



## ON THE TENSOR RANK OF THE $3 \times 3$ PERMANENT AND DETERMINANT\*

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**Abstract.** The tensor rank and border rank of the  $3 \times 3$  determinant tensor are known to be 5 if the characteristic is not two. In characteristic two, the existing proofs of both the upper and lower bounds fail. In this paper, we show that the tensor rank remains 5 for fields of characteristic two as well.

**Key words.** Tensor rank, Border rank, Permanent, Determinant.

**AMS subject classification.** 15A69.

**1. Introduction.** An alternate way of looking at the rank of a matrix  $A$  is as the smallest integer  $r$  such that you can write  $A$  as a sum of  $r$  rank 1 matrices. The definition of tensor rank is a generalization of this idea. We consider the tensor space  $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ , where  $V_i$  denote finite dimensional vector spaces over a field  $K$ . Tensors of the form  $v_1 \otimes \cdots \otimes v_n$  with  $v_i \in V_i$  are called simple (or rank 1) tensors.

**DEFINITION 1.1.** *The tensor rank  $\text{trk}(T)$  of a tensor  $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$  is defined as the smallest integer  $r$  such that  $T = T_1 + \cdots + T_r$  for simple tensors  $T_i$ .*

Let  $\mathcal{Z}_r \subset V_1 \otimes V_2 \otimes \cdots \otimes V_n$  denote the subspace of tensors of rank  $\leq r$ . Unfortunately,  $\mathcal{Z}_r$  is not Zariski-closed, giving rise to the notion of border rank. Let  $\overline{\mathcal{Z}_r}$  denote the Zariski-closure of  $\mathcal{Z}_r$ .

**DEFINITION 1.2.** *The border rank  $\text{brk}(T)$  of a tensor  $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$  is the smallest integer  $r$  such that  $T \in \overline{\mathcal{Z}_r}$ .*

The tensor and border rank of various tensors have been well studied. For example, the tensor and border rank of the matrix multiplication tensor is intimately related to the speed of an algorithm for matrix multiplication. Using this approach, Strassen gave an algorithm for matrix multiplication with a running time of  $O(n^{\log_2 7})$  (as opposed to the running time of  $O(n^3)$  for the naive algorithm). Various improvements have since been made, see for e.g., [17, 5, 3, 18]. We refer the interested reader to [2, 15] for an introduction to the subject.

In this paper, we are interested in the determinant and permanent tensors. Let  $\{e_i \mid 1 \leq i \leq n\}$  denote the standard basis for  $K^n$ , and let  $\Sigma_n$  denote the symmetric group on  $n$  letters. The determinant tensor is

$$\det_n = \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)} \in (K^n)^{\otimes n},$$

where  $\text{sgn}(\sigma)$  is the sign of the permutation  $\sigma$ .

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Similarly, the permanent tensor is defined as

$$\text{per}_n = \sum_{\sigma \in \Sigma_n} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)} \in (K^n)^{\otimes n}.$$

The determinant and permanent tensors have been studied before, see [6] for known upper and lower bounds for the tensor rank. For  $K = \mathbb{C}$ , the tensor rank of  $\det_3$  and  $\text{per}_3$  were precisely determined by Ilten and Teitler in [13] to be 5 and 4, respectively. Using linear algebraic techniques, Derksen and the second author showed in [9] that the border rank (and tensor rank) of  $\det_3$  and  $\text{per}_3$  are 5 and 4, respectively, for all algebraically closed fields of characteristic not equal to two.

Observe that in characteristic 2, the determinant and permanent tensors are equal. In this paper, we remove the dependence on the characteristic of the field for the tensor rank of the determinant. The main result of this paper is the following:

**THEOREM 1.3.** *For any field  $K$ , the tensor rank of  $\det_3$  is 5.*

This allows us to extend a result of Derksen in [6] to arbitrary characteristic.

