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RESISTANCE DISTANCE IN DIRECTED CACTUS GRAPHS*

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Abstract. Let $G = (V, E)$ be a strongly connected and balanced digraph with vertex set $V = \{1, \ldots, n\}$. The classical distance d_{ij} between any two vertices i and j in G is the minimum length of all the directed paths joining i and j. The resistance distance (or, simply the resistance) between any two vertices i and j in V is defined by $r_{ij} := l_{ii}^{\dagger} + l_{jj}^{\dagger} - 2l_{ij}^{\dagger}$, where $l_{pq}^{\dagger}$ is the $(p, q)^{\text{th}}$ entry of the Moore-Penrose inverse of L which is the Laplacian matrix of G. In practice, the resistance r_{ij} is more significant than the classical distance. One reason for this is, numerical examples show that the resistance distance between i and j is always less than or equal to the classical distance, i.e., $r_{ij} \leq d_{ij}$. However, no proof for this inequality is known. In this paper, it is shown that this inequality holds for all directed cactus graphs.

Key words. Strongly connected balanced digraph, Directed cactus graph, Laplacian matrix, Moore-Penrose inverse, Cofactor sums.

AMS subject classifications. 05C50.

1. Introduction.

Consider a simple undirected connected graph $H = (W, E)$, where $W := \{1, \ldots, n\}$ is the set of all vertices and E is the set of all edges. If i and j are adjacent in W, we write ij to denote an element in E and δ_i to denote the degree of the vertex i. There are several matrices associated with H. Define

$$l_{ij} := \begin{cases} \delta_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } ij \in E \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian matrix of H is then the matrix $L := [l_{ij}]$. If x and y are any two vertices, then the classical distance d_{xy} is defined as the length of the shortest path connecting x and y. If there are multiple paths connecting two distinct vertices, then in applications, those two vertices are interpreted as better communicated. Thus, it makes more sense to define a distance which is shorter than the classical distance. Let $L^{\dagger}$ denote the Moore-Penrose inverse of L and the $(i, j)^{\text{th}}$-entry of $L^{\dagger}$ be $l_{ij}^{\dagger}$. The resistance distance R_{xy} between vertices x and y is defined by

$$R_{xy} := l_{xx}^{\dagger} + l_{yy}^{\dagger} - 2l_{xy}^{\dagger}. \tag{1.1}$$

In order to address the drawbacks of classical distance, Klein and Randić introduced the resistance distance (1.1) in [8]. A connected graph is a formal representation of an electrical network with unit resistance placed on each of its edges. If i and j are any two vertices, and if current is allowed to enter the electrical circuit

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only at i and to leave at j, then the effective resistance between i and j is same as the resistance distance R_{ij}. The resistance matrix is now defined by $[R_{ij}]$. Resistance matrices of connected graphs have a wide literature. Klein and Randić [8] showed that $R_{ij} : W \times W \to \mathbb{R}$ is a metric. A formula for the inverse of a resistance matrix is obtained in [1]. This in turn extends the remarkable formula of Graham and Lovász to find the inverse of the distance matrix of a tree. All resistance matrices are Euclidean distance matrices (EDMs); see Bapat and Raghavan [2]. Hence, the wide theory of EDMs are applicable to resistance matrices. Our interest on resistance matrices in this paper is the following inequality: If u and v are any two distinct vertices, then $R_{uv} \leq d_{uv}$; see Theorem D in [8].

1.1. Extension of resistance to digraphs. In [5], the concept of resistance distance is extended for digraphs. Let $G = (V, \mathcal{E})$ be a simple digraph with vertex set $V = \{1, 2, \dots, n\}$ and edge set $\mathcal{E}$. For $i, j \in V$, we write $(i, j) \in \mathcal{E}$ whenever there is a directed edge from i to j. For a vertex $i \in V$, the indegree δ_i^{in} and the outdegree δ_i^{out} are defined as follows:

$$\delta_i^{in} := |\{j \in V |\ (j, i) \in \mathcal{E}\}| \quad \text{and} \quad \delta_i^{out} := |\{j \in V |\ (i, j) \in \mathcal{E}\}|.$$

The Laplacian matrix of G is $L(G) = (l_{ij})$, where for each $i, j \in V$

$$l_{ij} := \begin{cases} \delta_i^{out} & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

A digraph is strongly connected if there is a directed path between any two distinct vertices. If $\delta_x^{out} = \delta_x^{in}$, then the vertex x is said to be balanced. A digraph is called balanced if every vertex is balanced. In this paper, we consider only strongly connected and balanced digraphs. With this assumption, the Laplacian $L(G)$ will have the following properties: $\text{rank}(L(G)) = n - 1$, row and column sums of $L(G)$ are equal to zero; see [5]. As usual, let $L^\dagger = (l_{ij}^\dagger)$ be the Moore-Penrose inverse of $L(G)$. It can be noted that $L(G)$ is not a symmetric matrix in general. From now on, we will use L to denote the Laplacian matrix $L(G)$.

The resistance r_{ij} between any two vertices i and j in V is defined by

$$r_{ij} := l_{ii}^\dagger + l_{jj}^\dagger - 2l_{ij}^\dagger. \tag{1.2}$$

In [5], by using certain specialized results on **Z**-matrices and the Moore-Penrose inverse, it is shown that

$$r_{ij} \geq 0 \quad \forall\, i, j,$$

and for each $i, j, k \in V$

$$r_{ij} \leq r_{ik} + r_{kj}.$$

For each distinct pair of vertices i and j in V, let d_{ij} be the length of the shortest directed path from i to j and define $d_{ii} := 0$. The non-negative real number d_{ij} is the classical distance between i and j. By numerical experiments, we noted that the inequality $r_{ij} \leq d_{ij}$ always holds.

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Example 1.1. Consider the graph below.

