v)$$

for all rank k bivectors $u, v \in \bigwedge^2 \mathcal{U}$ if and only if ψ is an additive map. Examples showing the indispensability of the assumption on k are given.

Key words. Additive maps, Second exterior powers, Bivectors, Ranks, Alternate matrices.

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1. Introduction. Let $n \geqslant 2$ be an integer and let $\mathbb{F}$ be a field. We denote by $M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over $\mathbb{F}$. Given a nonempty subset S of $M_n(\mathbb{F})$, a map $\psi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is called *commuting on S* (respectively, *additive on S*) if $\psi(A)A = A\psi(A)$ for all $A \in S$ (respectively, $\psi(A + B) = \psi(A) + \psi(B)$ for all $A, B \in S$). In 2012, using Brešar's result [1, Theorem A], Franca [3] characterized commuting additive maps on invertible (respectively, singular) matrices of $M_n(\mathbb{F})$. He continued to characterize commuting additive maps on rank k matrices of $M_n(\mathbb{F})$ in [4], where $2 \leqslant k \leqslant n$ is a fixed integer, and commuting additive maps on rank one matrices of $M_n(\mathbb{F})$ in [5]. Later, Xu and Yi [8] improved the result of Franca [4] and showed that the same result holds when $\operatorname{char} \mathbb{F} = 2$ and $\operatorname{char} \mathbb{F} = 3$. Recently, Xu, Pei and Yi [7] initiated the study of additive maps on invertible matrices of $M_n(\mathbb{F})$ and showed that the additivity of a map on $M_n(\mathbb{F})$ can be determined by its invertible matrices. This work was continued by Xu and Liu [6] to study additive maps on rank k matrices $M_n(\mathbb{F})$ with $n/2 \leqslant k \leqslant n$. Motivated by these works, the authors studied additive maps on rank k tensors and rank k symmetric tensors in [2], which have slightly improved the result of [6] and provided an affirmative answer for the case of symmetric matrices. In this note we continue our investigation to study additive maps on rank k bivectors of the second exterior powers of linear spaces.

We now introduce some notation to describe our main result precisely and to present some examples for showing the indispensability of the assumption on k in the result. Let $\mathcal{U}$ be a linear space over a field. We denote by $\bigwedge^2 \mathcal{U}$ the *second exterior power* of $\mathcal{U}$, i.e., the quotient space $\bigwedge^2 \mathcal{U} := \bigotimes^2 \mathcal{U}/\mathcal{Z}$, where $\bigotimes^2 \mathcal{U}$ is the *tensor product* of $\mathcal{U}$ with itself, and $\mathcal{Z}$ is the subspace of $\bigotimes^2 \mathcal{U}$ spanned by tensors of the form $u \otimes u$. The elements of $\bigwedge^2 \mathcal{U}$ are referred to as *bivectors*. Bivectors in $\bigwedge^2 \mathcal{U}$ of the form $u_1 \wedge u_2$, for some $u_1, u_2 \in \mathcal{U}$, are called *decomposable*. Note that $u_1 \wedge u_2 \neq 0$ if and only if u_1, u_2 are linearly independent in $\mathcal{U}$. A bivector $u \in \bigwedge^2 \mathcal{U}$ is said to be of *rank* $2r$, denoted $\rho(u) = 2r$, provided that r is the least integer such that u can be represented as a sum of r decomposable bivectors. It is a known fact that $u \in \bigwedge^2 \mathcal{U}$ is of rank $2r$, with

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$r \geqslant 1$, if and only if

$$u = \sum_{i=1}^{r} u_{2i-1} \wedge u_{2i} \tag{1.1}$$

for some linearly independent vectors $u_1, \ldots, u_{2r} \in \mathcal{U}$. If u has another representation $u = \sum_{i=1}^{r} v_{2i-1} \wedge v_{2i}$ for some $v_1, \ldots, v_{2r} \in \mathcal{U}$, then $\langle u_1, \ldots, u_{2r} \rangle = \langle v_1, \ldots, v_{2r} \rangle$. Here $\langle u_1, \ldots, u_{2r} \rangle$ denotes the subspace of $\mathcal{U}$ spanned by $u_1, \ldots, u_{2r}$. Evidently, $\rho(u) = \dim \langle u_1, \ldots, u_{2r} \rangle$ if u is of rank $2r$ of the form (1.1). In what follows, $\mathcal{R}_k(\bigwedge^2 \mathcal{U})$ denotes the totality of rank k bivectors in $\bigwedge^2 \mathcal{U}$. We write it as $\mathcal{R}_k$ for brevity when it is clear from the context.

We can now state the main theorem.

Theorem 1.1. *Let $n \geqslant 2$ be an integer and let k be a fixed even integer such that $\frac{n-1}{2} \leqslant k \leqslant n$. Let $\mathcal{U}$ and $\mathcal{V}$ be linear spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, with $\dim \mathcal{U} = n$ and $|\mathbb{F}| \geqslant 3$. Then a map $\psi : \bigwedge^2 \mathcal{U} \to \bigwedge^2 \mathcal{V}$ satisfies $\psi(u+v) = \psi(u) + \psi(v)$ for all rank k bivectors $u, v \in \bigwedge^2 \mathcal{U}$ if and only if ψ is an additive map.*

Let $\mathcal{A}_n(\mathbb{F})$ denote the linear space of all $n \times n$ alternate matrices over a field $\mathbb{F}$. In matrix language, we obtain the corresponding result for additive maps on rank k alternate matrices on $\mathcal{A}_n(\mathbb{F})$.