**COROLLARY 1.4.** *For any field  $K$ , we have  $\text{brk}(\det_n) \leq \text{trk}(\det_n) \leq \left(\frac{5}{6}\right)^{\lfloor n/3 \rfloor} n!$ .*

For a matrix whose entries are either 0 or 1, it can be viewed as a matrix over any field. It is easy to see that the rank of such a matrix in positive characteristic is at most its rank in characteristic zero (it is easy to construct examples where it is indeed smaller). However, this phenomenon does not extend to higher order tensors, and the tensor  $\text{per}_3$  witnesses this phenomenon explicitly. Indeed by the above theorem, we have  $\text{trk}(\text{per}_3) = 5$  in characteristic two, whereas we know that  $\text{trk}(\text{per}_3) = 4$  in characteristic zero. Finally, we remark that the tensor  $\det_3$  is the structure tensor for the skew-symmetric matrix-vector product (up to a relabeling of coordinates) whose tensor rank was studied but not determined precisely [19, Proposition 12]. Theorem 1.3 fully resolves its tensor rank in all characteristics.

**1.1. Organization.** In Section 2, we present characteristic free decompositions of  $\text{per}_3$  and  $\det_3$  as a sum of 5 simple tensors. In Section 3, we prove tensor rank lower bounds for  $\det_3$ , and this completes the proof of Theorem 1.3. Finally, in Section 4, we discuss some lower and upper bounds for the tensor rank of the  $5 \times 5$  and  $7 \times 7$  determinant and permanent tensors in characteristic zero.

**2. Upper bounds.** An explicit expression for a tensor  $T$  in terms of simple tensors naturally gives us an upper bound for tensor rank and border rank of  $T$ . Glynn's formula (see [12]) for the permanent tensor is

$$\text{per}_n = \frac{1}{2^{n-1}} \sum_{v \in \{\pm 1\}^{n-1}} (e_1 + v_1 e_2 + \cdots + v_{n-1} e_n)^{\otimes n}.$$

In particular, this shows that

$$\text{brk}(\text{per}_n) \leq \text{trk}(\text{per}_n) \leq 2^{n-1},$$

as long as characteristic is not two. For the determinant tensor, known upper bounds are much weaker. The best known upper bound comes from Derksen's formula (see [6]) for  $\det_3$ .

$$\begin{aligned} \det_3 = \frac{1}{2} & \left( (e_3 + e_2) \otimes (e_1 - e_2) \otimes (e_1 + e_2) \right. \\ & + (e_1 + e_2) \otimes (e_2 - e_3) \otimes (e_2 + e_3) \\ & + 2e_2 \otimes (e_3 - e_1) \otimes (e_3 + e_1) \\ & + (e_3 - e_2) \otimes (e_2 + e_1) \otimes (e_2 - e_1) \\ & \left. + (e_1 - e_2) \otimes (e_3 + e_2) \otimes (e_3 - e_2) \right). \end{aligned}$$

Unfortunately, both Glynn’s and Derksen’s expressions fail in characteristic two because they have denominators that are multiples of two. Hence, the best known upper bound for the tensor rank of  $\det_3$  and  $\text{per}_3$  was 6, given by the defining expression.

We give expressions for both  $\det_3$  and  $\text{per}_3$  as a sum of 5 simple tensors that are valid over any field  $K$ . We have:

$$\begin{aligned} \det_3 = & (e_2 + e_3) \otimes e_1 \otimes e_2 \\ & - (e_1 + e_3) \otimes e_2 \otimes e_1 \\ & - e_2 \otimes (e_1 + e_3) \otimes (e_2 + e_3) \\ & + (e_2 - e_1) \otimes e_3 \otimes (e_1 + e_2 + e_3) \\ & + e_1 \otimes (e_2 + e_3) \otimes (e_1 + e_3), \end{aligned}$$

and

$$\begin{aligned} \text{per}_3 = & (e_2 + e_3) \otimes e_1 \otimes e_2 \\ & + (e_1 + e_3) \otimes e_2 \otimes e_1 \\ & + e_2 \otimes (e_1 + e_3) \otimes (e_3 - e_2) \\ & + (e_1 + e_2) \otimes e_3 \otimes (e_1 + e_2 - e_3) \\ & + e_1 \otimes (e_2 + e_3) \otimes (e_3 - e_1). \end{aligned}$$

COROLLARY 2.1.  $\text{brk}(\det_3) \leq \text{trk}(\det_3) \leq 5$ .