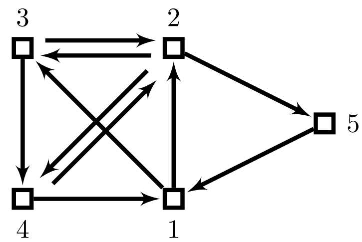

Figure 1.1. *A strongly connected and balanced digraph on* 5 *vertices.*

The Laplacian matrix and its Moore-Penrose inverse are

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L^\dagger = \begin{bmatrix} \frac{9}{35} & 0 & \frac{1}{35} & -\frac{3}{35} & -\frac{1}{5} \\ -\frac{4}{35} & \frac{1}{5} & -\frac{2}{35} & -\frac{1}{35} & 0 \\ -\frac{6}{35} & 0 & \frac{11}{35} & \frac{2}{35} & -\frac{1}{5} \\ -\frac{1}{35} & 0 & -\frac{4}{35} & \frac{12}{35} & -\frac{1}{5} \\ \frac{2}{35} & -\frac{1}{5} & -\frac{6}{35} & -\frac{2}{7} & \frac{3}{5} \end{bmatrix}.$$

The resistance and distance matrices of G are:

$$R = [r_{ij}] = \begin{bmatrix} 0 & \frac{16}{35} & \frac{18}{35} & \frac{27}{35} & \frac{44}{35} \\ \frac{24}{35} & 0 & \frac{22}{35} & \frac{3}{5} & \frac{4}{5} \\ \frac{32}{35} & \frac{18}{35} & 0 & \frac{19}{35} & \frac{46}{35} \\ \frac{23}{35} & \frac{19}{35} & \frac{31}{35} & 0 & \frac{47}{35} \\ \frac{26}{35} & \frac{6}{5} & \frac{44}{35} & \frac{53}{35} & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 3 & 0 \end{bmatrix}.$$

It is easily seen that $r_{ij} \leq d_{ij}$ for each i, j. Given a general strongly connected and balanced digraph, we do not know how to prove the above inequality. In this paper, when G is a directed cactus graph, we give a proof for this inequality.

1.2. Directed cactus graphs.

Definition 1.2. A directed cactus graph is a strongly connected digraph in which each edge is contained in exactly one directed cycle.

Here is an equivalent condition for a directed cactus: A digraph G is a directed cactus if and only if any two directed cycles of G share at most one common vertex. In a directed cactus, for each vertex i, $\delta_i^{in} = \delta_i^{out}$, and hence, it is balanced. The graph G given in Figure 1.2 is a directed cactus graph.

Distance matrices of directed cactus appear in [7]. An interesting formula for the determinant of the distance matrix $D := (d_{ij})$ of a cactoid graph and an expression for the inverse of D are computed in [7].

2. Preliminaries.

Definition 2.1. A directed cycle graph is a directed version of a cycle graph with all edges being oriented in the same direction. For $n > 1$, we shall use $C_n = (V, \mathcal{E})$ to denote a directed cycle on n vertices.

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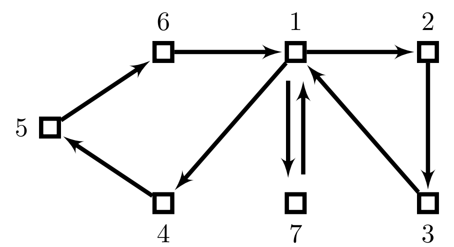

FIGURE 1.2. *A directed cactus graph on* 7 *vertices.*

An example of a directed cycle on 5 vertices is shown in Figure 2.1.

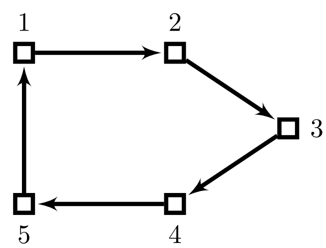

FIGURE 2.1. *Directed cycle Graph* C_5.

DEFINITION 2.2. Suppose $G = (V, \mathcal{E})$ is a digraph with vertex set $V = \{1, 2, ..., n\}$ and Laplacian matrix L. A spanning *tree* of G rooted at vertex i is a connected subgraph T with vertex set V such that

(i) Every vertex of T other than i has indegree 1.
(ii) The vertex i has indegree 0.
(iii) T has no directed cycles.

EXAMPLE 2.3. The graph H in Figure 2.2 has two spanning trees rooted at 1.

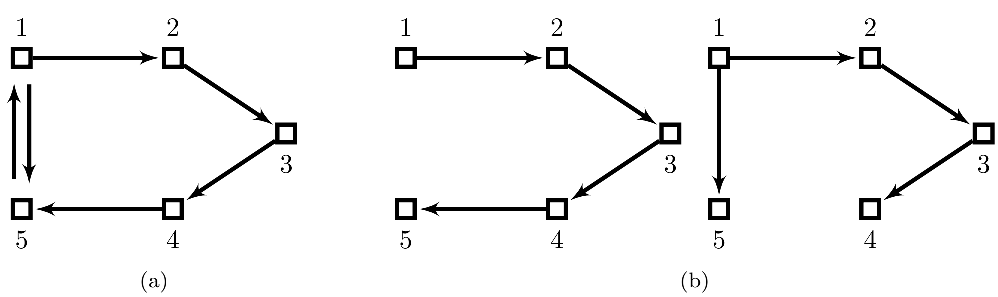

FIGURE 2.2. *(a) Digraph* H *(b) Spanning trees of* H *rooted at* 1.

We use the following notation. If Δ_1 and Δ_2 are non-empty subsets of $\{1, \ldots, n\}$ and $\pi : \Delta_1 \to \Delta_2$ is a bijection, then the pair $\{i, j\} \subset \Delta_1$ is called an inversion in π if $i < j$ and $\pi(i) > \pi(j)$. Let $n(\pi)$ be the number of inversions in π. For an $n \times n$ matrix A, $A[\Delta_1, \Delta_2]$ will denote the submatrix of A obtained

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by choosing rows and columns corresponding to Δ_1 and Δ_2, respectively. For $\Delta \subseteq \{1, 2, \ldots, n\}$, we define $\alpha(\Delta) = \sum_{i \in \Delta} i$. Our main tool will be the following theorem from [3].

THEOREM 2.4. (All minors matrix tree theorem) *Let $G = (V, E)$ be a digraph with vertex set $V = \{1, 2, \ldots, n\}$ and Laplacian matrix L. Let $\Delta_1, \Delta_2 \subset V$ be such that $|\Delta_1| = |\Delta_2|$. Then*

$$\det(L[\Delta_1^c, \Delta_2^c]) = (-1)^{\alpha(\Delta_1)+\alpha(\Delta_2)} \sum_F (-1)^{n(\pi)},$$

where the sum is over all spanning forests F such that

(a) *F contains exactly $|\Delta_1| = |\Delta_2|$ trees.*
(b) *each tree in F contains exactly one vertex in Δ_2 and exactly one vertex in Δ_1.*
(c) *each directed edge in F is directed away from the vertex in Δ_2 of the tree containing that directed edge (i.e., each vertex in Δ_2 is the root of the tree containing it).*

F defines a bijection $\pi : \Delta_1 \to \Delta_2$ such that $\pi(j) = i$ if and only if i and j are in the same oriented tree of F.

