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# THE LEAST LAPLACIAN EIGENVALUE OF THE UNBALANCED UNICYCLIC SIGNED GRAPHS WITH $K$ PENDANT VERTICES*

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**Abstract.** Let $\Gamma = (G, \sigma)$ be a signed graph and $L(\Gamma) = D(G) - A(\Gamma)$ be the Laplacian matrix of $\Gamma$, where $D(G)$ is the diagonal matrix of vertex degrees of the underlying graph $G$ and $A(\Gamma)$ is the adjacency matrix of $\Gamma$. It is well-known that the least Laplacian eigenvalue $\lambda_n$ is positive if and only if $\Gamma$ is unbalanced. In this paper, the unique signed graph (up to switching equivalence) which minimizes the least Laplacian eigenvalue among unbalanced connected signed unicyclic graphs with $n$ vertices and $k$ pendant vertices is characterized.

**Key words.** Unicyclic signed graph, Unbalanced signed graph, Pendant vertex, Least Laplacian eigenvalue.

**AMS subject classifications.** 05C50.

**1. Introduction.** All graphs in this paper are simple and connected. A *signed graph* $\Gamma$ is a pair $(G, \sigma)$, where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ is a *sign function* on the edges of $G$. The graph $G$ is called the *underlying graph* of $\Gamma$. The sign of a *cycle* $C$ is given by $sign(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is $+1$ (resp., $-1$) is called *positive* (resp., *negative*). A signed graph $\Gamma$ is *balanced* if all cycles of $\Gamma$ are positive. Otherwise $\Gamma$ is *unbalanced.* If all edges in $\Gamma$ are positive (resp., negative), then $\Gamma$ is denoted by $(G, +)$ (resp., $(G, -)$), and we say that such a signature is *all-positive* (resp., *all-negative*).

Most of the concepts defined for (unsigned) graphs can be directly extended to signed graphs. For example, the *degree* $d_v$ of a vertex $v$ in a signed graph $\Gamma$ is the number of edges incident with vertex $v$. A *pendant vertex* is a vertex of degree one. Furthermore, a *subgraph* of $\Gamma$ is a subgraph of $G$ with the signature induced by $\sigma$, which is of course a signed graph. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$. The *order* of $\Gamma$ is the order of $G$ and it is denoted by $|\Gamma|$. A signed graph is called *$k$-cyclic* if its underlying graph is $k$-cyclic, which means that $|E| = |G| + k - 1$. The *girth* $g(\Gamma)$ of $\Gamma$ is the length of the shortest cycle in $\Gamma$.

For a signed graph $\Gamma = (G, \sigma)$ and a subset $U \subset V(G)$, $\Gamma[U]$ denotes the *induced subgraph* by $U$ and $\Gamma - U = \Gamma[V(G) \setminus U]$. We also write $\Gamma - \Gamma[U]$ instead of $\Gamma - U$. Let $\Gamma^U$ be the signed graph obtained from $\Gamma$ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$. That is, $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for each edge $e$ between $U$ and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph $\Gamma^U$ is said to be *switching equivalent* to $\Gamma$. For a comprehensive bibliography on signed graphs, see [14].

The *adjacency matrix* of $\Gamma$ is $A(\Gamma) = (a^\sigma_{ij})$, where $a^\sigma_{ij} = \sigma(ij)$ if $v_i$ is adjacent to $v_j$, and $a^\sigma_{ij} = 0$ otherwise, where $\sigma(ij)$ is the signature of the edge $v_iv_j$. The *Laplacian matrix* of $\Gamma$ is $L(\Gamma) = D(G) - A(\Gamma)$, where $D(G)$ is the *diagonal matrix* of vertex degrees. The adjacency (resp., Laplacian) matrices of two switching equivalent signed graphs are similar. In fact, any switching arising from vertex subset $U$ can be described by a diagonal

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matrix $S_U = diag(s_i)$ having $s_i = 1$ for each $i \in U$ and $s_i = -1$ otherwise. Hence, $A(\Gamma) = S_U A(\Gamma^U) S_U$ and $L(\Gamma) = S_U L(\Gamma^U) S_U$. Thus, by studying the spectrum of a signed graph, we are really studying all the signed graphs in a switching isomorphism class. The eigenvalues $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$ of $L(\Gamma)$ are called the *Laplacian eigenvalues* of a signed graph $\Gamma$. There have been many investigations of the area of the Laplacian eigenvalues of signed graphs, one can see [2, 3, 8, 9, 10, 15, 16].