REMARK 2.2. Tensor rank over  $\mathbb{Z}$  is in general an undecidable problem, see [16]. However, the expressions above show that the tensor rank over  $\mathbb{Z}$  of both  $\det_3$  and  $\text{per}_3$  is  $\leq 5$ . On the other hand, Theorem 1.3 shows that the tensor rank over  $\mathbb{Z}$  cannot be less than 5, and so  $\text{trk}_{\mathbb{Z}}(\det_3) = \text{trk}_{\mathbb{Z}}(\text{per}_3) = 5$ .

**3. Lower bounds.** Rank methods are a popular technique to prove lower bounds on tensor rank.<sup>1</sup> Young flattenings are rank methods that come from representation theory and are useful in giving the best lower bounds we know. We refer the interested reader to [14, 8, 9]. The following result is straightforward and well known, see for example [14, 8, 9].

PROPOSITION 3.1 (Rank method). *Let  $\phi : V_1 \otimes V_2 \otimes \cdots \otimes V_n \rightarrow \text{Mat}_{m,m}$  be a linear map. Suppose that for all  $S \in \mathcal{Z}_1$  we have  $\text{rk}(\phi(S)) \leq r$ , then for any tensor  $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$ , we have*

$$\text{brk}(T) \geq \frac{\text{rk}(\phi(T))}{r}.$$

<sup>1</sup>An interesting line of research is to understand the power of rank methods (also known as lifting techniques) in proving lower bounds. In this context, barrier results have been shown in [10, 11]. Further, new techniques that work around these barriers have been developed in [4].

We will only use a special case of a rank method that we will recall. The case we need is a generalization of Strassen's equations for 3-slice tensors (see, e.g., [2, Chapter 19]). Note that  $K^m \otimes K^m$  can be naturally identified with  $\text{Mat}_{m,m}$ .

**THEOREM 3.2 (Strassen).** *Let  $T = (e_1 \otimes A + e_2 \otimes B + e_3 \otimes C) \in K^3 \otimes K^m \otimes K^m$  for  $A, B, C \in K^m \otimes K^m = \text{Mat}_{m,m}$ . If  $A$  is invertible, then*

$$\text{brk}(T) \geq m + \frac{1}{2} \text{rk}(BA^{-1}C - CA^{-1}B).$$

In essence, Strassen's Theorem says for any tensor  $T$  as above, if  $k$  is the rank of  $BA^{-1}C - CA^{-1}B$ , then the  $k \times k$  minors of  $BA^{-1}C - CA^{-1}B$  vanish on tensors of border rank less than  $m + \lceil k/2 \rceil$ . The following proposition is a modern interpretation of a (slight generalization) of Strassen's theorem (see Remark 3.4 below).

**PROPOSITION 3.3.** *Let  $T, A, B, C$  be as in Theorem 3.2, then*

$$\text{brk}(T) \geq \frac{1}{2} \text{rk} \begin{pmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{pmatrix}.$$

*Proof.* Consider  $\phi : K^3 \otimes K^m \otimes K^m \rightarrow \text{Mat}_{3m,3m}$  where

$$\phi(e_1 \otimes A + e_2 \otimes B + e_3 \otimes C) = \begin{pmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{pmatrix}.$$

We claim that  $\text{rk}(\phi(S)) = 2$  for any rank 1 tensor  $S \in K^3 \otimes K^m \otimes K^m$ . There are many ways to see this. For example, it follows from [8, Corollary 4.4]. Let us present a concrete approach. Any rank 1 tensor  $S$  is of the form  $S = (\alpha, \beta, \gamma) \otimes X$  where  $(\alpha, \beta, \gamma) \in K^3 \setminus \{0\}$  and  $X \in \text{Mat}_{m,m} = K^m \otimes K^m$  is of rank 1. Thus,

$$\phi(S) = \begin{pmatrix} 0 & \alpha X & \beta X \\ -\alpha X & 0 & \gamma X \\ -\beta X & -\gamma X & 0 \end{pmatrix}.$$

Let us assume without loss of generality that  $\alpha \neq 0$  (the cases  $\beta \neq 0$  and  $\gamma \neq 0$  are similar). Perform the following block row and column transformations:

- $C_3 \mapsto C_3 - \frac{\beta}{\alpha} C_2 + \frac{\gamma}{\alpha} C_1$ ;
- $R_3 \mapsto R_3 - \frac{\beta}{\alpha} R_2 + \frac{\gamma}{\alpha} R_1$ .