Corollary 1.2. *Let $\mathbb{F}$ and $\mathbb{K}$ be fields with $|\mathbb{F}| \geqslant 3$. Let m and n be integers such that $m, n \geqslant 2$ and let k be a fixed even integer such that $\frac{n-1}{2} \leqslant k \leqslant n$. Then a map $\psi : \mathcal{A}_n(\mathbb{F}) \to \mathcal{A}_m(\mathbb{K})$ satisfies $\psi(A+B) = \psi(A) + \psi(B)$ for all rank k matrices $A, B \in \mathcal{A}_n(\mathbb{F})$ if and only if ψ is an additive map.*

We give some examples to highlight that the condition $\frac{n-1}{2} \leqslant k \leqslant n$ in Theorem 1.1 is indispensable.

Example 1.3. Let $n \geqslant 6$ be an integer. Let $\mathcal{U}$ be an n-dimensional linear space and let $\mathcal{V}$ be a non-trivial linear space. Given a fixed nonzero vector $w \in \mathcal{V}$, we let $\varphi : \bigwedge^2 \mathcal{U} \to \bigwedge^2 \mathcal{V}$ be the map defined by

$$\varphi(u) = \left(\prod_{i=1}^{n-2} \rho(u) - i \right) w \quad \text{for all } u \in \bigwedge^2 \mathcal{U}.$$

Note that $\varphi(u+v) = 0 = \varphi(u) + \varphi(v)$ for every even rank k bivectors $u, v \in \bigwedge^2 \mathcal{U}$ with $1 \leqslant k < \frac{n-1}{2}$. Nevertheless, φ is not an additive map on $\bigwedge^2 \mathcal{U}$. To see this, we select $u_1 = b_1 \wedge b_2$ and

$$u_2 = \begin{cases} b_3 \wedge b_4 + \cdots + b_{n-1} \wedge b_n & \text{when } n \text{ is even,} \\ b_3 \wedge b_4 + \cdots + b_{n-2} \wedge b_{n-1} & \text{when } n \text{ is odd,} \end{cases}$$

where $\{b_1, \ldots, b_n\}$ is a basis of $\mathcal{U}$. Clearly, $\varphi(u_1) = 0 = \varphi(u_2)$ and

$$\varphi(u_1 + u_2) = \begin{cases} (n-1)!\, w & \text{when } n \text{ is even,} \\ (n-2)!\, w & \text{when } n \text{ is odd.} \end{cases}$$

Hence, $\varphi(u_1 + u_2) \neq \varphi(u_1) + \varphi(u_2)$.

Example 1.4. Let $\mathcal{U}$ be a 2-dimensional linear space over the Galois field of two elements and let $\mathcal{V}$ be a non-trival linear space over a field of characteristic not two. Notice that $\bigwedge^2 \mathcal{U} = \{0, b_1 \wedge b_2\}$ where $\{b_1, b_2\}$ is a basis of $\mathcal{U}$. Let z be a fixed nonzero bivector in $\bigwedge^2 \mathcal{V}$ and let $\varphi : \bigwedge^2 \mathcal{U} \to \bigwedge^2 \mathcal{V}$ be the map defined by

$$\varphi(u) = \begin{cases} z & \text{if } u = 0, \\ 2^{-1} z & \text{if } u = b_1 \wedge b_2. \end{cases}$$

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Clearly, φ is not additive since $\varphi(0) \neq 0$. However, $\varphi(u+v) = z = \varphi(u) + \varphi(v)$ for every $u, v \in \mathcal{R}_2$.

EXAMPLE 1.5. Let $\mathbb{F}$ be a field and let $n \geqslant 6$ be an integer. Let $E \in \mathcal{A}_n(\mathbb{F})$ be a fixed nonzero matrix and let $\varphi_i : \mathcal{A}_n(\mathbb{F}) \to \mathcal{A}_n(\mathbb{F})$, $i = 1, 2, 3, 4$, be the maps defined by

$$\varphi_1(A) = \operatorname{adj} A \quad \text{for all } A \in \mathcal{A}_n(\mathbb{F});$$

$$\varphi_2(A) = \begin{cases} 0 & \text{when } A \text{ is singular,} \\ E & \text{when } A \text{ is invertible,} \end{cases} \quad \text{with } n \text{ even;}$$

$$\varphi_3(A) = \begin{cases} A & \text{when } A \text{ is of rank less than } n-1, \\ 0 & \text{when } A \text{ is of rank } n-1, \end{cases} \quad \text{with } n \text{ odd;}$$

$$\varphi_4(A) = \begin{cases} \operatorname{tr}(A)E & \text{when } A \text{ is singular,} \\ 0 & \text{when } A \text{ is invertible,} \end{cases} \quad \text{with } n \text{ even,}$$

where $\operatorname{adj} A$ and $\operatorname{tr}(A)$ denote the classical adjoint of A and the trace of A, respectively. Note that no φ_i is an additive map on $\mathcal{A}_n(\mathbb{F})$, but $\varphi_i(A+B) = \varphi_i(A) + \varphi_i(B)$ for all even rank k matrices $A, B \in \mathcal{A}_n(\mathbb{F})$ with $1 \leqslant k < \frac{n-1}{2}$.

2. Results.

We start with three lemmas.

LEMMA 2.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be linear spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, such that $\dim \mathcal{U} = 2$ with $|\mathbb{F}| \geqslant 3$, or $\dim \mathcal{U} = n \geqslant 3$. Let r be a fixed even integer with $2 \leqslant r \leqslant n$. If $\psi : \bigwedge^2 \mathcal{U} \to \bigwedge^2 \mathcal{V}$ is a map satisfying $\psi(x+y) = \psi(x) + \psi(y)$ for every $x, y \in \mathcal{R}_r$, then the following hold.*

- **(i)** *$\psi(0) = 0$ and $\psi(-u) = -\psi(u)$ for every $u \in \mathcal{R}_r$.*
- **(ii)** *$\psi(u-v) = \psi(u) - \psi(v)$ for every $u, v \in \mathcal{R}_r$.*
- **(iii)** *Let $u, v \in \bigwedge^2 \mathcal{U}$ be such that $u, u+v \in \mathcal{R}_r$. Then $\psi(u+v) = \psi(u) + \psi(v)$.*

Proof. (i) Let $u \in \mathcal{R}_r$. We distinguish two cases.