Let $\mathcal{U}(n, g, k, \bar{\sigma})$ $(3 \leq g \leq n-k)$ denote the set of unbalanced unicyclic signed graphs of order $n$ having girth $g$ and $k$ pendant vertices such that there is a unique negative edge and that edge belongs to the cycle. We denote $\lambda(\Gamma)$ as the least Laplacian eigenvalue of a sign graph $\Gamma$. In this paper, we investigate the signed graph $\Gamma \in \mathcal{U}(n, g, k, \bar{\sigma})$ which has the smallest least Laplacian eigenvalue. For the analogous results for signless Laplacian eigenvalues of non-bipartite unicyclic graphs with $k$ pendant vertices see [11]. The remainder of the paper is organized as follows: In Section 2, we introduce some basic facts, and in Section 3, we identify the structure of unbalanced unicyclic singed graph with $n$ vertices and $k$ pendant vertices which arrives the smallest least Laplacian eigenvalue.

**2. Preliminaries.** For a signed graph $\Gamma$ with $n$ vertices, let $x = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector with respect to a Laplacian eigenvalue $\lambda$, we also call $x_i$ the value on the vertex $i$. The following expression is known as the eigenvector equation $\lambda x = Lx$ of $x$ at vertex $v$:

$$\lambda x_v = d_v x_v - \sum_{u \sim v} \sigma(uv) x_u, \tag{2.1}$$

where $u \sim v$ means that vertex $u$ is adjacent to vertex $v$.

The matrix $L(\Gamma) = D(G) - A(\Gamma)$ is symmetric and positive semi-definite. Furthermore, the following expression is well known:

$$x^T L(\Gamma) x = \sum_{vw \in E(G)} (x_v - \sigma(vw) x_w)^2. \tag{2.2}$$

The interlacing theorem, applied to signed graphs, gives the following.

Lemma 2.1. ([2]) *Let $\Gamma$ be an unbalanced unicyclic signed graph, $\Gamma - e$ denotes the signed graph obtained from $\Gamma$ by deleting the edge $e$. Then we have*

$$\lambda_n(\Gamma - e) \leq \lambda_n(\Gamma) \leq \lambda_{n-1}(\Gamma - e) \leq \lambda_{n-1}(\Gamma) \leq \cdots \leq \lambda_1(\Gamma - e) \leq \lambda_1(\Gamma).$$

If a signed graph $\Gamma$ is balanced, then the spectrum of $\Gamma$ is exactly that of $G$, and the least nonzero eigenvalue of $\Gamma$ is equal to $\lambda_{n-1}(\Gamma) = \alpha(G)$, which is the *algebraic connectivity* of $G$. The algebraic connectivity of a graph has received much attention. The eigenvectors corresponding to the algebraic connectivity, called *Fiedler vectors*, are also of interest and are investigated. See [6, 7] for more details.

Let $\Gamma = (G, \sigma) \in \mathcal{U}(n, g, k, \bar{\sigma})$, and the vertices of $\Gamma$ be labelled as $v_1, v_2, \ldots, v_n$, with all edges positive except a negative edge $e = v_{i_1} v_{i_2}$ on the cycle. Let $\Gamma'$ be a copy of $\Gamma$, in which we replace the label of vertex $v_i$ by $u_i$ for each $i = 1, \ldots, n$. Let $W$ be a (unsigned) unicyclic graph on $2n$ vertices which is obtained from the union $(\Gamma - e) \cup (\Gamma' - e')$ by adding two edges $v_{i_1} u_{i_2}$ and $v_{i_2} u_{i_1}$, where the edge $e' = u_{i_1} u_{i_2}$ (see Figure 1 on the right). Ordering the vertices of $W$ as $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Then we can obtain the following lemma by methods similar to [4].