Let $\kappa(G, i)$ be the number of spanning trees of G rooted at i. By Theorem 2.4, it immediately follows that

$$\kappa(G, i) = \det(L[\{i\}^c, \{i\}^c]). \tag{2.1}$$

Since $\text{rank}(L) = n - 1$ and $L\mathbf{1} = L'\mathbf{1} = 0$, all the cofactors of L are equal. From (2.1), we conclude that $\kappa(G, i)$ is independent of i. From here on, we shall denote $\kappa(G, i)$ simply by $\kappa(G)$. Let $i, j, k \in V$. We introduce the following two notation.

1. Let $\#(F[\{i \to\}, \{j \to\}])$ denote the number of spanning forests F of G such that (i) F contains exactly 2 trees, (ii) each tree in F contains either i or j, and (iii) vertices i and j are the roots of the respective trees containing them.
2. Let $\#(F[\{k \to\}, \{j \to, i\}])$ denote the number of spanning forests F of G such that (i) F contains exactly 2 trees, (ii) each tree in F exactly contains either k or both i and j, and (iii) vertices k and j are the roots of the respective trees containing them.

From Theorem 2.4, we deduce the following proposition which will be used to prove our main result.

PROPOSITION 2.5. *Let $i, j \in V$ be two distinct vertices. Then*

(a)

$$\det(L[\{i, j\}^c, \{i, j\}^c]) = \#(F[\{i \to\}, \{j \to\}]).$$

(b) *If $i \neq n$ and $j \neq n$, then*

$$\det(L[\{n, i\}^c, \{n, j\}^c]) = (-1)^{i+j} \#(F[\{n \to\}, \{j \to, i\}]).$$

(c) *If $i \neq 1$ and $j \neq 1$, then*

$$\det(L[\{1, i\}^c, \{1, j\}^c]) = (-1)^{i+j} \#(F[\{1 \to\}, \{j \to, i\}]).$$

Proof. Substituting $\Delta_1 = \Delta_2 = \{i, j\}$ in Theorem 2.4, we have

$$\det(L[\{i, j\}^c, \{i, j\}^c]) = (-1)^{2i+2j} \sum_F (-1)^{n(\pi)}, \tag{2.2}$$

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where the sum is over all forests F such that (i) F contains exactly 2 trees, (ii) each tree in F contains either i or j, and (iii) vertices i and j are the roots of the respective trees containing them. Since for each such forest F, $\pi(i) = i$ and $\pi(j) = j$, there are no inversions in π. Thus, $n(\pi) = 0$. Hence, from (2.2), we have

$$\det(L[\{i,j\}^c, \{i,j\}^c]) = \#(F[\{i \rightarrow\}, \{j \rightarrow\}]).$$

This completes the proof of (a).

To prove (b), we substitute $\Delta_1 = \{n, i\}$ and $\Delta_2 = \{n, j\}$ in Theorem 2.4 to obtain

$$\det(L[\{n,i\}^c, \{n,j\}^c]) = (-1)^{2n+i+j} \sum_F (-1)^{n(\pi)}, \tag{2.3}$$

where the sum is over all forests F such that (i) F contains exactly 2 trees, (ii) each tree in F exactly contains either n or both i and j, and (iii) vertices n and j are the roots of the respective trees containing them. For each such forest F, $\pi(n) = n$ and $\pi(i) = j$. Since $i, j < n$, there are no inversions in π and so $n(\pi) = 0$. From (2.3), we have

$$\det(L[\{n,i\}^c, \{n,j\}^c]) = (-1)^{i+j} \#(F[\{n \rightarrow\}, \{j \rightarrow, i\}]).$$

Hence, (b) is proved. The proof of (c) is similar to the proof of (b). $\square$

The following is one of the main results in [5].

Theorem 2.6. *For every distinct $i, j \in V$,*

$$r_{ij} + r_{ji} = \frac{2}{\kappa(G)} \det(L[\{i,j\}^c, \{i,j\}^c])$$

where r_{ij} is defined in (1.2).

The following lemma can be verified by direct computation and appears in [5].

Lemma 2.7. *Let L be a $\mathbf{Z}$-matrix such that $L\mathbf{1} = L'\mathbf{1} = 0$ and $\operatorname{rank}(L) = n - 1$. If e is the vector of all ones in $\mathbb{R}^{n-1}$, then L can be partitioned as*

$$L = \begin{bmatrix} B & -Be \\ -e'B & e'Be \end{bmatrix},$$

where B is a square matrix of order $n-1$ and

$$L^\dagger = \begin{bmatrix} B^{-1} - \frac{1}{n}ee'B^{-1} - \frac{1}{n}B^{-1}ee' & -\frac{1}{n}B^{-1}e \\ -\frac{1}{n}e'B^{-1} & 0 \end{bmatrix} + \frac{e'B^{-1}e}{n^2}\mathbf{1}\mathbf{1}'.$$

Lemma 2.8. *Let A be an $n \times n$ matrix. If u and v belong to $\mathbb{R}^n$, then*

$$\det(A + uv') = \det(A) + v' \operatorname{adj}(A) u.$$

Proof. See Lemma 1.1 in [4]. $\square$

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3. Result. Let $G = (V, E)$ be a strongly connected and balanced digraph with vertex set $V = \{1, 2, \dots, n\}$, Laplacian matrix L and resistance matrix $R = (r_{ij})$. First, we prove some lemmas which will be used later.

Lemma 3.1. *Let $i, j \in V$. If $(i, j) \in E$ or $(j, i) \in E$, then*

$$\det(L[\{i,j\}^c, \{i,j\}^c]) \le \kappa(G).$$

Proof. Let $i, j \in V$. Without loss of generality, assume that $(i, j) \in E$. By Proposition 2.5(a), $\det(L[\{i,j\}^c, \{i,j\}^c])$ is equal to the number of spanning forests F of G such that (i) F contains exactly 2 trees, (ii) each tree in F contains either i or j, and (iii) vertices i and j are the roots of the respective trees containing them. Let F be one such forest. Now, $F + (i, j)$ is a spanning tree of G rooted at i. Moreover, each such spanning forest will give a unique spanning tree rooted at i. Since $\kappa(G)$ is the number of spanning trees of G rooted at i,

$$\det(L[\{i,j\}^c, \{i,j\}^c]) \le \kappa(G). \qquad \square$$

As G is balanced, we know that $\delta_i^{in} = \delta_i^{out}$ for any i. This common value will be called the degree of i.