Case 1. $\operatorname{char} \mathbb{K} = 2$. We first claim that $\psi(-u) = -\psi(u)$. If in addition, $\operatorname{char} \mathbb{F} = 2$, then $\psi(-u) = \psi(u)$, and we have $\psi(-u) = -\psi(u)$. Now consider the case $\operatorname{char} \mathbb{F} \neq 2$. Notice that $\psi(2u) = \psi(u+u) = \psi(u) + \psi(u) = 0$. Thus, $-\psi(u) = \psi(u) = \psi(2u + (-u)) = \psi(2u) + \psi(-u) = \psi(-u)$ as claimed. For $\psi(0) = 0$, we note that $\psi(0) = \psi(u) + \psi(-u) = \psi(u) - \psi(u) = 0$.

Case 2. $\operatorname{char} \mathbb{K} \neq 2$. If in addition, $\operatorname{char} \mathbb{F} \neq 2$, we have $\psi(u) = \psi(2u + (-u)) = \psi(2u) + \psi(-u) = 2\psi(u) + \psi(-u)$. Hence, $\psi(-u) = -\psi(u)$. A similar argument as in the proof of Case 1 shows $\psi(0) = 0$. Now suppose that $\operatorname{char} \mathbb{F} = 2$. Then $2\psi(u) = \psi(u) + \psi(u) = \psi(u+u) = \psi(0)$, so $\psi(u) = 2^{-1}\psi(0)$. Therefore $\psi(v) = 2^{-1}\psi(0)$ for every $v \in \mathcal{R}_r$. We next claim that there exist $v_1, v_2 \in \mathcal{R}_r$ such that

$$v_1 + v_2 \in \mathcal{R}_r. \tag{2.2}$$

Let $\{b_1, \ldots, b_r, \ldots, b_n\}$ be a basis for $\mathcal{U}$. When $|\mathbb{F}| > 2$, let $v_1 = b_1 \wedge b_2 + \cdots + b_{r-1} \wedge b_r$ and $v_2 = \lambda v_1$ for some nonzero $\lambda \in \mathbb{F}$ with $\lambda + 1 \neq 0$. Clearly, $v_1, v_2, v_1 + v_2 \in \mathcal{R}_r$ as required. When $|\mathbb{F}| = 2$, we have $n \geqslant 3$. Let

$$v_1 = \begin{cases} b_1 \wedge b_2 & \text{when } r = 2, \\ b_1 \wedge b_r + b_2 \wedge b_3 + \cdots + b_{r-2} \wedge b_{r-1} & \text{when } 2 < r \leqslant n, \end{cases}$$

$$v_2 = \begin{cases} b_1 \wedge b_3 & \text{when } r = 2, \\ b_1 \wedge (b_2 + b_r) + b_3 \wedge b_4 + \cdots + b_{r-1} \wedge b_r & \text{when } 2 < r \leqslant n \end{cases}$$

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be two rank r bivectors in $\bigwedge^2 \mathcal{U}$. Then

$$v_1 + v_2 = \begin{cases} b_1 \wedge (b_2 + b_3) & \text{when } r = 2, \\ b_1 \wedge b_2 + b_3 \wedge (b_2 + b_4) + \cdots + b_{r-1} \wedge (b_{r-2} + b_r) & \text{when } 2 < r \leqslant n. \end{cases}$$

Clearly, $v_1 + v_2 \in \mathcal{R}_r$ as claimed. It follows from (2.2) that $2^{-1}\psi(0) + 2^{-1}\psi(0) = \psi(v_1) + \psi(v_2) = \psi(v_1 + v_2) = 2^{-1}\psi(0)$, so $\psi(0) = 0$. Hence, $\psi(-u) = 0 = -\psi(u)$ for every $u \in \mathcal{R}_r$.

(ii) Let $u, v \in \mathcal{R}_r$. Then $\psi(u - v) = \psi(u) + \psi(-v) = \psi(u) - \psi(v)$ by (i).

(iii) By (ii), $\psi(u + v) - \psi(u) = \psi((u + v) - u) = \psi(v)$. Thus, $\psi(u + v) = \psi(u) + \psi(v)$. □

Remark 2.2. Let $\mathcal{U}$ and $\mathcal{V}$ be linear spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, with $\dim \mathcal{U} = n$. As an immediate consequence of Lemma 2.1, we notice that when $n = 2$ with $|\mathbb{F}| \geqslant 3$, or $n = 3$, $\psi : \bigwedge^2 \mathcal{U} \to \bigwedge^2 \mathcal{V}$ is additive if and only if $\psi(u + v) = \psi(u) + \psi(v)$ for all rank two bivectors $u, v \in \bigwedge^2 \mathcal{U}$.

Let $\mathcal{U}$ be a linear space over a field $\mathbb{F}$. Then $u_1, \ldots, u_k \in \mathcal{U}$ are linearly independent if and only if so are

$$u_1, \ldots, u_i + \sum_{j=1,\, j \neq i}^{k} \alpha_j u_j, \ldots, u_k \tag{2.3}$$

for any $1 \leqslant i \leqslant k$ and any scalars $\alpha_j \in \mathbb{F}$, $j \neq i$. This fact will be frequently employed in our argument.

Lemma 2.3. *Let $n \geqslant 4$ be an even integer. Let $\mathcal{U}$ be an n-dimensional linear space over a field $\mathbb{F}$ with at least three elements. Then for each even integer $0 \leqslant k \leqslant n$ and each $u \in \mathcal{R}_k$ and $v \in \mathcal{R}_2$, there exists $z \in \mathcal{R}_n$ such that $u + z, v - z \in \mathcal{R}_n$.*

Proof. Let $v = x \wedge y \in \mathcal{R}_2$ for some linearly independent vectors $x, y \in \mathcal{U}$. If $k = 0$, we take a basis $\{x, y, b_3, \ldots, b_n\}$ for $\mathcal{U}$ and set

$$z_1 = x \wedge b_3 + y \wedge b_4 + \sum_{i=3}^{n/2} b_{2i-1} \wedge b_{2i}.$$

Then $z_1 \in \mathcal{R}_n$ and $z_1 - v = x \wedge (b_3 - y) + y \wedge b_4 + \sum_{i=3}^{n/2} b_{2i-1} \wedge b_{2i} \in \mathcal{R}_n$.