This transforms

$$\begin{pmatrix} 0 & \alpha X & \beta X \\ -\alpha X & 0 & \gamma X \\ -\beta X & -\gamma X & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \alpha X & 0 \\ -\alpha X & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The latter matrix clearly has rank 2 since  $X$  is of rank 1, and block column and row transformations preserve rank. Thus  $\text{rank}(\phi(S)) = 2$ .

Applying Proposition 3.1, we get the required conclusion. □

REMARK 3.4. When  $A$  is invertible, the following (block) Gaussian elimination procedure shows that we can recover Strassen’s result from Proposition 3.3:

$$\begin{pmatrix} 0 & A & B \\ -A & 0 & C \\ -B & -C & 0 \end{pmatrix} \xrightarrow[\substack{R_2 \mapsto A^{-1}R_2 \\ R_1 \mapsto A^{-1}R_1}]{\substack{R_2 \mapsto A^{-1}R_2 \\ R_1 \mapsto A^{-1}R_1}} \begin{pmatrix} 0 & I & A^{-1}B \\ -I & 0 & A^{-1}C \\ -B & -C & 0 \end{pmatrix} \\ \xrightarrow[\substack{R_3 \mapsto R_3 + CR_1 - BR_2}]{\substack{C_3 \mapsto C_3 - C_2(A^{-1}B) + C_1(A^{-1}C)}} \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & CA^{-1}B - BA^{-1}C \end{pmatrix}.$$

3.1. Lower bounds for  $\det_3$ . In every characteristic other than two, a direct application of Proposition 3.3 gives us that  $\text{rk}(\det_3) \geq 5$ , see [9]. Let us recall the determinant tensor

$$\det_3 = \sum_{\sigma \in \Sigma_3} \text{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)} \in K^3 \otimes K^3 \otimes K^3.$$

Identifying  $K^3 \otimes K^3$  with  $\text{Mat}_{3,3}$  via  $e_i \otimes e_j \mapsto E_{i,j}$ , we identify  $K^3 \otimes (K^3 \otimes K^3)$  with  $K^3 \otimes \text{Mat}_{3,3}$ . Under this identification, we have

$$\det_3 = e_1 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + e_2 \otimes \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + e_3 \otimes \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We briefly recall the proof of the following proposition from [9], as we will modify the proof to remove the dependence on characteristic.

PROPOSITION 3.5 ([9]). *If  $\text{char } K \neq 2$ , then  $\text{trk}(\det_3) = \text{brk}(\det_3) = 5$ .*

*Proof.* Applying Proposition 3.3, we get that

$$\text{brk}(\det_3) \geq \frac{1}{2} \text{rk} \left( \begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

This matrix contains only 12 nonzero entries. Six of these entries (with gray background) are in a column or a row with no other nonzero entry, reducing our computation to a  $3 \times 3$  minor  $\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ .

This minor has rank 3 as long as characteristic is not two, and hence we have  $\text{brk}(\det_3) \geq \frac{9}{2} = 4.5$ . But since border rank is an integer, we have  $\text{brk}(\det_3) \geq 5$ . On the other hand, we have  $\text{brk}(\det_3) \leq 5$  by the expression in Section 2, giving us the required conclusion.  $\square$

The problem with this argument in characteristic two is that the aforementioned  $3 \times 3$  minor has rank 2 instead of 3. This only gives that  $\text{trk}(\det_3) \geq \text{brk}(\det_3) \geq 4$ . Nevertheless, we are able to modify the argument to show that the tensor rank of the  $3 \times 3$  determinant is 5. First, we need a simple lemma.

