



M-COMMUTING MAPS ON TRIANGULAR AND STRICTLY TRIANGULAR INFINITE MATRICES*

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Abstract. Let $N_\infty(F)$ be the ring of infinite strictly upper triangular matrices with entries in an infinite field. The description of the commuting maps defined on $N_\infty(F)$, i.e. the maps $f: N_\infty(F) \rightarrow N_\infty(F)$ such that $[f(X), X] = 0$ for every $X \in N_\infty(F)$, is presented. With the use of this result, the form of m -commuting maps defined on $T_\infty(F)$ – the ring of infinite upper triangular matrices, i.e. the maps $f: T_\infty(F) \rightarrow T_\infty(F)$ such that $[f(X), X^m] = 0$ for every $X \in T_\infty(F)$, is found.

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1. Introduction. Denote by $[a, b]$ the standard *commutator* of a, b , i.e. $[a, b] = ab - ba$. A map $f: R \rightarrow R$ is called *m-commuting* if $[f(x), x^m] = 0$ for all $x \in R$. In particular, if $m = 1$, then f is simply *commuting*. The study of such maps was inspired by Posner [17] who proved that if a prime ring has a nonzero centralizing derivation, then it must be commutative. This theorem was generalized in many ways (see for instance [5, 13, 15, 16, 19]). The first general result regarding commuting maps comes from Brešar [7] who showed that additive commuting maps f over a simple unital ring R are of the form $f(x) = \lambda x + \mu(x)$ for some $\lambda \in Z(R)$ and additive $\mu: R \rightarrow Z(R)$ where $Z(R)$ denotes the center of R . This form is usually called a *standard form* for the commuting map. There are plenty of results on commuting maps, and the reader is referred to the survey paper [8] for acquaintance with the development of the theory of commuting maps and the various results that have been established.

In this article, we will be interested in upper triangular matrices.

Commuting maps on triangular algebras were first studied in [10], and further in [11, 12]. In 2000, Beidar, Brešar, and Chebotar [3] proved that any linear commuting map on $T_r(F)$ – the algebra of $r \times r$ upper triangular matrices over F , $f: T_r(F) \rightarrow T_r(F)$ is of the standard form: $f(x) = \lambda x + \mu(x)$ for some $\lambda \in F$ and linear map $\mu: T_r(F) \rightarrow Z(T_r(F))$. Recently, in [6], J. Bounds extended some of these results to the case $N_r(F)$ – the ring of strictly upper triangular matrices over a field F of characteristic zero. He proved that if $f: N_r(F) \rightarrow N_r(F)$ is a commuting linear map, then there exists $\lambda \in F$ and an additive map $\mu: N_r(F) \rightarrow \Omega$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in N_r(F)$, where $\Omega = \{ae_{1,r-1} + be_{1,r} + ce_{2,r} : a, b, c \in F\}$ and $e_{i,j}$ denotes the standard matrix unit.

Following these results, we would like to examine m -commuting additive maps defined on two rings of $\mathbb{N} \times \mathbb{N}$ matrices. The first of them is the ring of all upper triangular matrices; the second one is its subring consisting of all matrices with zero main diagonal. We will assume that these rings are defined over an infinite field. Our rings will be denoted by $T_\infty(F)$, $N_\infty(F)$. First, we will prove the theorem that describes commuting maps on $N_\infty(F)$.

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THEOREM 1.1. *Let F be an infinite field. If $f: N_\infty(F) \rightarrow N_\infty(F)$ is an additive, commuting map, then there exists $\lambda \in F$ such that $f(X) = \lambda X$ for all $X \in N_\infty(F)$.*

For some classes of rings R (see for instance [2, 4]), all m -commuting maps are commuting. For instance, in [9], Brešar and Hvala studied 2-power commuting additive maps and showed that if R is a prime ring of characteristic 2 with the extended centroid, then every such map is commuting. Later, Beidar, Fong, Lee and Wong [4] (see also [2, 14]) extended this result to m -power commuting additive maps and proved that if a ring R of characteristic either equal to 0 or greater than m , also with extended centroid, then each such map is also commuting. However, it should be mentioned that this property does not hold for arbitrary R . A nice example of a ring without this property is $M_2(GF(2))$ (see [14]). In this paper, we will show that if F is a field of appropriate characteristic, then the m -commuting maps on $T_\infty(F)$ and the commuting maps on $N_\infty(F)$ coincide.