Suppose that $2 \leqslant k \leqslant n$. Let $u = \sum_{i=1}^{k/2} u_{2i-1} \wedge u_{2i} \in \mathcal{R}_k$ for some linearly independent vectors $u_1, \ldots, u_k$ in $\mathcal{U}$. Four cases are consided below.

Case I. $\{x, y, u_1, \ldots, u_k\}$ is linearly independent. Let $\{x, y, u_1, \ldots, u_k, b_{k+1}, \ldots, b_{n-2}\}$ be a basis for $\mathcal{U}$. We set

$$z_2 = x \wedge u_2 + u_{k-1} \wedge y + \sum_{i=1}^{\frac{k}{2}-1} u_{2i-1} \wedge u_{2i+2} + \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} b_{2i-1} \wedge b_{2i} \in \mathcal{R}_n.$$

Note that

$$z_2 - v = x \wedge (u_2 - y) + u_{k-1} \wedge y + \sum_{i=1}^{\frac{k}{2}-1} u_{2i-1} \wedge u_{2i+2} + \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} b_{2i-1} \wedge b_{2i} \in \mathcal{R}_n,$$

since $x, y, u_1, u_2 - y, u_3, \ldots, u_k, b_{k+1}, \ldots, b_{n-2}$ are linearly independent by (2.3). Also, when $k = 2$, $u + z_2 =$

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$(x+u_1)\wedge u_2 + u_1\wedge y + \sum_{i=2}^{\frac{n}{2}-1} b_{2i-1}\wedge b_{2i} \in \mathcal{R}_n$. When $k \geqslant 4$, we have

$$\begin{aligned} u+z_2 &= (x+u_1)\wedge u_2 + u_{k-1}\wedge(y+u_k) + u_1\wedge u_4 \\ &\quad + \sum_{i=2}^{\frac{k}{2}-1} u_{2i-1}\wedge(u_{2i+2}+u_{2i}) + \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} b_{2i-1}\wedge b_{2i} \in \mathcal{R}_n, \end{aligned}$$

as $x+u_1, y+u_k, u_1, u_2, u_3, u_4, u_5, u_6+u_4, \ldots, u_{k-3}, u_{k-2}+u_{k-4}, u_{k-1}, u_k+u_{k-2}$, $b_{k+1}, \ldots, b_{n-2}$ are linearly independent by (2.3).

Case II. $\{x, u_1, \ldots, u_k\}$ is linearly independent and $y \in \langle x, u_1, \ldots, u_k\rangle$. Then $k+2 \leqslant n$. By a suitable rearrangement of subscripts, we assume $y = \alpha_{k+1}x + \sum_{i=1}^{k}\alpha_i u_i$ for some $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{F}$ with $\alpha_{k-1} \neq 0$. We extend $\{x, u_1, \ldots, u_k\}$ to a basis $\{x, q, u_1, \ldots, u_k, b_{k+1}, \ldots, b_{n-2}\}$ for $\mathcal{U}$. Let

$$z_3 = \begin{cases} x\wedge u_2 + u_{k-1}\wedge q + w & \text{when } \alpha_2 = 0, \\ x\wedge(\alpha_2 u_2) + (u_2+u_{k-1})\wedge q + w & \text{when } \alpha_2 \neq 0, \end{cases}$$

where $w := \sum_{i=1}^{\frac{k}{2}-1} u_{2i-1}\wedge u_{2i+2} + \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} b_{2i-1}\wedge b_{2i}$ with $\sum_{i=1}^{\frac{k}{2}-1} u_{2i-1}\wedge u_{2i+2} = 0$ when $k=2$. Clearly, $z_3 \in \mathcal{R}_n$. Note that

$$z_3 - v = \begin{cases} x\wedge\left(u_2 - \sum_{i=1, i\neq 2}^{k}\alpha_i u_i\right) + u_{k-1}\wedge q + w & \text{when } \alpha_2 = 0, \\ -x\wedge\left(\sum_{i=1, i\neq 2}^{k}\alpha_i u_i\right) + (u_2+u_{k-1})\wedge q + w & \text{when } \alpha_2 \neq 0. \end{cases}$$

Note that $z_3 - v \in \mathcal{R}_n$ as $\{x, q, u_1, u_2 - \sum_{i=1, i\neq 2}^{k}\alpha_i u_i, u_3, \ldots, u_{k-1}, u_k, b_{k+1}, \ldots, b_n\}$ and $\{x, q, u_1, u_2+u_{k-1}, u_3, \ldots, u_{k-2}, \sum_{i=1, i\neq 2}^{k}\alpha_i u_i, u_k, b_{k+1}, \ldots, b_n\}$ are linearly independent sets by (2.3). Note also that if $k=2$, then

$$u+z_3 = \begin{cases} (x+u_1)\wedge u_2 + u_1\wedge q + \sum_{i=2}^{\frac{n}{2}-1} b_{2i-1}\wedge b_{2i} & \text{when } \alpha_2 = 0, \\ (u_1+\alpha_2 x)\wedge u_2 + (u_2+u_1)\wedge q + \sum_{i=2}^{\frac{n}{2}-1} b_{2i-1}\wedge b_{2i} & \text{when } \alpha_2 \neq 0 \end{cases}$$