**LEMMA 3.6.** *Let  $T \in V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ . Suppose  $\text{trk}(T - S) \geq r$  for every rank 1 tensor  $S \in V$ , then we have  $\text{trk}(T) \geq r + 1$ .*

*Proof.* Suppose  $\text{trk}(T) \leq r$ , then we have  $T = T_1 + \cdots + T_k$  with  $k \leq r$ , where  $T_i$  are rank 1 tensors. Now, take  $S = T_1$  to see that  $\text{trk}(T - S) \leq k - 1 \leq r - 1$  contradicting the hypothesis.  $\square$

Now, we are ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* We want to prove that  $\text{trk}(\det_3) \geq 5$ . By Lemma 3.6, it suffices to prove that  $\text{trk}(\det_3 - S) \geq 4$  for every rank 1 tensor  $S$ . Observe that  $\text{SL}_3$  acts on  $K^3 \otimes K^3 \otimes K^3$  by  $g \cdot (v_1 \otimes v_2 \otimes v_3) = gv_1 \otimes gv_2 \otimes gv_3$  for  $g \in \text{SL}_3$  and  $v_i \in K^3$ . The action of  $g \in \text{SL}_3$  preserves tensor rank and border rank since it is a linear map preserving the set of rank 1 tensors. There is also an action of  $C_3$ , the cyclic group on three letters that cyclically rotates the tensor factors. This action too preserves tensor rank and border rank, and further it commutes with the action of  $\text{SL}_3$ . Thus, we have an action of  $\text{SL}_3 \times C_3$  on  $K^3 \otimes K^3 \otimes K^3$  that preserves tensor rank and border rank. Further, the tensor  $\det_3$  is invariant under this action.

Now, let  $S = v_1 \otimes v_2 \otimes v_3$  be a rank 1 tensor. We want to show  $\text{trk}(\det_3 - S) \geq 4$ . There are 3 cases.

- **Case 1:**  $v_1, v_2, v_3$  are linearly independent. Then without loss of generality, we can assume  $S = \lambda e_1 \otimes e_2 \otimes e_3$ , by applying the action of an appropriate  $g \in \text{SL}_3$ . Now, apply Proposition 3.3 to  $T = (\det_3 - \lambda e_1 \otimes e_2 \otimes e_3)$  to get

$$\text{brk}(T) \geq \frac{1}{2} \text{rk} \left( \begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 + \lambda & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Once again observe that the 5 gray entries are in a column or row with no other nonzero entries, reducing our computation to a  $4 \times 4$  minor. This  $4 \times 4$  minor clearly has rank  $\geq 2$ . So, this gives  $\text{brk}(T) \geq \lceil 7/2 \rceil = 4$  in all characteristic.

- **Case 2:** The span  $\langle v_1, v_2, v_3 \rangle$  is 2-dimensional. In this case, without loss of generality, we can assume  $S = e_1 \otimes e_2 \otimes (ae_1 + be_2)$ , by using the action of  $\text{SL}_3 \times C_3$ . Now, apply Proposition 3.3 to  $T = (\det_3 - e_1 \otimes e_2 \otimes (ae_1 + be_2))$  to get

$$\text{brk}(T) \geq \frac{1}{2} \text{rk} \left( \begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -a & -b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a & b & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Applying the row transformation  $R_5 \mapsto R_5 + aR_9 - bR_6$  and the column transformations  $C_5 \mapsto C_5 + bC_6$  and  $C_4 \mapsto C_4 + aC_6$ , we see that we are back to computing the rank of the matrix in Proposition 3.5, which as we have seen is at least 8 in all characteristics. Hence,  $\text{brk}(T) \geq 8/2 = 4$  as required.