THEOREM 1.2. *Let m be a natural number and let F be an infinite field whose characteristic is not a divisor of m . If $f: T_\infty(F) \rightarrow T_\infty(F)$ is an additive m -commuting map, then there exists $\lambda \in F$ and an additive map $\mu: T_\infty(F) \rightarrow F$ such that $f(X) = \lambda X + \mu(X)I_\infty$ for all $X \in T_\infty(F)$.*

2. Notation and basic information. Before we start, let us introduce the notation used in the paper.

For every $i, j \in \mathbb{N}$, we will use E_{ij} for the matrix unit – the matrix with 1 in the position (i, j) and 0 in every other position. It is known that $E_{ij} \cdot E_{kl} = \delta_{jk}E_{il}$, where δ is the Kronecker delta.

If $A = [a_{ij}] \in T_\infty(F)$, then we will write

$$A = \sum_{i \leq j} a_{ij} E_{ij}.$$

Note that this is not a sum, but only a notation.

The symbol I_∞ will stand for the $\mathbb{N} \times \mathbb{N}$ identity matrix, whereas J_∞ will be equal to $\sum_{i=1}^{\infty} E_{i,i+1}$.

By $D_\infty(F)$, we will mean the subring of $T_\infty(F)$ consisting of all diagonal matrices.

Now let us present some auxiliary results that hold for $UT_n(F)$ and $UT_\infty(F)$ – the rings of $n \times n$, and infinite, respectively, unitriangular matrices.

LEMMA 2.1 (Lemma 2.1, [18]). *Suppose that K is an arbitrary field and $n \in \mathbb{N}$.*

1. *If $g, h \in UT_n(K)$ are such that $g_{i,i+1} = h_{i,i+1} \neq 0$ for all $1 \leq i \leq n-1$, then g and h are conjugates in $UT_n(K)$.*
2. *If $g, h \in UT_\infty(K)$ are such that $g_{i,i+1} = h_{i,i+1} \neq 0$ for all $1 \leq i$, then g and h are conjugates in $UT_\infty(K)$.*

Here, $UT_\infty(F)$ ($UT_n(F)$, respectively) is the group of all infinite ($n \times n$) upper triangular matrices with only 1's in the main diagonal.

From the above lemma, we obtain the corollary.

COROLLARY 2.2. *Let F be a field. For every $A = \sum_{i < j} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$ for all $i \in \mathbb{N}$, there exists $B \in T_\infty(F)$ such that $B^{-1}AB = J_\infty$.*

Proof. Let A satisfy the assumption given in the lemma. First, let us notice that there exists $B_1 \in D_\infty(F)$ such that $(B_1^{-1}AB_1)_{i,i+1} = 1$ for all $i \in \mathbb{N}$. Namely, one can define $B_1 \in D_\infty(F)$ inductively:

$$(B_1)_{11} = 1, \quad (B_1)_{i+1,i+1} = (B_1)_{ii} \cdot (A_{i,i+1})^{-1} \quad \text{for } i \geq 1.$$

The obtained matrix $B_1^{-1}AB_1$ fulfills the assumption of Lemma 2.1. Hence, there exists $B_2 \in T_\infty(F)$ such that $B_2^{-1}(B_1^{-1}AB_1)B_2 = J_\infty$. Therefore, the desired matrix B equals B_1B_2 . \square

3. Proof of Theorem 1.1. We start with characterizing centralizers of matrices of some simple form.

LEMMA 3.1. *For any field F , we have $C(J_\infty) = \left\{ \sum_{i=1}^{\infty} \alpha_i (J_\infty)^i : \alpha_i \in F \right\}$.*

Note that $\sum_{i=1}^{\infty} \alpha_i (J_\infty)^i$ is an abbreviation for $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_i E_{k,k+i}$. (For more information about writing the matrices in the form of series, we recommend article [1].)