is of rank n. Now consider $k \geqslant 4$. When $\alpha_2 = 0$, we have

$$\begin{aligned} u+z_3 &= (x+u_1)\wedge u_2 + u_{k-1}\wedge(q+u_k) + u_1\wedge u_4 \\ &\quad + \sum_{i=2}^{\frac{k}{2}-1} u_{2i-1}\wedge(u_{2i+2}+u_{2i}) + \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} b_{2i-1}\wedge b_{2i} \in \mathcal{R}_n, \end{aligned}$$

as $x+u_1, q+u_k, u_1, u_2, u_3, u_4, u_5, u_6+u_4, \ldots, u_{k-3}, u_{k-2}+u_{k-4}, u_{k-1}, u_k+u_{k-2}$, $b_{k+1}, \ldots, b_{n-2}$ are linearly independent by (2.3). When $\alpha_2 \neq 0$, we have

$$\begin{aligned} u+z_3 &= (\alpha_2 x + u_1 - \wedge u_2 + u_{k-1}\wedge(q+u_k) + u_1\wedge u_4 \\ &\quad + \sum_{i=2}^{\frac{k}{2}-1} u_{2i-1}\wedge(u_{2i+2}+u_{2i}) + \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} b_{2i-1}\wedge b_{2i} \in \mathcal{R}_n, \end{aligned}$$

as $\alpha_2 x + u_1 - q, q+u_k, u_1, u_2, u_3, u_4, u_5, u_6+u_4, \ldots, u_{k-3}, u_{k-2}+u_{k-4}, u_{k-1}, u_k+u_{k-2}$, $b_{k+1}, \ldots, b_{n-2}$ are linearly independent by (2.3).

Case III. $\{y, u_1, \ldots, u_k\}$ is linearly independent and $x \in \langle y, u_1, \ldots, u_k\rangle$. Repeating the argument for Case II but interchanging x with y, we can find $z \in \mathcal{R}_n$ such that $u+z$, $v-z \in \mathcal{R}_n$.

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Case IV. $\{x,u_1,\dots,u_k\}$ and $\{y,u_1,\dots,u_k\}$ are linearly dependent sets. Since x,y are linearly independent, by a suitable rearrangement of subscripts, we may assume $x=\sum_{i=1}^k \alpha_i u_i$ for some scalars $\alpha_1,\dots,\alpha_k\in\mathbb{F}$ with $\alpha_1\neq 0$, and $y=\sum_{i=1}^k \beta_i u_i$ for some scalars $\beta_1,\dots,\beta_k\in\mathbb{F}$ with $\beta_2\neq 0$. Then $u_1=\alpha_1^{-1}x-\sum_{i=2}^k \alpha_1^{-1}\alpha_i u_i$ and

$$y=\beta_1\alpha_1^{-1}x+\sum_{i=2}^k(\beta_i-\beta_1\alpha_1^{-1}\alpha_i)u_i.$$

Note that $\{x,u_2,u_3,\dots,u_k\}$ is linearly independent. We argue in the following two subcases.

Case IV-1. $k<n$. We extend $\{x,u_2,u_3,\dots,u_k\}$ to a basis $\{x,u_2,u_3,\dots,u_k,\, b_{k+1},\dots,b_n\}$ for $\mathcal{U}$. Let

$$z_4=x\wedge b_{k+1}+u_2\wedge b_{k+2}+\sum_{i=2}^{k/2}\eta_{2i-1}u_{2i-1}\wedge u_{2i}+\sum_{i=\frac{k}{2}+2}^{n/2}b_{2i-1}\wedge b_{2i}$$

for some scalars $\eta_{2i-1}\in\mathbb{F}\setminus\{0,-1\}$, $i=2,\dots,k/2$. Then

$$\begin{aligned}z_4-v&=x\wedge(b_{k+1}-y)+u_2\wedge b_{k+2}+\sum_{i=2}^{k/2}\eta_{2i-1}u_{2i-1}\wedge u_{2i}+\sum_{i=\frac{k}{2}+2}^{n/2}b_{2i-1}\wedge b_{2i}\\&=x\wedge\left(b_{k+1}-\sum_{i=2}^k(\beta_i-\beta_1\alpha_1^{-1}\alpha_i)u_i\right)+u_2\wedge b_{k+2}\\&\quad+\sum_{i=2}^{k/2}\eta_{2i-1}u_{2i-1}\wedge u_{2i}+\sum_{i=\frac{k}{2}+2}^{n/2}b_{2i-1}\wedge b_{2i}\in\mathcal{R}_n,\end{aligned}$$

since $x,u_2,u_3,\dots,u_k,\, b_{k+1}-\sum_{i=2}^k(\beta_i-\beta_1\alpha_1^{-1}\alpha_i)u_i,b_{k+2},\dots,b_n$ are linearly independent, and

$$\begin{aligned}z_4+u&=x\wedge b_{k+1}+u_2\wedge(b_{k+2}-u_1)+\sum_{i=2}^{k/2}(\eta_{2i-1}+1)u_{2i-1}\wedge u_{2i}+\sum_{i=\frac{k}{2}+2}^{n/2}b_{2i-1}\wedge b_{2i}\\&=x\wedge b_{k+1}+u_2\wedge\left(b_{k+2}-\alpha_1^{-1}x+\sum_{i=3}^k\alpha_1^{-1}\alpha_i u_i\right)\\&\quad+\sum_{i=2}^{k/2}(\eta_{2i-1}+1)u_{2i-1}\wedge u_{2i}+\sum_{i=\frac{k}{2}+2}^{n/2}b_{2i-1}\wedge b_{2i}\in\mathcal{R}_n,\end{aligned}$$

since $x,u_2,u_3,\dots,u_k,\, b_{k+1},b_{k+2}-\alpha_1^{-1}x+\sum_{i=3}^k\alpha_1^{-1}\alpha_i u_i,b_{k+3},\dots,b_n$ are linearly independent as desired.