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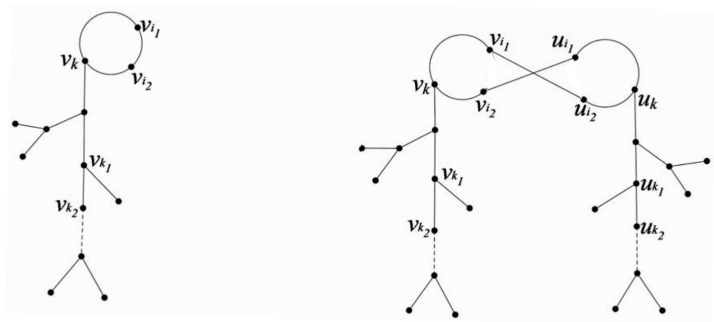


FIGURE 1. $\Gamma$ *and* $W = (\Gamma - e) \cup (\hat{\Gamma} - \hat{e}) + v_{i_1}u_{i_2} + v_{i_2}u_{i_1}$.

LEMMA 2.2. $\lambda(\Gamma) = \alpha(W)$. *Moreover, if* $x \in \mathbb{R}^n$ *is an eigenvector of* $\Gamma$ *corresponding to* $\lambda(\Gamma)$*, then* $(x^T, -x^T)^T \in \mathbb{R}^{2n}$ *is an eigenvector of* $W$ *corresponding to* $\alpha(W)$.

*Proof.* Let $x \in \mathbb{R}^n$ be an eigenvector of $\Gamma$ corresponding to $\lambda(\Gamma)$. Then it is easy to verify that $(x^T, -x^T)^T \in \mathbb{R}^{2n}$ is an eigenvector of $W$ corresponding to eigenvalue $\lambda(\Gamma)$. Noting that $W$ is unsigned and connected, so $\lambda(\Gamma) \geq \alpha(W) > 0$.

Next we will show that $\lambda(\Gamma) = \alpha(W)$. Let $(y^T, z^T)^T \in \mathbb{R}^{2n}$ be an eigenvector of $W$ corresponding to $\alpha(W)$, where $y, z \in \mathbb{R}^n$. We can check that $(z^T, y^T)^T$ is also an eigenvector of $W$ corresponding to $\alpha(W)$. Thus, if $y \neq z$, then $(y^T - z^T, z^T - y^T)^T$ is also an eigenvector of $W$ respect to $\alpha(W)$. Hence, $y - z$ is an eigenvector of $\Gamma$ corresponding to the eigenvalue $\alpha(W)$ by (2.1), and hence, $\alpha(W) \geq \lambda(\Gamma)$. If $y = z$, then $y$ is an eigenvector of $G$ corresponding to the eigenvalue $\alpha(W)$. Note that the edge $v_{i_1}v_{i_2}$ in $G$ is positive and now is denoted it by $\tilde{e}$, and $G - \tilde{e} = \Gamma - e$. By Lemma 2.1, we have $\alpha(W) \geq \alpha(G) \geq \alpha(G - \tilde{e}) = \alpha(\Gamma - e) \geq \lambda(\Gamma)$. So we have $\alpha(W) \geq \lambda(\Gamma)$ in both cases. Hence, $\lambda(\Gamma) = \alpha(W)$, and $(x^T, -x^T)^T$ is an eigenvector of $W$ corresponding to $\alpha(W)$ by (2.1), namely, $(x^T, -x^T)^T$ is a Fiedler vector of $W$. ∎

A path in $G$ is *pure* provided it does not contain more than two cut vertices in any block of $G$. The following important lemma is from [7]:

LEMMA 2.3. ([7]) *Let* $G$ *be a graph and* $y$ *be a Fiedler vector of* $G$*. Then exactly one of the following two cases occurs:*