Proof. Suppose that $BJ_\infty = J_\infty B$, where $B = \sum_{i < j} b_{ij} E_{ij}$. From this equality, we obtain a system of equations:

$$b_{n,m-1} - b_{n+1,m} = 0 \quad \text{for all } n, m \in \mathbb{N}.$$

From this system, it follows that for every $i \in \mathbb{N}$, there exists $\alpha_i \in F$ such that for all $k \in \mathbb{N}$ we have $b_{k,k+i} = \alpha_i$. Thus, $B = \sum_{i=1}^{\infty} \alpha_i (J_\infty)^i$. \square

Lemma 3.1 can be extended to the following.

LEMMA 3.2. *Let $A = \sum_{i=1}^{\infty} a_{i,i+1} E_{i,i+1}$, where $a_{i,i+1} \neq 0$ for all $i \geq 1$. Then, $C(A) = \left\{ \sum_{i=1}^{\infty} \alpha_i A^i : \alpha_i \in F \right\}$.*

Proof. From Corollary 2.2, we know that there exists $S \in D_\infty(F)$ such that $A = SJ_\infty S^{-1}$. Substituting this relation to $BA = AB$, we get $BSJ_\infty S^{-1} = SJ_\infty S^{-1}B$, and hence $(S^{-1}BS)J_\infty = J_\infty(S^{-1}BS)$. From Lemma 3.1, we then get $S^{-1}BS = \sum_{i=1}^{\infty} \alpha_i (J_\infty)^i$. Therefore, $B = S \left[\sum_{i=1}^{\infty} \alpha_i (J_\infty)^i \right] S^{-1}$. Since S is diagonal, for every $n, m \in \mathbb{N}$, we have

$$\begin{aligned} B_{nm} &= S_{nn} \left[\sum_{i=1}^{\infty} \alpha_i (J_\infty)^i \right] S_{mm}^{-1} = S_{nn} \cdot \alpha_{m-n} [(J_\infty)^{m-n}]_{nm} \cdot S_{mm}^{-1} \\ &= \alpha_{m-n} [S(J_\infty)^{m-n} S^{-1}]_{nm}, \end{aligned}$$

so we can write that

$$B = \sum_{i=1}^{\infty} \alpha_i S (J_\infty)^i S^{-1} = \sum_{i=1}^{\infty} \alpha_i (S^{-1}J_\infty S)^i = \sum_{i=1}^{\infty} \alpha_i (A)^i.$$

This completes the proof. \square

PROPOSITION 3.3. *Let F be an infinite field and let $f: N_\infty(F) \rightarrow N_\infty(F)$ be a commuting map. For every $x \in F \setminus \{0\}$, there exists $\lambda_x \in F$ such that $f(xJ_\infty) = \lambda_x J_\infty$.*

Proof. Since F is infinite, there exists a set $A = \{\alpha_i : i \in \mathbb{N}\}$ such that α_i are pairwise different and $\alpha_i \neq x$ for all $i \in \mathbb{N}$.

Let $W_1 = \sum_{i=1}^{\infty} \alpha_i E_{i,i+1}$. Then, by Lemma 3.2

$$f(W_1) = \sum_{i=1}^{\infty} b_i (W_1)^i, \quad \text{where } W_1^k = \sum_{i=1}^{\infty} \alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1} E_{i,i+k}.$$

Define W_2 as $\sum_{i=1}^{\infty} (x - \alpha_i) E_{i,i+1}$. Using Lemma 3.2 again, we get

$$f(W_2) = \sum_{i=1}^{\infty} c_i (W_2)^i,$$

where

$$W_2^k = \sum_{i=1}^{\infty} (x - \alpha_i) (x - \alpha_{i+1}) \cdots (x - \alpha_{i+k-1}) E_{i,i+k}.$$