Case IV-2. $k=n$. Then $\{x,u_2,u_3,\dots,u_n\}$ is a basis for $\mathcal{U}$. Set

$$z_5=\mu x\wedge u_4+\eta u_2\wedge u_3+\sum_{i=3}^{n/2}\lambda_{2i-1}u_{2i-1}\wedge u_{2i},$$

where $\mu,\eta,\lambda_5,\dots,\lambda_{n-1}\in\mathbb{F}$ are nonzero scalars such that $\mu\neq\beta_4-\beta_1\alpha_1^{-1}\alpha_4$, $\eta\neq-\alpha_1^{-1}(\mu^{-1}+\alpha_3)$ and

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$\lambda_{2i-1}+1\neq 0$ for $i=3,\dots,n/2$. Then

$$\begin{aligned}
z_5-v &= x\wedge(\mu u_4-y)+\eta u_2\wedge u_3+\sum_{i=3}^{n/2}\lambda_{2i-1}u_{2i-1}\wedge u_{2i}\\
&= x\wedge\left((\mu+\beta_1\alpha_1^{-1}\alpha_4-\beta_4)u_4+\sum_{i=2,\, i\neq 4}^{n}(\beta_1\alpha_1^{-1}\alpha_i-\beta_i)u_i\right)\\
&\quad+\eta u_2\wedge u_3+\sum_{i=3}^{n/2}\lambda_{2i-1}u_{2i-1}\wedge u_{2i}\in\mathcal{R}_n,
\end{aligned}$$

because $x,u_2,u_3,(\mu+\beta_1\alpha_1^{-1}\alpha_4-\beta_4)u_4+\sum_{i=2,\, i\neq 4}^{n}(\beta_1\alpha_1^{-1}\alpha_i-\beta_i)u_i,u_5,\dots,u_n$ are linearly independent by (2.3); and

$$\begin{aligned}
z_5+u &= \mu x\wedge u_4+\eta u_2\wedge u_3+\sum_{i=3}^{n/2}\lambda_{2i-1}u_{2i-1}\wedge u_{2i}+\sum_{i=1}^{n/2}u_{2i-1}\wedge u_{2i}\\
&= \left(\alpha_1^{-1}x-(\eta+\alpha_1^{-1}\alpha_3)u_3-\sum_{i=4}^{n}\alpha_1^{-1}\alpha_i u_i\right)\wedge u_2+(\mu x+u_3)\wedge u_4\\
&\quad+\sum_{i=3}^{n/2}(\lambda_{2i-1}+1)u_{2i-1}\wedge u_{2i}\in\mathcal{R}_n,
\end{aligned}$$

because $\eta\neq-\alpha_1^{-1}(\mu^{-1}+\alpha_3)$ and $\alpha_1^{-1}x-(\eta+\alpha_1^{-1}\alpha_3)u_3-\sum_{i=4}^{n}\alpha_1^{-1}\alpha_i u_i,\mu x+u_3,u_2,u_4,\ u_5,\dots,u_n$ are linearly independent by (2.3). ∎

LEMMA 2.4. *Let $n\geqslant 4$ be an integer and let k and $r\geqslant 2$ be even integers such that $0\leqslant k\leqslant r<n$. Let $\mathcal{U}$ be an n-dimensional linear space over a field $\mathbb{F}$ with at least three elements. Then for any $u\in\mathcal{R}_k$ and $v\in\mathcal{R}_2$, there exists $z\in\mathcal{R}_r$ such that $u+z,\ v-z\in\mathcal{R}_r$.*

Proof. Let $v=x\wedge y$ for some linearly independent vectors $x,y\in\mathcal{U}$, and $u=\sum_{i=1}^{k/2}u_{2i-1}\wedge u_{2i}$ for some linearly independent subset $X=\{u_1,\dots,u_k\}$ of $\mathcal{U}$ with the convention that $u=0$ and $X=\varnothing$ when $k=0$. If $0\leqslant k<r$ or $\dim\langle X\cup\{x,y\}\rangle=r$, then we let $\mathcal{W}$ be an r-dimensional subspace of $\mathcal{U}$ containing $X\cup\{x,y\}$. Evidently, $v\in\mathcal{R}_2(\bigwedge^2\mathcal{W})$ and $u\in\mathcal{R}_k(\bigwedge^2\mathcal{W})$. By Lemma 2.3, there exists $z\in\mathcal{R}_r(\bigwedge^2\mathcal{W})$ such that $u+z,u-z\in\mathcal{R}_r(\bigwedge^2\mathcal{W})$. Since any linearly independent set in $\mathcal{W}$ is linearly independent in $\mathcal{U}$, $\mathcal{R}_r(\bigwedge^2\mathcal{W})\subseteq\mathcal{R}_r(\bigwedge^2\mathcal{U})$ by (1.1). So the result follows.

Suppose that $k=r$ and $\dim\langle X\cup\{x,y\}\rangle>r$. We only consider the case that $X\cup\{x\}$ is linearly independent and $y\in\langle X\rangle$ as the other cases can be argued similarly. Let $y=\sum_{i=1}^{k}\beta_i u_i$ for some scalars $\beta_1,\dots,\beta_k\in\mathbb{F}$. Without loss of generality, we assume $\beta_1\neq 0$. Let $z=x\wedge u_2+\sum_{i=2}^{k/2}\alpha_i u_{2i-1}\wedge u_{2i}$ for some scalars $\alpha_2,\dots,\alpha_{k/2}\in\mathbb{F}\backslash\{0,-1\}$. Clearly, $z\in\mathcal{R}_r$ and

$$u+z=(x+u_1)\wedge u_2+\sum_{i=2}^{k/2}(\alpha_i+1)u_{2i-1}\wedge u_{2i}\in\mathcal{R}_r,$$

$$z-v=x\wedge(u_2-y)+\sum_{i=2}^{k/2}\alpha_i u_{2i-1}\wedge u_{2i}\in\mathcal{R}_r,$$

as desired. ∎

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We are now ready to prove the main theorem.