- **Case 3:** The span  $\langle v_1, v_2, v_3 \rangle$  is 1-dimensional. Once again, without loss of generality, we can assume  $S = \lambda e_1 \otimes e_1 \otimes e_1$ . We are reduced to computing the rank of the matrix

$$\left( \begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ \hline \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

But again, the row transformations  $R_4 \mapsto R_4 + \lambda R_9$  and  $R_1 \mapsto R_1 - \lambda R_8$  put us back to computing the rank of the matrix in Proposition 3.5. The rest of the analysis is as in the previous case.  $\square$

While we have successfully computed the tensor rank, the border rank still remains undetermined.

**PROBLEM 3.7.** *What is the border rank of  $\det_3$  over an algebraically closed field of characteristic two?*

**4.  $5 \times 5$  and  $7 \times 7$  determinant and permanent tensors.** In this section, we study the ranks of the  $5 \times 5$  and  $7 \times 7$  determinant and permanent tensors. For this section, we assume that  $K$  is a field of characteristic 0. From the results in [9], we know that  $\det_3$  has strictly larger tensor rank and border rank than  $\text{per}_3$ , i.e.,

$$\text{brk}(\text{per}_3) = \text{trk}(\text{per}_3) = 4 < 5 = \text{trk}(\det_3) = \text{brk}(\det_3).$$

We would like to separate  $\text{per}_n$  and  $\det_n$  for larger  $n$ . The upper bounds we know for the tensor rank and border rank for  $\text{per}_n$  are stronger than the ones we know for  $\det_n$ . On the other hand, the best known lower bounds for both are the same, see [6]. Koszul flattenings are powerful enough to separate  $\det_5$  and  $\text{per}_5$ .

**4.1. Koszul flattenings.** We will only recall the Koszul flattening we need and very briefly. Let  $\bigwedge^i K^n$  denote the  $i^{\text{th}}$  exterior power of  $K^n$ . For any integer  $p$ , consider the map

$$\psi_p : K^{2p+1} \rightarrow \text{Hom}\left(\bigwedge^p K^{2p+1}, \bigwedge^{p+1} K^{2p+1}\right) = \text{Mat}_{\binom{2p+1}{p}, \binom{2p+1}{p+1}}.$$

where  $\psi_p(v)$  is the map that sends  $w \in \bigwedge^p K^{2p+1}$  to  $v \wedge w$ . Using  $\psi_2$ , we define the following composite map.

$$L : K^5 \otimes (K^5 \otimes K^5) \otimes (K^5 \otimes K^5) = K^5 \otimes \text{Mat}_{25,25} \xrightarrow{\psi_2 \otimes \text{id}} \text{Mat}_{10,10} \otimes \text{Mat}_{25,25} = \text{Mat}_{250,250}.$$

Observe that  $\dim \bigwedge^2 K^5 = \dim \bigwedge^3 K^5 = 10$ , so  $\text{Hom}(\bigwedge^2 K^5, \bigwedge^3 K^5) = \text{Mat}_{10,10}$ . The last equality is in the above is just the equality  $\text{Mat}_{10,10} \otimes \text{Mat}_{25,25} = \text{Mat}_{250,250}$  given by Kronecker product of matrices. It follows from [8, Corollary 4.4] that  $\text{trk}(L(S)) = 6$  for all rank 1 tensors in  $K^5 \otimes K^5 \otimes K^5 \otimes K^5 \otimes K^5$ . This along with Proposition 3.1 gives the following:

LEMMA 4.1. *For any tensor  $T \in K^5 \otimes K^5 \otimes K^5 \otimes K^5 \otimes K^5$ , we have  $\text{brk}(T) \geq \text{rk}(L(T))/6$ .*

Note that both  $\det_5$  and  $\text{per}_5$  are in  $K^5 \otimes K^5 \otimes K^5 \otimes K^5 \otimes K^5$ . So, we get the following:

PROPOSITION 4.2. *Assume  $\text{char}(K) = 0$ . Then, we have*

$$13 \leq \text{brk}(\text{per}_5) \leq \text{trk}(\text{per}_5) \leq 16 < 17 \leq \text{brk}(\det_5) \leq \text{trk}(\det_5) \leq 20.$$

*Proof.* The upper bounds are due to Glynn and Derksen as mentioned in Section 2. The lower bounds come from Lemma 4.1. This requires finding the rank of a large matrix, which we do with the help of a computer. We omit the details, referring the interested reader to the Python code available at [1].  $\square$

Using a similar argument, we get the following bounds for the tensor rank and border rank of  $\text{per}_7$  and  $\det_7$ .

