



## ON THE EXISTENCE OF HURWITZ POLYNOMIALS WITH NO HADAMARD FACTORIZATION\*

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**Abstract.** A Hurwitz stable polynomial of degree  $n \geq 1$  has a Hadamard factorization if it is a Hadamard product (i.e., element-wise multiplication) of two Hurwitz stable polynomials of degree  $n$ . It is known that Hurwitz stable polynomials of degrees less than four have a Hadamard factorization. It is shown that, for arbitrary  $n \geq 4$ , there exists a Hurwitz stable polynomial of degree  $n$  which does not have a Hadamard factorization.

**Key words.** Hadamard factorization, Hadamard product of polynomials, Hurwitz stable polynomials.

**AMS subject classifications.** 26C10, 30C15, 93D99.

**1. Introduction.** A polynomial is said to be (Hurwitz) stable if all its zeros lie in the open left-half of the complex plane. In the entire class of polynomials, stable polynomials occupy a special place and stability of polynomials is a classical topic having both theoretical and applied significance.

In this short note, we deal with the Hadamard factorization problem. Recall, that a Hurwitz stable polynomial of degree  $n \geq 1$  has (or *admits*) a Hadamard factorization if it is a Hadamard product (i.e., element-wise multiplication) of two Hurwitz stable polynomials of degree  $n$ . A problem of the existence of a Hadamard factorization for a given stable polynomial has been taken by many authors. It is obvious that  $\mathcal{F}_1^+ = \mathcal{H}_1^+$  and  $\mathcal{F}_2^+ = \mathcal{H}_2^+$ , where  $\mathcal{H}_n^+$  denotes the entire family of Hurwitz stable polynomials of degree  $n$  with positive coefficients and  $\mathcal{F}_n^+$  is its subset containing only polynomials admitting a Hadamard factorization. Moreover, as follows from Garloff and Shrinivasan [3], we know that  $\mathcal{F}_3^+ = \mathcal{H}_3^+$  but  $\mathcal{F}_4^+ \neq \mathcal{H}_4^+$ . Some necessary and sufficient conditions for the existence of a Hadamard factorization for a polynomial of arbitrary degree can be found in Loredó-Villalobos and Aguirre-Hernández [6, 7] (unfortunately, their conditions cannot be effectively applied in practice) and some topological properties of the set  $\mathcal{F}_n^+$  (as openness, non-convexity and arc-connectedness) in Aguirre-Hernández et al. [1].

Note also, that there are some issues in which polynomials admitting a Hadamard factorization play an important role. For example, in [2], the authors of this paper have considered the generalized Hadamard product of polynomials and have shown, among others, that if  $f$  is stable and  $g$  has a Hadamard factorization then the generalized Hadamard product of  $f$  and  $g$  is stable (in [2], one can also find a very simple sufficient condition for the Hadamard factorizability of a polynomial). Taking into account all these results it seems to be quite surprising that we still do not know if there exists a polynomial of an arbitrary degree greater than four which does not admit a Hadamard factorization. We will show that this is the case.

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**2. Preliminary results.** In this part, we introduce the basic notation and remind some results which will be used in the sequel.

**2.1. Basic notations.** We use standard notation:  $\mathbb{N}$  and  $\mathbb{R}$  stand for the set of positive integers and real numbers, respectively;  $\mathbb{R}^{n \times n}$  stands for the set of real matrices of order  $n \times n$ ;  $\pi_n^+$  denotes the family of  $n$ -th degree polynomials with positive coefficients.

**2.2. Stable polynomials.** A polynomial  $f \in \pi_n^+$  ( $n \geq 1$ ),

$$(2.1) \quad f(s) = a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + a_n s^n,$$

is *Hurwitz stable* (or shortly *stable*) if all its zeros have negative real parts. It is well known (and easily verified) that a necessary condition for the stability of a real polynomial is that its coefficients are all of the same sign; without losing generality we will assume in the sequel that they are positive.

Let  $H_f \in \mathbb{R}^{n \times n}$  be the Hurwitz matrix associated with polynomial (2.1), i.e.,

$$(2.2) \quad H_f = \begin{pmatrix} a_{n-1} & a_n & 0 & 0 & \cdots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & \cdots & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & a_{n-5} & a_{n-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}.$$

From among many interesting properties of the Hurwitz matrix we recall the following one given by Kemperman [5] (see Theorem 2 therein).

**THEOREM 2.1.** *If  $f \in \mathcal{H}_n^+$  then every square submatrix of the Hurwitz matrix  $H_f$  has positive determinant if and only if all its diagonal elements are positive.*

Note that as long as  $n \geq 4$ , the matrix

$$(2.3) \quad \begin{pmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_{n-1} \\ 0 & a_0 & a_{n-3} \end{pmatrix}$$

is a submatrix of matrix (2.2). Thus, in view of Theorem 2.1, it allows us to conclude that if polynomial (2.1) of degree  $n \geq 4$  is stable then

$$(2.4) \quad a_3 a_2 - a_1 a_4 > 0, \quad a_{n-3} a_2 - a_{n-1} a_0 > 0$$

and

$$(2.5) \quad a_{n-3} a_3 a_2 - a_{n-3} a_4 a_1 - a_{n-1} a_3 a_0 > 0.$$

These inequalities will be crucial in our further considerations.