Proof of Theorem 1.1. The sufficiency is trivial. We consider the necessity. The result is clear when $n \leqslant 3$ by Remark 2.2. Suppose that $n \geqslant 4$. We claim, for any even integer $0 \leqslant h \leqslant n$, that

$$\psi(u+v) = \psi(u) + \psi(v) \quad \text{for every } u \in \mathcal{R}_h \text{ and } v \in \mathcal{R}_2. \tag{2.4}$$

The discussion is split into two cases.

Case I. $\frac{n-1}{2} \leqslant k < n$. We first prove (2.4) for $0 \leqslant h \leqslant k$. By Lemma 2.4, there exists $z \in \mathcal{R}_k$ such that $u+z,\ v-z \in \mathcal{R}_k$. It follows from Lemma 2.1 (ii) and (iii) that $\psi(u+z) = \psi(u) + \psi(z)$ and $\psi(v-z) = \psi(v) - \psi(z)$. So

$$\begin{aligned}
\psi(u+v) &= \psi(u+z+v-z) \\
&= \psi(u+z) + \psi(v-z) \\
&= \psi(u) + \psi(z) + \psi(v) - \psi(z) \\
&= \psi(u) + \psi(v).
\end{aligned}$$

Consider now $k < h \leqslant n$. We use induction on h and assume (2.4) holds for each $0, 2, \ldots, h-2$. Let $u = \sum_{i=1}^{h/2} u_{2i-1} \wedge u_{2i} \in \mathcal{R}_h$ and $v = u_{h+1} \wedge u_{h+2} \in \mathcal{R}_2$. Let $\mathcal{H} = \{u_1, \ldots, u_h, u_{h+1}, u_{h+2}\}$. We distinguish two cases.

Case I-A. $\mathcal{H}$ is linearly dependent. Note that u_{h+1}, u_{h+2} are linearly independent. By a suitable rearrangement of subscripts, we may assume $u_{h+1} = a_{h+2}u_{h+2} + \sum_{i=1}^{h} a_i u_i$ for some $a_1, \ldots, a_h, a_{h+2} \in \mathbb{F}$ with $a_1 \neq 0$. Then $u_1 = a_1^{-1}u_{h+1} - a_1^{-1}a_{h+2}u_{h+2} - \sum_{i=2}^{h} a_1^{-1}a_i u_i$. We thus obtain

$$u = (a_1^{-1}u_{h+1} - a_1^{-1}a_{h+2}u_{h+2}) \wedge u_2 + \sum_{i=2}^{h/2} \Lambda_i,$$

where $\Lambda_i = -a_1^{-1}(a_{2i-1}u_{2i-1} + a_{2i}u_{2i}) \wedge u_2 + u_{2i-1} \wedge u_{2i}$ for $i = 2, \ldots, h/2$. For each $2 \leqslant i \leqslant h/2$, we note that

$$\Lambda_i = \begin{cases} a_1^{-1}(a_{2i-1}u_{2i-1} + a_{2i}u_{2i}) \wedge (a_1 a_{2i-1}^{-1}u_{2i} - u_2) & \text{if } a_{2i-1} \neq 0, \\ (a_1^{-1}a_{2i}u_2 + u_{2i-1}) \wedge u_{2i} & \text{if } a_{2i-1} = 0. \end{cases}$$

Since $u \in \mathcal{R}_h$, we must have $(a_1^{-1}u_{h+1} - a_1^{-1}a_{h+2}u_{h+2}) \wedge u_2 \in \mathcal{R}_2$ and $\sum_{i=2}^{h/2} \Lambda_i \in \mathcal{R}_{h-2}$. Set

$$z = (a_1^{-1}u_{h+1} - a_1^{-1}a_{h+2}u_{h+2}) \wedge u_2.$$

Then $z \in \mathcal{R}_2$, $u - z \in \mathcal{R}_{h-2}$ and

$$v + z = (a_1^{-1}u_{h+1} - a_1^{-1}a_{h+2}u_{h+2}) \wedge (a_1 u_{h+2} + u_2) \in \mathcal{R}_2 \cup \{0\}.$$

Then, by the induction hypothesis, $\psi(u+v) = \psi(u-z+v+z) = \psi(u-z) + \psi(z+v) = \psi(u-z) + \psi(z) + \psi(v) = \psi((u-z)+z) + \psi(v) = \psi(u) + \psi(v)$. So claim (2.4) holds true for h.

Case I-B. $\mathcal{H}$ is linearly independent. Then $\frac{n-1}{2} \leqslant k \leqslant h < h+2 \leqslant n$. We extend $\mathcal{H}$ to a basis $\{u_1, \ldots, u_{h+2}, \ldots, u_n\}$ for $\mathcal{U}$. Let

$$x = \begin{cases} \sum_{i=1}^{k/2} u_{2i-1} \wedge u_{n-k+2i} & \text{when } n \text{ is even}, \\ \sum_{i=1}^{k/2} u_{n-k-2+2i} \wedge u_{2i} & \text{when } n \text{ is odd}, \end{cases}$$

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and

$$y = \sum_{i=\frac{k}{2}+1}^{h/2} u_{2i-1} \wedge u_{2i},$$

with $y = 0$ when $h = k$. It is easily seen that $y \in \mathcal{R}_{h-k}$, and $x \in \mathcal{R}_k$ as $n > k$. Note that

$$u - y = \sum_{i=1}^{k/2} u_{2i-1} \wedge u_{2i} \in \mathcal{R}_k \quad \text{and} \quad v + y = \sum_{i=\frac{k}{2}+1}^{\frac{h}{2}+1} u_{2i-1} \wedge u_{2i} \in \mathcal{R}_{h-k+2}.$$

Let us proceed to verify that both bivectors $u - x - y$ and $v - x - y$ are of rank k.