Since f is additive, we have $f(W_1) + f(W_2) = f(xJ_{\infty})$. We use Lemma 3.2 one more time, and we see that

$$\sum_{k=1}^{\infty} s_k x^k (J_{\infty})^k = \sum_{k=1}^{\infty} b_k (W_1)^k + \sum_{k=1}^{\infty} c_k (W_2)^k,$$

for some $b_k, c_k, s_k \in F$.

Expanding the above expressions, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} s_k x^k \left(\sum_{i=1}^{\infty} E_{i,i+k} \right) &= \sum_{k=1}^{\infty} b_k \sum_{i=1}^{\infty} \alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1} E_{i,i+k} \\ &\quad + \sum_{k=1}^{\infty} c_k \sum_{i=1}^{\infty} (x - \alpha_i) (x - \alpha_{i+1}) \cdots (x - \alpha_{i+k-1}) E_{i,i+k}, \\ \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} s_k x^k E_{i,i+k} &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} b_k \alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1} E_{i,i+k} \\ &\quad + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} c_k (x - \alpha_i) (x - \alpha_{i+1}) \cdots (x - \alpha_{i+k-1}) E_{i,i+k}. \end{aligned}$$

Comparing the coefficients from the positions $(i, i+k)$, we conclude that for $i, k \in \mathbb{N}$, the following equations hold:

$$s_k x^k = b_k \alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1} + c_k (x - \alpha_i) (x - \alpha_{i+1}) \cdots (x - \alpha_{i+k-1}).$$

In particular, for $i = 1, 2, 3$, we get

$$(3.1) \quad \begin{cases} s_k x^k = b_k \alpha_1 \alpha_2 \cdots \alpha_k + c_k (x - \alpha_1) (x - \alpha_2) \cdots (x - \alpha_k) \\ s_k x^k = b_k \alpha_2 \alpha_3 \cdots \alpha_{k+1} + c_k (x - \alpha_2) (x - \alpha_3) \cdots (x - \alpha_{k+1}) \\ s_k x^k = b_k \alpha_3 \alpha_4 \cdots \alpha_{k+2} + c_k (x - \alpha_3) (x - \alpha_4) \cdots (x - \alpha_{k+2}). \end{cases}$$

For $k = 1$, system (3.1) takes the form:

$$\begin{cases} s_1 x = b_1 \alpha_1 + c_1 (x - \alpha_1) \\ s_1 x = b_1 \alpha_2 + c_1 (x - \alpha_2) \\ s_1 x = b_1 \alpha_3 + c_1 (x - \alpha_3), \end{cases}$$

so we can conclude that $b_1 = c_1$.

Consider now $k \geq 2$. From equating the first two and the last two equations of (3.1), we get

$$(3.2) \quad \begin{cases} b_k(\alpha_{k+1} - \alpha_1)\alpha_2 \cdots \alpha_k = c_k(\alpha_{k+1} - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k) \\ b_k(\alpha_{k+2} - \alpha_2)\alpha_3 \cdots \alpha_{k+1} = c_k(\alpha_{k+2} - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_{k+1}). \end{cases}$$

Suppose that $b_k \neq 0$. Then, according to (3.2) and the assumptions, c_k is also nonzero. Thus, we should have

$$\frac{\alpha_{k+1}}{\alpha_2} = \frac{x - \alpha_{k+1}}{x - \alpha_2} \Rightarrow (\alpha_{k+1} - \alpha_2)x = 0.$$

However, as all the elements α_i are pairwise different, this would imply that $x = 0$ - contrary to the assumption. Thus, we must have $b_k = 0 = c_k$ for $k \geq 2$. This yields

$$\begin{aligned} f(xJ_\infty) &= \sum_{i=1}^{\infty} s_i x^i (J_\infty)^i = \sum_{i=1}^{\infty} b_i (W_1)^i + \sum_{i=1}^{\infty} c_i (W_2)^i = b_1 W_1 + c_1 W_2 \\ &= b_1 (W_1 + W_2) = b_1 x J_\infty. \end{aligned} \quad \square$$

Using the lemmas about conjugacy, we can now prove the following.