*Case A: There is a single block* $E_0$ *in* $G$ *which contains both positively and negatively valued vertices. Each other block has either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every pure path* $P$ *starting in* $E_0$ *and containing just one vertex* $k$ *in* $E_0$ *has the property that the values at the cut vertices contained in* $P$ *form either an increasing, or decreasing, or a zero sequence along this path according to whether* $y_k > 0$, $y_k < 0$ *or* $y_k = 0$*; in the last case all vertices in* $P$ *have value zero.*

*Case B: No block of* $G$ *contains both positively and negatively valuated vertices. There exists a single vertex* $z$ *which has value zero and has a neighbour with a non-zero valuation. This vertex is a cut vertex. Each block contains (with the exception of* $z$*) either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every pure path* $P$ *starting in* $z$ *has the property that the values at its cut vertices either increase, and then all values in vertices of* $P$ *are (with the exception of* $z$*) positive, or decrease, and then all values (up to that of* $z$*) are negative, or all values in vertices of* $P$

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*are equal to zero. Every path containing both positively and negatively valuated vertices passes through $z$.*

The following lemma gives a property of the eigenvector respect to the least Laplacian eigenvalue $\lambda(\Gamma)$ of the signed graph $\Gamma$.

LEMMA 2.4. *Let $\Gamma \in \mathcal{U}(n, g, k, \bar{\sigma})$. Let $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in \mathbb{R}^n$ be an eigenvector of $\Gamma$ corresponding to the least Laplacian eigenvalue $\lambda(\Gamma)$. Then there exists a vertex $w$ on the cycle in $\Gamma$ with $x_w \neq 0$, and for each vertex $v$ on the cycle with $d_v \geq 3$, every path $P$ which starts from $v$ and contains no vertices of the cycle except $v$ has the property that the values at the vertices of $P$ form either an increasing, or decreasing, or a zero sequence along this path according to whether $x_v > 0$, $x_v < 0$, or $x_v = 0$; in the last case all vertices in $P$ have value zero.*

*Proof.* Suppose that the vertices of $\Gamma$ are labelled as $v_1, v_2, \ldots, v_n$ and the cycle in $\Gamma$ is $C$. Let $W$ be an all positive unicyclic graph which is obtained from $\Gamma$ as defined in Lemma 2.2. By Lemma 2.2, $\xi = (x^T, -x^T)^T \in \mathbb{R}^{2n}$ is an eigenvector of $W$ corresponding to $\alpha(W)$.

Assume that the values of the vertices on the cycle $C$ are all zero. Then the values on the cycle in $W$ given by $\xi$ are also zero. If $W$ is the first case of Lemma 2.3, then the single block of $W$ with both positively and negatively valuated vertices is an edge not on the cycle. Without loss of generality, let this edge be $v_{k_1}v_{k_2}$. Observing the structures of $W$ and $\xi$, the edge $u_{k_1}u_{k_2}$ has the same property as that of $v_{k_1}v_{k_2}$, which is a contradiction. If $W$ is the second case of Lemma 2.3, there is a unique vertex which has value zero and is adjacent to a non-zero valuated vertex. Without loss of generality, let the unique zero valuated vertex be $v_{k_1}$ and let $v_{k_2}$ be the non-zero valued vertex adjacent to $v_{k_1}$. Consequently, $u_{k_1}$ is a zero valued vertex and is adjacent to non-zero valued vertex $u_{k_2}$, which is also a contradiction.

By the above discussion, there exists a vertex $w$ on the cycle $C$ with $x_w \neq 0$. By (2.1) and from the structures of $W$ and $\xi$, the cycle of $W$ has both positive and negative valued vertices. Hence, $W$ is the first case of Lemma 2.3 and the result follows from Case A of Lemma 2.3. ◻

Let $\mathcal{U}_n(\bar{\sigma})$ denote the class of unbalanced unicyclic signed graphs of order $n$ (note, $\bar{\sigma}$ denotes an unbalanced signature). The following lemmas will be used in the next section.