Lemma 3.2. *Let $(i, j) \in E$. If either i or j has degree 1, then $r_{ij} \le 1$.*

Proof. Without loss of generality, let $i = 1$ and $j = n$. From Lemma 2.7, we have

$$L^\dagger = \begin{bmatrix} B^{-1} - \frac{1}{n}ee'B^{-1} - \frac{1}{n}B^{-1}ee' & -\frac{1}{n}B^{-1}e \\ -\frac{1}{n}e'B^{-1} & 0 \end{bmatrix} + \frac{e'B^{-1}e}{n^2}\mathbf{11}', \tag{3.1}$$

where $B = \det(L[\{n\}^c, \{n\}^c])$. Let $C = B^{-1}$, $C = (c_{ij})$, $x = Ce$ and $y = C'e$. By a well-known result on **Z**-matrices (Theorem 2.3 in [6]), we note that C is a non-negative matrix. Using (3.1), we have

$$\begin{aligned} r_{1n} &= l_{11}^\dagger + l_{nn}^\dagger - 2l_{1n}^\dagger \\ &= c_{11} - \frac{1}{n}y_1 - \frac{1}{n}x_1 + \frac{2}{n}x_1 \\ &= c_{11} - \frac{1}{n}(y_1 - x_1). \end{aligned} \tag{3.2}$$

We claim that $x_1 \le y_1$. To see this, we consider the following cases:

(i) Suppose degree of vertex 1 is one. For $k \in \{2, 3, \dots, n-1\}$,

$$\begin{aligned} c_{1k} &= \frac{(-1)^{1+k}}{\det(B)}\det(B[\{k\}^c, \{1\}^c]) \\ &= \frac{(-1)^{1+k}}{\det(L[\{n\}^c, \{n\}^c])}\det(L[\{n,k\}^c, \{n,1\}^c]). \end{aligned} \tag{3.3}$$

Using (2.1) and Proposition 2.5(b) in (3.3), we get

$$c_{1k} = \frac{\#(F[\{n \to\}, \{1 \to, k\}])}{\kappa(G)}, \tag{3.4}$$

where $\#(F[\{n \to\}, \{1 \to, k\}])$ is the number of spanning forests F of G such that (i) F contains exactly 2 trees, (ii) each tree in F exactly contains either n or both 1 and k, and (iii) vertices n and

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1 are the roots of the respective trees containing them. As degree of vertex 1 is one, $(1,n)$ is the only edge directed away from 1. So, it is not possible for a forest to have a tree such that the tree does not contain the vertex n but contains both the vertices 1 and k with 1 as the root. Therefore, no such forest F exists, and hence, by (3.4), $c_{1k}=0$ for each $k\in\{2,3,\dots,n-1\}$. Using the fact that C is a non-negative matrix, we have

$$x_1=\sum_{k=1}^{n-1}c_{1k}=c_{11}\leq\sum_{k=1}^{n-1}c_{k1}=y_1. \tag{3.5}$$

Hence, $x_1\leq y_1$.

(ii) Suppose degree of n is one. Let e_i be the vector $(0,\dots,1,\dots,0)'\in\mathbb{R}^{n-1}$ with 1 as its i^{th} coordinate. Then the Laplacian matrix L can be partitioned as

$$L=\begin{bmatrix} B & -e_1 \\ & \\ -e_p' & 1\end{bmatrix}, \tag{3.6}$$

for some $p\in\{1,2,\dots,n-1\}$. Let $\widetilde{B}=B-E_p$, where E_p is the $(n-1)\times(n-1)$ matrix with p^{th} column equal to e_1 and the remaining columns equal to zero. It can be seen that $\widetilde{B}$ has all row and column sums zero and so all its cofactors are identical. For $k\in\{2,3,\dots,n-1\}$, we have

$$\begin{aligned} c_{k1}&=\frac{(-1)^{1+k}}{\kappa(G)}\det(B[\{1\}^c,\{k\}^c])\\ &=\frac{(-1)^{1+k}}{\kappa(G)}\det\big((\widetilde{B}+E_p)[\{1\}^c,\{k\}^c]\big)\\ &=\frac{(-1)^{1+k}}{\kappa(G)}\det(\widetilde{B}[\{1\}^c,\{k\}^c])\\ &=\frac{1}{\kappa(G)}\frac{\text{cofsum}(\widetilde{B})}{(n-1)^2}.\end{aligned} \tag{3.7}$$

If $p=1$, then for each $k\in\{2,3,\dots,n-1\}$,

$$\begin{aligned} c_{1k}&=\frac{(-1)^{1+k}}{\kappa(G)}\det(B[\{k\}^c,\{1\}^c])\\ &=\frac{(-1)^{1+k}}{\kappa(G)}\det\big((\widetilde{B}+E_1)[\{k\}^c,\{1\}^c]\big)\\ &=\frac{(-1)^{1+k}}{\kappa(G)}\det(\widetilde{B}[\{k\}^c,\{1\}^c])\\ &=\frac{1}{\kappa(G)}\frac{\text{cofsum}(\widetilde{B})}{(n-1)^2}.\end{aligned} \tag{3.8}$$

Therefore, from (3.7) and (3.8), $c_{k1}=c_{1k}$ for each $k\in\{2,3,\dots,n-1\}$. Thus,

$$x_1=c_{11}+\sum_{k=2}^{n-1}c_{1k}=c_{11}+\sum_{k=2}^{n-1}c_{k1}=y_1. \tag{3.9}$$

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If $p \neq 1$, then for each $k \in \{2, 3, \ldots, n-1\}$, we have

$$\begin{aligned} c_{1k} &= \frac{(-1)^{1+k}}{\kappa(G)} \det(B[\{k\}^c, \{1\}^c]) \\ &= \frac{(-1)^{1+k}}{\kappa(G)} \det\big((\widetilde{B} + E_p)[\{k\}^c, \{1\}^c]\big) \\ &= \frac{(-1)^{1+k}}{\kappa(G)} \det\big(\widetilde{B}[\{k\}^c, \{1\}^c] + E_p[\{k\}^c, \{1\}^c]\big). \end{aligned} \tag{3.10}$$

Let u_ν be the vector $(0, \ldots, 0, 1, 0, \ldots, 0)' \in \mathbb{R}^{n-2}$ with 1 as its ν^{th} coordinate. From (3.10) and Lemma 2.8, we have

$$\begin{aligned} c_{1k} &= \frac{(-1)^{1+k}}{\kappa(G)} \det\big(\widetilde{B}[\{k\}^c, \{1\}^c] + u_1 u_{p-1}'\big) \\ &= \frac{(-1)^{1+k}}{\kappa(G)} \big(\det(\widetilde{B}[\{k\}^c, \{1\}^c]) + u_{p-1}' \operatorname{adj}(\widetilde{B}[\{k\}^c, \{1\}^c]) u_1\big) \\ &= \frac{1}{\kappa(G)} \frac{\operatorname{cofsum}(\widetilde{B})}{(n-1)^2} + \frac{(-1)^{1+k}}{\kappa(G)} (\operatorname{adj}(\widetilde{B}[\{k\}^c, \{1\}^c]))_{p-1,1} \\ &= \frac{1}{\kappa(G)} \frac{\operatorname{cofsum}(\widetilde{B})}{(n-1)^2} + \frac{(-1)^{1+k+p}}{\kappa(G)} \det(\widetilde{B}[\{1, k\}^c, \{1, p\}^c]). \end{aligned} \tag{3.11}$$