LEMMA 3.4. Assume that F is an infinite field and that $f: N_\infty(F) \rightarrow N_\infty(F)$ is a commuting map. If $A = \sum_{i < j} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$ for all $i \in \mathbb{N}$, then there exists $\lambda_A \in F$ such that $f(A) = \lambda_A A$.

Proof. From Corollary 2.2, we know that $A = T^{-1}xJ_\infty T$ for some $T \in T_\infty(F)$ and some $x \in F \setminus \{0\}$. We define $h: N_\infty(F) \rightarrow N_\infty(F)$ by $h(X) = Tf(T^{-1}XT)T^{-1}$.

Applying the commutativity of f , we see that

$$\begin{aligned} 0 &= T[f(T^{-1}XT), T^{-1}XT]T^{-1} \\ &= T(f(T^{-1}XT)T^{-1}XT - T^{-1}XTf(T^{-1}XT))T^{-1} \\ &= Tf(T^{-1}XT)T^{-1}X - XTf(T^{-1}XT)T^{-1} = [h(X), X]. \end{aligned}$$

Hence, h is commuting and $h(xJ_\infty) = \lambda_x J_\infty$ by Proposition 3.3. Then,

$$Tf(A)T^{-1} = Tf(T^{-1}(TAT^{-1})T)T^{-1} = h(xJ_\infty) = \lambda_{xJ_\infty} J_\infty = \lambda_{J_\infty} TAT^{-1}.$$

Multiplying the left and right sides by T^{-1} and T , respectively, yields $f(A) = \lambda_A A$. \square

Now, we wish to extend Lemma 3.4 to all elements of $N_\infty(F)$. In order to do so, let us introduce the set that we will denote by \mathcal{S} :

$$\mathcal{S} = \{B = (b_{ij}) \in N_\infty(F) : b_{i,i+1} \neq 0\}.$$

This set has a very nice property that is established below.

LEMMA 3.5. Let F be a field. Every element of $N_\infty(F)$ can be written as a sum of at most two elements of \mathcal{S} .

Proof. If $a_{i,i+1} \neq 0$ for all $i \in \mathbb{N}$, then A is in \mathcal{S} , so there is nothing to prove. If A is not in \mathcal{S} , then we can define B_1 and B_2 as follows:

$$(B_1)_{ij} = \begin{cases} a_{i,i+1} - b_i & \text{if } j = i + 1 \\ a_{ij} & \text{if } j > i + 1, \end{cases} \quad (B_2)_{ij} = \begin{cases} b_i & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where b_i is a nonzero element of F distinct from $a_{i,i+1}$. Obviously, since $|F| > 2$, for every $a_{i,i+1}$, such b_i exists. One can see that B_1, B_2 are in \mathcal{S} , and $A = B_1 + B_2$, so we are done. \square

Now, we can present the proof of the main theorem.

Proof of Theorem 1.1. First take noncommuting $A, B \in \mathcal{S}$. Then, by Lemma 3.4, $f(A) = \lambda_A A$, $f(B) = \lambda_B B$ for some $\lambda_A, \lambda_B \in F$. Since f is commuting and additive, the following holds:

$$\begin{aligned} 0 &= [f(A+B), A+B] = [\lambda_A A + \lambda_B B, A+B] = [\lambda_A A, B] + [\lambda_B B, A] \\ &= (\lambda_A - \lambda_B)[A, B]. \end{aligned}$$

As A, B do not commute, we must have $\lambda_A = \lambda_B$. Consider now A and B from \mathcal{S} that do commute. Then, there exists $C \in \mathcal{S}$ such that the pairs A, C and A, B are not commuting, so we have $\lambda_A = \lambda_C$ and $\lambda_B = \lambda_C$. Thus, there exists $\lambda \in F$ such that $f(A) = \lambda A$ for all $A \in \mathcal{S}$. Now, we use Lemma 3.5 and additivity, and we get $f(X) = \lambda X$ for all $X \in N_\infty(F)$. \square

4. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. First, we will make some observation about triangular matrices. After that, we will make use of the results from previous section.