LEMMA 2.5. ([3]) *For a signed graph $\Gamma = (G, \sigma) \in \mathcal{U}_n(\bar{\sigma})$, let $\bar{C}$ be the cycle in $\Gamma$, and $\lambda(\Gamma)$ be the least Laplacian eigenvalue of $\Gamma$ with corresponding eigenvector $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T$. Assume that there is a tree $T$ attached to $\bar{C}$ and a vertex $u \in T$ such that $x_u = 0$. Then $x_v = 0$ for every vertex $v \in T$.*

LEMMA 2.6. ([3]) *Let $\Gamma = (G, \sigma) \in \mathcal{U}_n(\bar{\sigma})$ and $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T$ be an eigenvector corresponding to the least Laplacian eigenvalue $\lambda(\Gamma)$ of $\Gamma$ and let $\bar{C}$ be the cycle in $\Gamma$. If $\sigma$ is taken such that all edges are positive with the exception of the edge $pq$ which minimizes $|x_p x_q|$, then $x$ can be chosen so that:*
- *$x_v \geq 0$ for all $v \in \Gamma$;*
- *if $x_p x_q = 0$ then either $x_q = 0$ and $x_v > 0$ for all $v \in V(\bar{C} - q)$ or $x_p = 0$ and $x_v > 0$ for all $v \in V(\bar{C} - p)$;*
- *if $x_p x_q > 0$ then $x_v > 0$ for all $v \in \Gamma$.*

For an unbalanced unicyclic signed graph $\Gamma$, by appropriate switching one can take the only negative edge to be anywhere on the cycle. So, in view of Lemma 2.6, the least Laplacian eigenvalue $\lambda(\Gamma)$ of $\Gamma$ has a nonnegative eigenvector.

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**3. Main results.** Now we discuss the minimal least Laplacian eigenvalue among signed graphs among $\in \mathcal{U}(n,g,k,\bar{\sigma})$. Let $\hat{\Gamma} \in \mathcal{U}(n,g,k,\bar{\sigma})$ be a signed graph minimizing the least Laplacian eigenvalue, and $\hat{\lambda}$ denote the least Laplacian eigenvalue of $\hat{\Gamma}$. In this section, we will identify the structure of $\hat{\Gamma}$.

Lemma 2.1 can be used to prove the following simple observation which proof is analogous in [5].

Lemma 3.1. *Let $\Gamma \in \mathcal{U}(n,g,k,\bar{\sigma})$ and $\bar{C}$ denote the unbalanced signed cycle with $g$ vertices. Then $\lambda(\Gamma) \le \lambda(\bar{C})$, with equality if and only if $\Gamma$ is an unbalanced cycle on $g$ vertices.*

Lemma 3.2. *Let $\Gamma \in \mathcal{U}(n,g,k,\bar{\sigma})$ with the cycle $\bar{C}_g$ and $T$ be the unique tree attaching on $\bar{C}_g$ with root $v_i$. Suppose that $x$ is a nonnegative eigenvector of $\Gamma$ corresponding to $\lambda(\Gamma)$. Then $x_{v_i} > 0$.*

*Proof.* If $x_{v_i} = 0$, then $x_v = 0$ for all $v \in V(T)$ by Lemma 2.4. Without loss of generality, the vertices of $\bar{C}_g$ are written as $v_1, v_2, \ldots, v_g$. So $\bar{x} = (x_{v_1}, x_{v_2}, \ldots, x_{v_g})^\top$ is an eigenvector of the cycle $\bar{C}_g$ corresponding to the eigenvalue $\lambda(\Gamma)$. Thus, $\lambda(\Gamma) \ge \lambda(\bar{C}_g)$. This contradicts Lemma 3.1. ◻