Note that $\widetilde{B}$ is a matrix with off-diagonal entries as 0 or -1, has rank $n-1$ and has row and column sums zero. Therefore, $\widetilde{B}$ will be the Laplacian of a strongly connected and balanced digraph. Using Proposition 2.5(c) for the Laplacian matrix $\widetilde{B}$, we get

$$\det(\widetilde{B}[\{1, k\}^c, \{1, p\}^c]) = (-1)^{k+p} \lambda,$$

where λ is a non-negative number representing the number of a particular type of forests as described in Proposition 2.5(c). Substituting the above expression in (3.11), we get

$$\begin{aligned} c_{1k} &= \frac{1}{\kappa(G)} \frac{\operatorname{cofsum}(\widetilde{B})}{(n-1)^2} + \frac{(-1)^{1+2k+2p}}{\kappa(G)} \lambda \\ &= \frac{1}{\kappa(G)} \frac{\operatorname{cofsum}(\widetilde{B})}{(n-1)^2} - \frac{\lambda}{\kappa(G)} \\ &\leq \frac{1}{\kappa(G)} \frac{\operatorname{cofsum}(\widetilde{B})}{(n-1)^2}. \end{aligned} \tag{3.12}$$

From (3.7) and (3.12), we get $c_{1k} \leq c_{k1}$ for each $k \in \{2, 3, \ldots, n-1\}$. Thus,

$$x_1 = c_{11} + \sum_{k=2}^{n-1} c_{1k} \leq c_{11} + \sum_{k=2}^{n-1} c_{k1} = y_1. \tag{3.13}$$

So, $x_1 \leq y_1$.

This proves our claim. In view of (3.2), we now obtain

$$r_{1n} \leq c_{11} = \frac{\det(L[\{1, n\}^c, \{1, n\}^c])}{\kappa(G)}. \tag{3.14}$$

By Lemma 3.1, it follows that $r_{1n} \leq 1$. The proof is complete. ∎

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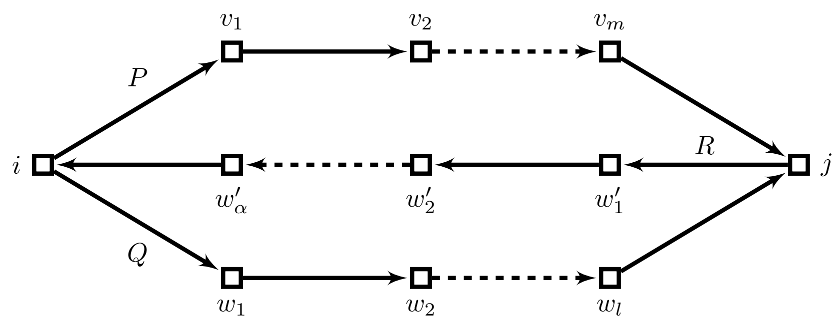
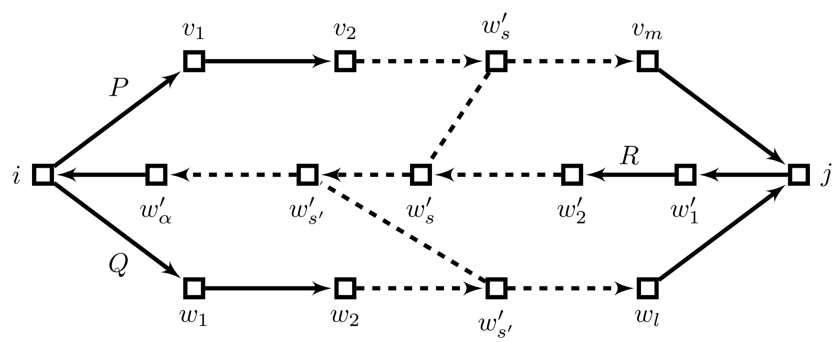
Lemma 3.3. *Let $G = (V, E)$ be a directed cactus graph on n vertices. Then there is a unique directed path from i to j.*

Proof. Let $i, j \in V$. Since G is strongly connected there exists a directed path from i to j. Let $P : i \to v_1 \to v_2 \to \cdots \to v_m \to j$ be one such path. If possible, let $Q : i \to w_1 \to w_2 \to \cdots \to w_l \to j$ be another directed path from i to j.

First, we assume all the internal vertices of P and Q are distinct. Since G is strongly connected, there will be a path from j to i. Let $R : j \to w'_1 \to w'_2 \to \cdots \to w'_\alpha \to i$ be a directed path from j to i. If R has no internal vertex in common with P and Q, then each edge of R will become a part of two distinct cycles $j \to w'_1 \to \cdots \to w'_\alpha \to i \to v_1 \to \cdots \to v_m \to j$ and $j \to w'_1 \to \cdots \to w'_\alpha \to i \to w_1 \to \cdots \to w_l \to j$ (see Figure 3.1(a)). This is a contradiction to the assumption that G is a directed cactus graph. Suppose R has some internal vertices in common with P and Q. Let w'_s and $w'_{s'}$ be the vertices in R such that no vertex of R after w'_s is a vertex of P and no vertex of R after $w'_{s'}$ is a vertex of Q. Without loss of generality, we assume $s' > s$. This makes each edge of the path $w'_{s'} \to w'_{s'+1} \to \cdots \to w'_\alpha \to i$ to be a part of two distinct cycles $w'_{s'} \to w'_{s'+1} \to \cdots \to w'_\alpha \to i \to v_1 \to \cdots \to w'_s \to \cdots \to w'_{s'}$ and $w'_{s'} \to w'_{s'+1} \to \cdots \to w'_\alpha \to i \to w_1 \to \cdots \to w'_{s'}$ (see Figure 3.1(b)), which is a contradiction.



(a)



(b)

Figure 3.1.