LEMMA 4.1. *Let $m \in \mathbb{N}$ and let F be a field whose characteristic is not a divisor of m . If $f: T_\infty(F) \rightarrow T_\infty(F)$ is an m -commuting additive map, then $f(Z(T_\infty(F))) \subseteq Z(T_\infty(F))$.*

Proof. Clearly, since f is additive, $f(0) = 0$.

Let $\alpha, \beta \in F \setminus \{0\}$. Moreover, let $i < j$. From

$$f(\beta I_\infty + \alpha E_{ij})(\beta I_\infty + \alpha E_{ij})^m = (\beta I_\infty + \alpha E_{ij})^m f(\beta I_\infty + \alpha E_{ij}),$$

it follows

$$[f(\beta I_\infty + \alpha E_{ij})](\beta^m I_\infty + m\alpha\beta^{m-1}E_{ij}) = (\beta^m I_\infty + m\alpha\beta^{m-1}E_{ij})[f(\beta I_\infty + \alpha E_{ij})],$$

and further

$$f(\beta I_\infty + \alpha E_{ij})E_{ij} = E_{ij}f(\beta I_\infty + \alpha E_{ij}).$$

Obviously, for every nonzero $\theta \in F$, we also have

$$f((\beta + \theta)I_\infty + \alpha E_{ij})E_{ij} = E_{ij}f((\beta + \theta)I_\infty + \alpha E_{ij}).$$

The difference of the two above equations yields

$$(4.3) \quad f(\beta I_\infty)E_{ij} = E_{ij}f(\beta I_\infty).$$

As (4.3) holds for any i, j , we have $f(\beta I_\infty) \subseteq Z(T_\infty(F))$. \square

LEMMA 4.2. *Let $m \in \mathbb{N}$, $m \geq 2$, and let F be a field such that $\text{char}(F) \nmid m$ and F contain at least $m+1$ elements. If $f: T_\infty(F) \rightarrow T_\infty(F)$ is an additive m -commuting map with $f(Z(T_\infty(F))) \subseteq Z(T_\infty(F))$, then f is commuting.*

Proof. Let $\alpha_1 = 1, \alpha_2, \dots, \alpha_m$ be distinct nonzero elements of F . For every i , $1 \leq i \leq m$, and every $X \in T_\infty(F)$, we have

$$[f(X + \alpha_i I_\infty), (X + \alpha_i I_\infty)^m] = 0.$$

As f is additive and $f(Z(T_\infty(F))) \subseteq Z(T_\infty(F))$, the latter implies

$$[f(X), (X + \alpha_i I_\infty)^m] = 0 \quad \text{for all } 1 \leq i \leq m.$$

Expansion of $(X + \alpha_i I_\infty)^m$ yields

$$\sum_{k=1}^m \binom{m}{k} \alpha_i^{m-k} [f(X), X^k] = 0 \quad \text{for all } 1 \leq i \leq m.$$

Using the matrix notation, we can rewrite the above system as follows:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_2^{m-1} & \alpha_2^{m-2} & \alpha_2^{m-3} & \cdots & \alpha_2 \\ \alpha_3^{m-1} & \alpha_3^{m-2} & \alpha_3^{m-3} & \cdots & \alpha_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_m^{m-1} & \alpha_m^{m-2} & \alpha_m^{m-3} & \cdots & \alpha_m \end{bmatrix} \begin{bmatrix} \binom{m}{1} [f(X), X] \\ \binom{m}{2} [f(X), X^2] \\ \binom{m}{3} [f(X), X^3] \\ \vdots \\ \binom{m}{m} [f(X), X^m] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since the coefficient matrix of the above system is the Vandermonde matrix, and $\alpha_i \neq \alpha_j$, its determinant is nonzero. Thus,

$$\left[\binom{m}{1} [f(X), X] \quad \binom{m}{2} [f(X), X^2] \quad \cdots \quad \binom{m}{m} [f(X), X^m] \right]^T = [0 \ 0 \ \cdots \ 0]^T.$$