PROPOSITION 4.3. *Assume  $\text{char}(K) = 0$ . Then, we have*

$$42 \leq \text{brk}(\text{per}_7) \leq \text{trk}(\text{per}_7) \leq 64,$$

and

$$62 \leq \text{brk}(\det_7) \leq \text{trk}(\det_7) \leq 100.$$

Koszul flattenings do not seem powerful enough to separate  $\text{per}_7$  and  $\det_7$ . Moreover, we point out that Koszul flattenings are helpful only for finding lower bounds for border rank of  $\det_n$  and  $\text{per}_n$  when  $n$  is odd.

## REFERENCES

- [1] <https://github.com/siddharth-krishna/tensor-rank>
- [2] P. Bürgisser, M. Clausen, and M.A. Shokrollahi. Algebraic Complexity Theory, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* 315, With the collaboration of Thomas Lickteig, Springer-Verlag, Berlin, 1997.
- [3] H. Cohn and C. Umans. A group-theoretic approach to fast matrix multiplication. In: *FOCS 2003*, 438–449, 2003.
- [4] A. Conner, A. Harper, and J.M. Landsberg. New lower bounds for matrix multiplication and the  $3 \times 3$  determinant. [arXiv:1911.07981](https://arxiv.org/abs/1911.07981), [math.AG], 2019.
- [5] D. Coppersmith and S. Winograd. Matrix multiplication via arithmetic progressions, *J. Symbolic Comput.*, 9(3):251–280, 1990.

- [6] H. Derksen. On the nuclear norm and singular value decomposition of tensors. *Found. Comput. Math.*, 16(3):779–811, 2016.
- [7] H. Derksen and V. Makam. Polynomial degree bounds for matrix semi-invariants. *Adv. Math.*, 310:44–63, 2017.
- [8] H. Derksen and V. Makam. On non-commutative rank and tensor rank. *Linear Multilinear Algebra*, 66(6):1069–1084, 2018.
- [9] H. Derksen and V. Makam. Explicit tensor of border rank at least  $2d - 2$  in  $K^d \otimes K^d \otimes K^d$  in arbitrary characteristic. *Linear Multilinear Algebra*, 67(10):2104–2116, 2019.
- [10] K. Efremenko, A. Garg, R. Oliveira, and A. Wigderson. Barriers for rank methods in arithmetic complexity. In: *ITCS 2018*, 1:1–1:19.
- [11] A. Garg, V. Makam, R. Oliveira, and A. Wigderson. More barriers for rank methods, via a “numeric to symbolic” transfer. In: *FOCS 2019*, 824–844.
- [12] D.G. Glynn. The permanent of a square matrix. *European J. Combin.*, 31(7):1887–1891, 2010.
- [13] N. Ilten and Z. Teitler. Product ranks of  $3 \times 3$  determinant and permanent. *Canad. Math. Bull.*, 59(2):311–319, 2016.
- [14] J.M. Landsberg. Nontriviality of equations and explicit tensors in  $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$  of border rank at least  $2m - 2$ . *J. Pure Appl. Algebra*, 219:3677–3684, 2015.
- [15] J.M. Landsberg. *Tensors: Geometry and Applications*, Graduate Studies in Mathematics 128. American Mathematical Society, Providence, RI, 2012.
- [16] Y. Shitov. How hard is tensor rank? [arXiv:1611.01559](https://arxiv.org/abs/1611.01559) [math.CO], 2016.
- [17] V. Strassen. Gaussian elimination is not optimal. *Numerische Mathematik*, 13:354–356, 1969.
- [18] V.V. Williams. Multiplying matrices faster than Coppersmith-Winograd. In: *STOC 2012*, 887–898.
- [19] K. Ye and L. Lim. Fast structured matrix computations: Tensor rank and Cohn–Umans method. *Found. Comput. Math.*, 18(1):45–95, 2018.