Suppose P and Q have some internal vertices in common. Let v_s be the first vertex of P in common with Q. This means there are two internally vertex disjoint directed paths from i to v_s. By a similar argument as above we get a contradiction. Hence, the directed path from i to j is unique. ∎

Lemma 3.4. *Let $V := \{1, \ldots, n\}$ and $G = (V, E)$ be a directed cactus graph. Suppose $(i, j) \in E$. If both*

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i and j have degree greater than one, then V can be partitioned into three disjoint sets

(a) $\{i,j\}$,
(b) $V_j(i \rightarrow)$,
(c) $V_i(j \rightarrow)$,

where $V_\nu(\delta \rightarrow) = \{k \in V \setminus \{\delta,\nu\} : \exists$ a directed path from δ to k which does not pass through $\nu\}$ (see Figure 3.2).

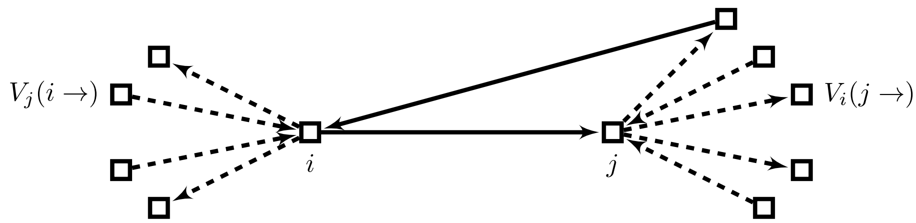

FIGURE 3.2. *Partition of a directed cactus graph.*

Proof. As i and j have degree greater than one, $V_j(i \rightarrow)$ and $V_i(j \rightarrow)$ are non empty. By definition,

$$\{i,j\} \cap V_j(i \rightarrow) = \emptyset \quad \text{and} \quad \{i,j\} \cap V_i(j \rightarrow) = \emptyset.$$

We now see that $V_j(i \rightarrow) \cap V_i(j \rightarrow) = \emptyset$. If possible, let $k \in V_j(i \rightarrow) \cap V_i(j \rightarrow)$. Let $P : i \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m \rightarrow k$ be a directed path from i to k which does not pass through j and $R : j \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_\alpha \rightarrow k$ be a directed path from j to k which does not pass through i (see Figure 3.3). Since $v_1 \neq j$, the edges (i,j) and (i,v_1) are not same. Hence, there are two different directed paths P and $i \rightarrow j \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_\alpha \rightarrow k$ from i to k. This contradicts Lemma 3.3. Thus, $V_j(i \rightarrow) \cap V_i(j \rightarrow) = \emptyset$.

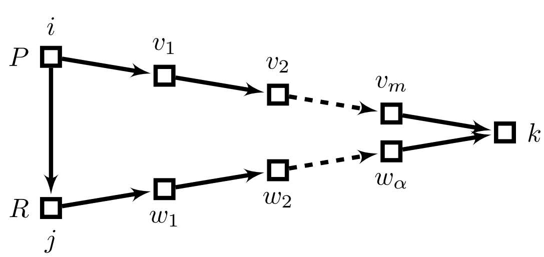

FIGURE 3.3. *Directed paths P and R.*

Let $k \in V$ be such that $k \notin \{i,j\}$ and $k \notin V_j(i \rightarrow)$. Since G is strongly connected there exists a directed path, say P from i to k. However, P must pass through j. Thus, the part of P between vertices j and k is a directed path from j to k which does not pass through i. So, $k \in V_i(j \rightarrow)$. Hence, $V = \{i,j\} \cup V_j(i \rightarrow) \cup V_i(j \rightarrow)$ is a disjoint partition of V. ∎

For a subgraph $\widetilde{G}$ of G, we use $V(\widetilde{G})$ to denote the vertex set of $\widetilde{G}$. We now prove our main result.

THEOREM 3.5. *Let $G = (V,E)$ be a directed cactus graph with $V = \{1,2,\ldots,n\}$. If $R = (r_{ij})$ and $D = (d_{ij})$ are the resistance and distance matrices of G, respectively, then $r_{ij} \leq d_{ij}$ for each $i,j \in \{1,2,\ldots,n\}$.*

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Proof. By triangle inequality, it suffices to show that if $(i,j) \in E$, then $r_{ij} \leq 1$. Let $(i,j) \in E$. In view of Lemma 3.2, it suffices to show this inequality when both i and j have degree greater than one. Without loss of generality, assume $i = 1$ and $j = n$. By Lemma 3.4, the vertex set V can be partitioned into three disjoint sets

(a) $\{1, n\}$,
(b) $V_n(1 \rightarrow)$,
(c) $V_1(n \rightarrow)$.

Let L be the Laplacian matrix of G. From Lemma 2.7, we have

$$L^\dagger = \begin{bmatrix} B^{-1} - \frac{1}{n}ee'B^{-1} - \frac{1}{n}B^{-1}ee' & -\frac{1}{n}B^{-1}e \\ -\frac{1}{n}e'B^{-1} & 0 \end{bmatrix} + \frac{e'B^{-1}e}{n^2}\mathbf{11'}. \tag{3.15}$$

where $B = \det(L[\{n\}^c, \{n\}^c])$. Let $C = B^{-1}$, $C = (c_{ij})$, $x = Ce$ and $y = C'e$. Note that C is a non-negative matrix. Using (3.15), we have

$$\begin{aligned} r_{1n} &= l^\dagger_{11} + l^\dagger_{nn} - 2l^\dagger_{1n} \\ &= c_{11} - \frac{1}{n}y_1 - \frac{1}{n}x_1 + \frac{2}{n}x_1 \\ &= c_{11} - \frac{1}{n}(y_1 - x_1). \end{aligned} \tag{3.16}$$

As in proof of Lemma 3.2, it suffices to show that $x_1 \leq y_1$. Let $k \in \{2, 3, \ldots, n-1\}$. Then by (3.4)

$$c_{1k} = \frac{\#(F[\{n \rightarrow\}, \{1 \rightarrow, k\}])}{\kappa(G)}, \tag{3.17}$$

where $\#(F[\{n \rightarrow\}, \{1 \rightarrow, k\}])$ is the number of spanning forests of G such that (i) F contains exactly 2 trees, (ii) each tree in F exactly contains either n or both k and 1, and (iii) vertices n and 1 are the roots of the respective trees containing them. We shall show that for each $k \in V_n(1 \rightarrow)$, such a forest F exists and is unique. Fix $k \in V_n(1 \rightarrow)$.