Since $\text{char}(F) \nmid m$, we get now $[f(X), X] = 0$. □

Note that analogously as in the previous section, the following is true.

LEMMA 4.3. *For an arbitrary field F , we have*

$$C_{T_\infty(F)}(J_\infty) = \left\{ \sum_{i=0}^{\infty} \alpha_i J_\infty^i : \alpha_i \in F \right\}.$$

Lemma 4.3 can be used to prove the following proposition.

PROPOSITION 4.4. *Let F be an infinite field. If $f : T_\infty(F) \rightarrow T_\infty(F)$ is an additive commuting map, then there exists $\delta \in F$ such that for every $X \in N_\infty(F)$ $f(X) = \delta X + \mu_X I_\infty$ for some $\mu_X \in F$.*

Proof. From Lemma 4.3, we know that $f(J_\infty) = Y + \lambda_J I_\infty$, where $Y = \sum_{i=1}^{\infty} \alpha_i J_\infty^i \in N_\infty(F)$.

By Corollary 2.2 for any $A \in N_\infty(F)$ satisfying condition $a_{i,i+1} \neq 0$, there exists T such that $A = T^{-1} J_\infty T$. Consider the map $h : T_\infty(F) \rightarrow T_\infty(F)$, $h(X) = T f(T^{-1} X T) T^{-1}$. For this h , we have

$$\begin{aligned} [h(X), X] &= T f(T^{-1} X T) T^{-1} \cdot X - X \cdot T f(T^{-1} X T) T^{-1} \\ &= T [f(T^{-1} X T), T^{-1} X T] T^{-1}, \end{aligned}$$

i.e. h is commuting. This means that

$$f(A) = T^{-1} h(J_\infty) T = T^{-1} (Y + \mu_J I_\infty) T = T^{-1} Y T + \mu_J I_\infty.$$

Since $T \in N_\infty(F)$, the matrix $T^{-1} Y T$ is in $N_\infty(F)$ as well. Thus, $f(A) = B + \mu_J I_\infty$ for some $B \in N_\infty(F)$. Now notice that, by Lemma 3.5, for every $A \in N_\infty(F)$, we have $f(A) = B + \mu_A I_\infty$ for some $B \in N_\infty(F)$.

Consider now the map $f|_{N_\infty(F)}$. From the above result, it follows that this map can be written as a sum $f|_{N_\infty(F)} = g_1 + g_2$, where if $f(A) = B + \mu_A I_\infty$, then $g_1(A) = B$ and $g_2(A) = \mu_A I_\infty$. Thus, g_1 is a commuting map defined on $N_\infty(F)$. From Theorem 1.1, we know that each such map is given by the formula $g_1(X) = \delta X$ for some fixed $\delta \in F$. Hence, $f|_{N_\infty(F)}(A) = \delta A + \mu_A I_\infty$. \square

Now, we will focus on the subset of $T_\infty(F)$, namely $D_\infty(F)$.

PROPOSITION 4.5. *Let F be an infinite field. If $f: T_\infty(F) \rightarrow T_\infty(F)$ is an additive commuting map, then there exists $\delta \in F$ such that for every $X = \sum_{i=1}^\infty x_i E_{ii}$ with $x_i \neq x_j$ for $i \neq j$, and $x_i \neq 0$, $f(X) = \delta X + \mu_X I_\infty$ for some $\mu_X \in F$.*

Proof. We use our result from Proposition 4.4, namely, the fact that $f(J_\infty) = \delta J_\infty + \mu_J I_\infty$.