When n is even, we see that $n - k \geqslant 2$ is even and $\frac{n}{2} \leqslant k$. Thus, $n - k + 2 \leqslant k + 2$. Therefore $v + x + y \in \mathcal{R}_k$ and

$$u - x - y = \sum_{i=1}^{k/2} u_{2i-1} \wedge (u_{2i} - u_{n-k+2i}) \in \mathcal{R}_k,$$

because $u_1, u_3, \ldots, u_{k-1}, u_2 - u_{n-k+2}, u_4 - u_{n-k-4}, \ldots, u_k - u_n$ are linearly independent.

When n is odd, we see that $k < h + 2 < n$ since k and h are even. So $n - k \geqslant 3$ is odd. Then

$$u - x - y = \sum_{i=1}^{k/2} (u_{2i-1} - u_{n-k-2+2i}) \wedge u_{2i} \in \mathcal{R}_k,$$

since $u_2, u_4, \ldots, u_k, u_1 - u_{n-k}, u_3 - u_{n-k+2}, \ldots, u_{k-1} - u_{n-2}$ are linearly independent. Moreover, we note that $h + 1 \leqslant n - 2$ as $h + 2 < n$, and $k + 1 \geqslant n - k$ since $\frac{n-1}{2} \leqslant k$. Consequently, $v + x + y \in \mathcal{R}_k$ as desired.

Now, $\psi(u + v) = \psi((u - x - y) + (v + x + y)) = \psi(u - x - y) + \psi(v + x + y)$. Note that $\psi(u - x - y) = \psi(u - y) - \psi(x)$ by Lemma 2.1 (ii), and $\psi(v + x + y) = \psi(v + y) + \psi(x)$ by Lemma 2.1 (iii). It follows that $\psi(u + v) = \psi(u - y) - \psi(x) + \psi(v + y) + \psi(x) = \psi(u - y) + \psi(v + y)$. The claim follows when $y = 0$. Now consider $y \neq 0$. Since $h - k \leqslant h - 2$, we infer from the induction hypothesis that $\psi(v + y) = \psi(v) + \psi(y)$. Again, by the induction hypothesis, we have

$$\begin{aligned}
\psi(u + v) &= \psi(u - y) + \psi(v) + \psi(y) \\
&= \psi(v) + \psi(u - y) + \psi\left(u_{k+1} \wedge u_{k+2} + \sum_{i=\frac{k}{2}+2}^{h/2} u_{2i-1} \wedge u_{2i} \right) \\
&= \psi(v) + \psi(u - y) + \psi(u_{k+1} \wedge u_{k+2}) + \psi\left(\sum_{i=\frac{k}{2}+2}^{h/2} u_{2i-1} \wedge u_{2i} \right).
\end{aligned}$$

Proceeding in this fashion, we obtain

$$\psi(u + v) = \psi(v) + \psi(u - y) + \sum_{i=\frac{k}{2}+1}^{h/2} \psi(u_{2i-1} \wedge u_{2i}).$$

Since $y \neq 0$, we get $k \leqslant h - 2$. So

$$\psi(u - y) + \sum_{i=\frac{k}{2}+1}^{h/2} \psi(u_{2i-1} \wedge u_{2i}) = \psi(u - y + u_{k+1} \wedge u_{k+2}) + \sum_{i=\frac{k}{2}+2}^{h/2} \psi(u_{2i-1} \wedge u_{2i}).$$

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Next, note that if $\sum_{i=\frac{k}{2}+2}^{h/2}\psi(u_{2i-1}\wedge u_{2i})\neq 0$, then $h-k-2\geqslant 2$, and hence, $k+2\leqslant h-2$. It follows that

$$\psi(u-y+u_{k+1}\wedge u_{k+2})+\sum_{i=\frac{k}{2}+2}^{h/2}\psi(u_{2i-1}\wedge u_{2i})=\psi(u-y+u_{k+1}\wedge u_{k+2}+u_{k+3}\wedge u_{k+4})+\sum_{i=\frac{k}{2}+3}^{h/2}\psi(u_{2i-1}\wedge u_{2i}).$$

Continuing in this way, we get

$$\psi(u+v)=\psi(v)+\psi\left(u-y+\sum_{i=\frac{k}{2}+1}^{h/2}u_{2i-1}\wedge u_{2i}\right)=\psi(u)+\psi(v).$$

Hence, claim (2.4) holds for h.

Consequently, by induction, claim (2.4) is proved.

Case II. $k=n$. By Lemma 2.3, there exists $z\in\mathcal{R}_n$ such that $u+z,\ v-z\in\mathcal{R}_n$. We thus obtain $\psi(u+z)=\psi(u)+\psi(z)$ and $\psi(v-z)=\psi(v)-\psi(z)$ by Lemma 2.1 (i) and (iii). Then $\psi(u+v)=\psi(u+z)+\psi(v-z)=\psi(u)+\psi(v)$. So claim (2.4) is proved.

We continue to prove

$$\psi(s+t)=\psi(s)+\psi(t)\quad\text{for every } s,t\in\bigwedge^2\mathcal{U}. \tag{2.5}$$

The result clearly holds if $t=0$. Let $t\in\mathcal{R}_{2\ell}$ for some integer $0<2\ell\leqslant n$. Then $t=\sum_{i=1}^{\ell}d_i$ for some nonzero decomposable bivectors $d_1,\ldots,d_\ell$ in $\bigwedge^2\mathcal{U}$. So

$$\psi(s+t)=\psi((s+d_1+\cdots+d_{\ell-1})+d_\ell)=\psi(s+t_1+\cdots+t_{\ell-1})+\psi(t_\ell)$$

by (2.4). Proceeding in this manner we arrive at $\psi(s+t)=\psi(s)+\sum_{i=1}^{\ell}\psi(t_i)$. Since $\sum_{i=1}^{\ell}\psi(t_i)=\psi(\sum_{i=1}^{\ell}t_i)=\psi(t)$ by (2.4), we infer that (2.5) holds. Hence, ψ is additive. ∎

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