Existence: Since $\kappa(G) = \det(B) \neq 0$, there is a spanning tree T of G rooted at 1. Since the edge $(1,n)$ is the only directed path from 1 to n in G, it must be an edge in T. By removing the edge $(1,n)$ from T, we obtain a spanning forest $\overline{F}$ containing exactly two trees $\overline{T_1}$ and $\overline{T_n}$ rooted at 1 and n, respectively. It remains to show that $k \in V(\overline{T_1})$. In order to do this, we prove that

$$V(\overline{T_1}) = V_n(1 \rightarrow) \cup \{1\} \quad \text{and} \quad V(\overline{T_n}) = V_1(n \rightarrow) \cup \{n\}.$$

Let $v \in V(\overline{T_1}) \smallsetminus \{1\}$. Then there is a directed path from 1 to v in G which does not pass through n. This implies $V(\overline{T_1}) \subseteq V_n(1 \rightarrow) \cup \{1\}$. Similarly, if $w \in V(\overline{T_n}) \backslash \{n\}$ then there is a directed path from n to w in G which does not pass through 1 and so $V(\overline{T_n}) \subseteq V_1(n \rightarrow) \cup \{n\}$. Let $v \in V_n(1 \rightarrow)$ be such that $v \notin V(\overline{T_1})$. Since $V = V(\overline{T_1}) \cup V(\overline{T_n})$, we have $v \in V(\overline{T_n}) \subseteq V_1(n \rightarrow) \cup \{n\}$. Thus,

$$v \in V_n(1 \rightarrow) \cap V_1(n \rightarrow),$$

which is a contradiction. Hence, $V(\overline{T_1}) = V_n(1 \rightarrow) \cup \{1\}$. Similarly, $V(\overline{T_n}) = V_1(n \rightarrow) \cup \{n\}$. Thus, $k \in V(\overline{T_1})$ and so

$$\#(F[\{n \rightarrow\}, \{1 \rightarrow, k\}]) \neq 0.$$

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Hence, a forest with the required properties exists.

Uniqueness: Let $\widetilde{F}$ be another forest other than $\overline{F}$. Suppose $\widetilde{T_1}$ and $\widetilde{T_n}$ be the trees in $\widetilde{F}$ rooted at 1 and n, respectively. It can be easily seen

$$V(\widetilde{T_1}) = V_n(1 \rightarrow) \cup \{1\} = V(\overline{T_1}) \quad \text{and} \quad V(\widetilde{T_n}) = V_1(n \rightarrow) \cup \{n\} = V(\overline{T_n}).$$

If the trees $\overline{T_1}$ and $\widetilde{T_1}$ are not identical, then there will be a vertex $v \in V(\overline{T_1})$ such that there are two different directed paths from 1 to v. This contradicts Lemma 3.3. So, the trees $\overline{T_1}$ and $\widetilde{T_1}$ are identical. The same happens with $\overline{T_n}$ and $\widetilde{T_n}$. Thus, the forest $\overline{F}$ is unique and so

$$\#(F[\{n \rightarrow\}, \{1 \rightarrow, k\}]) = 1 \tag{3.18}$$

for each $k \in V_n(1 \rightarrow)$.

Since for every $k \notin V_n(1 \rightarrow)$, there is no directed path from 1 to k that does not pass through n, $\#(F[\{n \rightarrow\}, \{1 \rightarrow, k\}]) = 0$. From (3.17) and (3.18), for each $k \in \{2, 3, \ldots, n-1\}$

$$c_{1k} = \begin{cases} \frac{1}{\kappa(G)} & \text{if } k \in V_n(1 \rightarrow) \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

Let $V_n(\rightarrow 1) = \{k \in V \setminus \{1, n\} : \exists \text{ directed path from } k \text{ to } 1 \text{ which does not pass through } n\}$. Now, we shall show that $V_n(1 \rightarrow) \subset V_n(\rightarrow 1)$. Let $k \in V_n(1 \rightarrow)$ and $P : 1 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m \rightarrow k$ be a directed path from 1 to k which does not pass through n. If possible, assume $k \notin V_n(\rightarrow 1)$, i.e., every directed path from k to 1 contains the vertex n. Since G is strongly connected, there is at least one such path say $Q : k \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_l \rightarrow n \rightarrow w'_1 \rightarrow \cdots \rightarrow w'_\alpha \rightarrow 1$ (see Figure 3.4). Since $v_1 \neq n$, the edges $(1, n)$ and $(1, v_1)$ are not the same. Hence, there are two different directed paths $1 \rightarrow n$ and $1 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m \rightarrow k \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_l \rightarrow n$ from 1 to n. This contradicts Lemma 3.3. Hence, $V_n(1 \rightarrow) \subset V_n(\rightarrow 1)$.

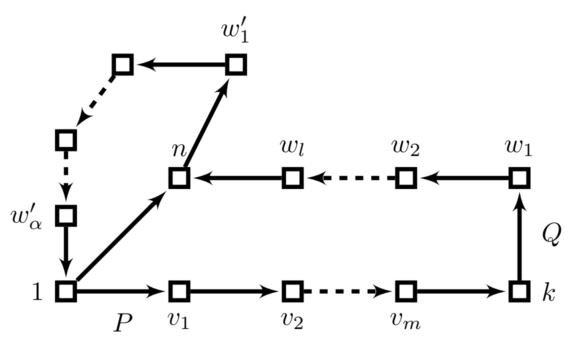

Figure 3.4. *Directed paths P and Q.*

If $k \in \{2, 3, \ldots, n-1\}$, then

$$\begin{aligned} c_{k1} &= \frac{(-1)^{1+k}}{\det(B)} \det(B[\{1\}^c, \{k\}^c]) \\ &= \frac{(-1)^{1+k}}{\det(L[\{n\}^c, \{n\}^c])} \det(L[\{n, 1\}^c, \{n, k\}^c]). \end{aligned} \tag{3.20}$$

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Using (2.1) and Proposition 2.5(b) in (3.20), we get

$$c_{k1} = \frac{\#(F[\{n \to\}, \{k \to, 1\}])}{\kappa(G)}, \tag{3.21}$$

where $\#(F[\{n \to\}, \{k \to, 1\}])$ is the number of spanning forests of G such that (i) F contains exactly 2 trees, (ii) each tree in F exactly contains either n or both k and 1, and (iii) vertices n and k are the roots of the respective trees containing them. We shall show that for each $k \in V_n(1 \to)$, $\#(F[\{n \to\}, \{k \to, 1\}]) \geq 1$.