As f is commuting, we have

$$[f(X + J_\infty), X + J_\infty] = 0.$$

Since $[f(X), X] = [f(J_\infty), J_\infty] = [\mu_J I_\infty, X] = 0$, the above equality simplifies to

$$(f(X) - \delta X)J_\infty = J_\infty(f(X) - \delta X).$$

Thus, for some $\alpha_i \in F$ ($i \in \mathbb{N}$), we have $f(X) = \delta X + \sum_{i=0}^\infty \alpha_i J_\infty^i$. Clearly, $f(X)$ commutes with X , so

$$\left(\delta_J X + \sum_{i=0}^\infty \alpha_i J_\infty^i \right) X = X \left(\delta_J X + \sum_{i=0}^\infty \alpha_i J_\infty^i \right),$$

and therefore

$$\sum_{i=0}^\infty \alpha_i J_\infty^i \cdot \sum_{i=1}^\infty x_i E_{ii} = \sum_{i=1}^\infty x_i E_{ii} \cdot \sum_{i=0}^\infty \alpha_i J_\infty^i.$$

From the above equality, we get that

$$\alpha_{k-n} x_k = x_n \alpha_{k-n} \quad \text{for all } n \leq k.$$

Since $x_k \neq x_n$, the latter forces $\alpha_i = 0$ for all i . This way we obtain $f(X) = \delta X + \alpha_0 I_\infty$. \square

To generalize Proposition 4.5 to all diagonal matrices, let us introduce the set, denoted by \mathcal{D} , that is the subset of $D_\infty(F)$ consisting of all matrices with diagonal entries that are distinct nonzero elements of F . For this set, the following lemma holds.

LEMMA 4.6. *Let F be an infinite field. For every $X \in D_\infty(F)$, there exist $A, B \in \mathcal{D}$ such that $X = A + B$.*

Proof. Assume that $X = \sum_{i=1}^\infty x_i E_{ii}$ and

$$A = \sum_{i=1}^\infty a_i E_{ii}, \quad X = \sum_{i=1}^\infty (x_i - a_i) E_{ii}.$$

The elements a_i can be found inductively as follows.

Element a_1 can be simply any element of F that is not equal to 0 and a_1 . Suppose now that a_1, a_2, \dots, a_n are already chosen. Then, a_{n+1} can be any element of F satisfying conditions

$$\begin{aligned} a_{n+1} &\neq 0, \quad a_{n+1} \neq x_{n+1}, \\ a_{n+1} - x_{n+1} &\neq a_1 - x_1, a_2 - x_2, a_3 - x_3, \dots, a_n - x_n. \end{aligned}$$

Since F is infinite, it is possible to find such a_{n+1} . \square

Now, we will prove our second main result.

Proof of Theorem 1.2. From Proposition 4.4, we know that for every $X \in N_\infty(F)$, we have $f(X) = \lambda X + \mu_X I_\infty$, whereas from Proposition 4.5 and Lemma 4.6, it follows that $f(Y) = \delta Y + \mu_Y I_\infty$.

Let X, Y be arbitrary noncommuting elements of $N_\infty(F), D_\infty(F)$, respectively. From $[f(X + Y), X + Y] = 0$, additivity of f , and the form of f , one gets $(\lambda - \delta)[X, Y]$. Since X, Y do not commute, this forces $\lambda = \delta$. Thus, $f(X) = \lambda X + \mu_X I_\infty$ for every $X \in N_\infty(F) \cup D_\infty(F)$.

Now, it suffices to notice that as every element of $T_\infty(F)$ is a sum of one element from $N_\infty(F)$ and one from $D_\infty(F)$, for any $Z \in T_\infty(F)$, we have $f(Z) = \lambda Z + \mu_Z I_\infty$. The latter can be written as $f(Z) = \lambda Z + \mu(Z) I_\infty$. Moreover, since f is additive, μ is additive as well. \square

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