**2.3. Polynomials admitting a Hadamard factorization.** Together with polynomial  $f$  of the form (2.1) we will consider a polynomial  $g \in \pi_n^+$ ,

$$(2.6) \quad g(s) = b_0 + b_1s + \cdots + b_{n-1}s^{n-1} + b_ns^n,$$

their Hadamard product  $f \circ g \in \pi_n^+$  defined as an element-wise multiplication, i.e.,

$$(f \circ g)(s) = a_0b_0 + a_1b_1s + \cdots + a_{n-1}b_{n-1}s^{n-1} + a_nb_ns^n$$

and their Hadamard quotient  $f \diamond g \in \pi_n^+$  defined as an element-wise division, i.e.,

$$(f \diamond g)(s) = \frac{a_0}{b_0} + \frac{a_1}{b_1}s + \cdots + \frac{a_{n-1}}{b_{n-1}}s^{n-1} + \frac{a_n}{b_n}s^n.$$

Following Garloff and Wagner [4], we say that the polynomial  $f \in \mathcal{H}_n^+$  has a Hadamard factorization if there exist two polynomials  $f_1, f_2 \in \mathcal{H}_n^+$  such that  $f = f_1 \circ f_2$ . Equivalently, the polynomial  $f \in \mathcal{H}_n^+$  has a Hadamard factorization if there exists a polynomial  $g \in \mathcal{H}_n^+$  such that  $f \diamond g \in \mathcal{H}_n^+$ .

**3. Main result.** Let  $f \in \pi_n^+$  be as in (2.1), and let  $\delta_1(f)$  and  $\delta_2(f)$  be two positive numbers given by

$$(3.7) \quad \delta_1(f) = \frac{a_1a_4}{a_3a_2} \quad \text{and} \quad \delta_2(f) = \frac{a_{n-1}a_0}{a_{n-3}a_2}.$$

We start with two simple observations.

LEMMA 3.1. For every  $n \geq 4$  and for the polynomial  $f \in \mathcal{H}_n^+$  we have

$$(3.8) \quad \delta_1(f) < 1, \quad \delta_2(f) < 1$$

and

$$(3.9) \quad \delta_1(f) + \delta_2(f) < 1.$$

*Proof.* The above inequalities follow from the stability of  $f$ : conditions (3.8) and (3.9) are equivalent to (2.4) and (2.5), respectively.  $\square$

LEMMA 3.2. If for  $f, g \in \pi_n^+$  we have  $f \diamond g \in \mathcal{H}_n^+$ , then

$$\delta_1(f) < \delta_1(g) \quad \text{and} \quad \delta_2(f) < \delta_2(g).$$

*Proof.* It is easy to see that

$$\delta_1(f \diamond g) = \frac{\delta_1(f)}{\delta_1(g)} \quad \text{and} \quad \delta_2(f \diamond g) = \frac{\delta_2(f)}{\delta_2(g)}.$$

Then, the thesis is a simple consequence of Lemma 3.1.  $\square$

We are now ready to prove the main result of this work.

**THEOREM 3.3.** For every  $n \geq 4$  there exists a stable polynomial of degree  $n$  that does not have a Hadamard factorization.

*Proof.* Fix any  $n \geq 4$  and suppose, by contradiction, that every stable polynomial of degree  $n$  admits a Hadamard factorization. It implies that for an arbitrary  $g_0 \in \mathcal{H}_n^+$ , there exists  $g_1 \in \mathcal{H}_n^+$  such that  $g_0 \diamond g_1 \in \mathcal{H}_n^+$ . Repeating this reasoning one can show that for  $g_i \in \mathcal{H}_n^+$  there exists  $g_{i+1} \in \mathcal{H}_n^+$  such that  $g_i \diamond g_{i+1} \in \mathcal{H}_n^+$ . In this way we obtain a sequence of stable polynomials  $\{g_i\}$  which generates two sequences of positive numbers  $\{\delta_1(g_i)\}$  and  $\{\delta_2(g_i)\}$  (as in (3.7)). It follows from Lemma 3.2 that sequences  $\{\delta_1(g_i)\}$  and  $\{\delta_2(g_i)\}$  are increasing. Moreover, by Lemma 3.1, they are bounded from above and thus convergent. They are also bounded from below by zero, and thus, their limits are positive. It means that

$$(3.10) \quad \frac{\delta_1(g_i)}{\delta_1(g_{i+1})} + \frac{\delta_2(g_i)}{\delta_2(g_{i+1})} \rightarrow 2$$

as  $i \rightarrow \infty$ . On the other hand, according to Lemma 3.1 it follows from the stability of  $g_i \diamond g_{i+1}$  that

$$\frac{\delta_1(g_i)}{\delta_1(g_{i+1})} + \frac{\delta_2(g_i)}{\delta_2(g_{i+1})} = \delta_1(g_i \diamond g_{i+1}) + \delta_2(g_i \diamond g_{i+1}) < 1$$

for every  $i \in \mathbb{N}$ . This contradicts to (3.10) completing the proof.  $\square$

Garloff and Shrinivasan proved in [3] that there exists a polynomial of degree 4 which is not a Hadamard product of stable polynomials of degree 4. Theorem 3.3 shows that similar examples can be constructed for every  $n \geq 4$ .

**4. Concluding remarks and open problems.** We have considered the Hadamard factorization problem for real polynomials. We have proved that for an arbitrary  $n \geq 4$  there exists a Hurwitz stable polynomial of degree  $n$  which does not have a Hadamard factorization. Unfortunately, the proof of our main result is not constructive, that is, it does not tell us how to construct such a polynomial. An effective method of finding polynomials that do not admit a Hadamard factorization seems to be a challenging open problem.

Other open problems in this area are the development of necessary and sufficient conditions for the Hadamard factorizability of a polynomial as well as to develop factorization (in the Hadamard sense) algorithms for stable polynomials admitting a Hadamard factorization.

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