Consider the induced subgraph $\widetilde{G}$ of G with vertex set $V(\widetilde{G}) = V_n(1 \to) \cup \{1\}$. Note that for each vertex $x \in V(\widetilde{G})$ there is a directed path from 1 to x in $\widetilde{G}$. Since $V(\widetilde{G}) \smallsetminus \{1\} \subset V_n(\to 1)$, for every $x \in V(\widetilde{G})$ there is a directed path from x to 1 in G which does not pass through n. For fixed $x \in V(\widetilde{G}) \smallsetminus \{1\}$, let $P : x \to v_1 \to v_2 \to \cdots \to v_m \to 1$ be one such path. We claim that each internal vertex of P is a vertex in $\widetilde{G}$. If possible, let v_s be an internal vertex of P which is not in $V(\widetilde{G})$. Since $v_s \notin V_n(1 \to)$, there is a directed path say $Q : 1 \to w_1 \to w_2 \to \cdots \to w_l \to n \to w_{l+1} \to \cdots \to w_{l'} \to v_s$ from 1 to v_s which passes through n (see Figure 3.5). Also there will be a directed path say $R : 1 \to w'_1 \to w'_2 \to \cdots \to w'_\alpha \to x$ from 1 to x which does not pass through n. Note that, there are two different paths Q and $1 \to w'_1 \to w'_2 \to \cdots \to w'_\alpha \to x \to v_1 \to v_2 \to \cdots \to v_s$ from 1 to v_s. This contradicts Lemma 3.3. Hence, each internal vertex of P is a vertex in $\widetilde{G}$. Thus, P is a directed path from x to 1 in $\widetilde{G}$.

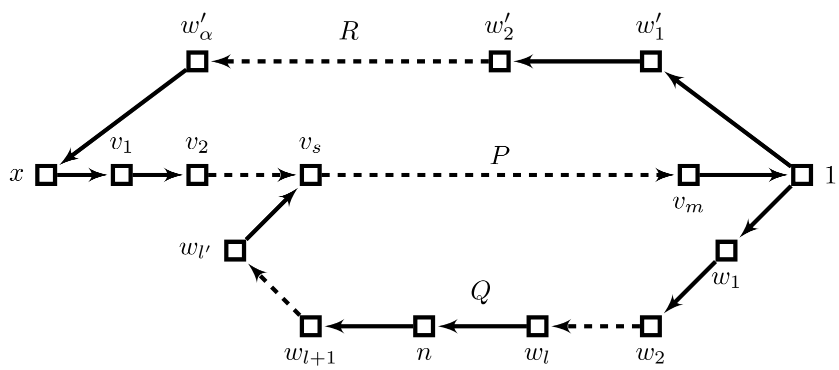

FIGURE 3.5. *Directed paths P,Q and R in $\widetilde{G}$.*

This means $\widetilde{G}$ is a strongly connected digraph. Let $\widetilde{L}$ be the Laplacian matrix of $\widetilde{G}$. Then $\text{rank}(\widetilde{L}) = |V(\widetilde{G})| - 1$. Let $k \in V_n(1 \to)$. Then $\kappa(\widetilde{G}, k) = \det(\widetilde{L}[\{k\}^c, \{k\}^c]) \neq 0$. Hence, there exists an oriented spanning tree of $\widetilde{G}$ rooted at k. Let $\widetilde{T}$ be a spanning tree of $\widetilde{G}$ rooted at k and $\overline{F}$ be the spanning forest of G with trees $\overline{T_1}$ and $\overline{T_n}$ obtained as before. Let F' be the forest consisting of trees $\widetilde{T}$ and $\overline{T_n}$. Since $\widetilde{T}$ and $\overline{T_n}$ are rooted at k and n, respectively, and $V(\widetilde{T}) = V_n(1 \to) \cup \{1\}$ and $V(\overline{T_n}) = V_1(n \to) \cup \{n\}$, it follows that F' is a required spanning forest. Hence, for each $k \in V_n(1 \to)$, we have $\#(F[\{n \to\}, \{k \to, 1\}]) \geq 1$. From (3.21), we have

$$c_{k1} \geq \frac{1}{\kappa(G)}, \quad \text{whenever } k \in V_n(1 \to). \tag{3.22}$$

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Since C is a non-negative matrix, from (3.19) and (3.22), we have

$$
\begin{aligned}
x_1 &= \sum_{k=1}^{n-1} c_{1k} \\
&= c_{11} + \sum_{k \in V_n(1\to)} c_{1k} \\
&= c_{11} + \sum_{k \in V_n(1\to)} \frac{1}{\kappa(G)} \\
&\le c_{11} + \sum_{k \in V_n(1\to)} c_{k1} \le \sum_{k=1}^{n-1} c_{k1} = y_1.
\end{aligned}
\tag{3.23}
$$

Hence, $r_{1n} \le 1$. This completes the proof. ∎

We illustrate our result by the following example.

EXAMPLE 3.6. Consider the strongly connected and directed cactus graph G. The resistance and dis-

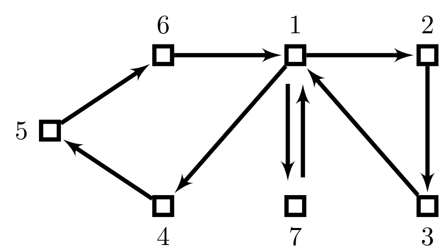

FIGURE 3.6. *The directed cactus graph G.*

tance matrices of G are:

$$
R = \begin{bmatrix}
0 & \frac{6}{7} & \frac{8}{7} & \frac{5}{7} & 1 & \frac{9}{7} & 1 \\
\frac{8}{7} & 0 & \frac{2}{7} & \frac{13}{7} & \frac{15}{7} & \frac{17}{7} & \frac{15}{7} \\
\frac{6}{7} & \frac{12}{7} & 0 & \frac{11}{7} & \frac{13}{7} & \frac{15}{7} & \frac{13}{7} \\
\frac{9}{7} & \frac{15}{7} & \frac{17}{7} & 0 & \frac{2}{7} & \frac{4}{7} & \frac{16}{7} \\
1 & \frac{13}{7} & \frac{15}{7} & \frac{12}{7} & 0 & \frac{2}{7} & 2 \\
\frac{5}{7} & \frac{11}{7} & \frac{13}{7} & \frac{10}{7} & \frac{12}{7} & 0 & \frac{12}{7} \\
1 & \frac{13}{7} & \frac{15}{7} & \frac{12}{7} & 2 & \frac{16}{7} & 0
\end{bmatrix}
\quad \text{and} \quad
D = \begin{bmatrix}
0 & 1 & 2 & 1 & 2 & 3 & 1 \\
2 & 0 & 1 & 3 & 4 & 5 & 3 \\
1 & 2 & 0 & 2 & 3 & 4 & 2 \\
3 & 4 & 5 & 0 & 1 & 2 & 4 \\
2 & 3 & 4 & 3 & 0 & 1 & 3 \\
1 & 2 & 3 & 2 & 3 & 0 & 2 \\
1 & 2 & 3 & 2 & 3 & 4 & 0
\end{bmatrix}.
$$

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It can be seen that for each pair of vertices, the resistance is less than the shortest distance.

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