The *coalescence* of $G_1$ and $G_2$ denoted by $G_1(v_1) \diamond G_2(v_2)$, is obtained from $G_1$ and $G_2$ by identifying $v_1 \in V(G_1)$ with $v_2 \in V(G_2)$, see [13]. For convenience, we use $G_1(v_1) \diamond G_2(v_2, v_3) \diamond G_3(v_4)$ to denote the coalescence of $G_1(v_1) \diamond G_2(v_2)$ and $G_3$ by identifying $v_3 \in V(G_2)$ and $v_4 \in V(G_3)$. We use $S_{(k,v)}$ to denote a star with $k$ vertices and center vertex $v$. and $P_{(v_1,v_n)}$ is a path of length $n-1$ from vertex $v_1$ to vertex $v_n$ with consecutive vertices $v_1, v_2, \ldots, v_n$. In particular, the path $P_{(v_1,v_n)}$ is just the vertex $v_1$ in the case of $v_1 = v_n$. Additionally, $d_G(v)$ denotes the degree of vertex $v$ in the graph $G$ and $N_G(v)$ denotes the neighborhood of vertex $v$ in the graph $G$.

Lemma 3.3. *Let $\Gamma = \Gamma_1(v_2) \diamond \Gamma_2(u)$ and $\Gamma^* = \Gamma_1(v_1) \diamond \Gamma_2(u)$ be signed graphs, where $\Gamma_1$ is a connected signed graph containing distinct vertices $v_1, v_2$, and $\Gamma_2$ is a connected graph containing a vertex $u$. If $x$ is a nonnegative eigenvector of $\Gamma$ corresponding the least Laplacian eigenvalue $\lambda(\Gamma)$ and $x_{v_1} \ge x_{v_2}$, then $\lambda(\Gamma^*) \le \lambda(\Gamma)$ with equality only if $x_{v_1} = x_{v_2}$ and $d_{\Gamma_2}(u)x_u = \sum_{v \in N_{\Gamma_2}(u)} x_v$.*

*Proof.* Without loss of generality, assume that $x$ is a unit vector. Let $\widetilde{x}$ be a vector defined on the vertices of $\Gamma^*$ such that

$$\widetilde{x}_v = \begin{cases} x_v, & \text{if } v \in V(\Gamma_1); \\ x_v + x_{v_1} - x_{v_2}, & \text{otherwise.} \end{cases}$$

There is a one-to-one correspondence between the edges set $E(\Gamma)$ and $E(\Gamma^*)$, that is, the edge $vv_2 \in E(\Gamma_2)$ of $\Gamma$ corresponds to the edge $vv_1 \in E(\Gamma_2)$ of $\Gamma^*$ for each $v \in N_{\Gamma_2}(u)$, and every common edge of $E(\Gamma)$ and $E(\Gamma^*)$ corresponds to itself. Note that $\sigma(vv_2) = 1$ for each edge $vv_2 \in E(\Gamma)$ with $v \in V(\Gamma_2)$, then $(x_v - \sigma(vv_2)x_{v_2})^2 = (\widetilde{x}_v - \sigma(vv_1)\widetilde{x}_{v_1})^2$. On the other hand, it follows that $(x_v - \sigma(vv')x_{v'})^2 = (\widetilde{x}_v - \sigma(vv')\widetilde{x}_{v'})^2$ for each edge $vv' \in E(\Gamma_2 - u) \cup E(\Gamma_1)$. Hence, we have

$$\widetilde{x}^T L(\Gamma^*)\widetilde{x} = \sum_{vv' \in E(\Gamma^*)} (\widetilde{x}_v - \sigma(vv')\widetilde{x}_{v'})^2 = \sum_{vv' \in E(\Gamma)} (x_v - \sigma(vv')x_{v'})^2 = x^T L(\Gamma)x = \lambda(\Gamma).$$

Furthermore, as $x$ is nonnegative and $x_{v_1} \ge x_{v_2}$,

$$\|\widetilde{x}\|^2 = \sum_{v \in V(\Gamma^*)} \widetilde{x}_v^2 = \sum_{v \in V(\Gamma_1)} x_v^2 + \sum_{v \in V(\Gamma_2)\setminus u} (x_v + x_{v_1} - x_{v_2})^2 \ge \sum_{v \in V(\Gamma)} x_v^2 = 1.$$

By the above discussion, we have

$$\lambda(\Gamma^*) \le \|\widetilde{x}\|^{-2}\widetilde{x}^T L(\Gamma^*)\widetilde{x} \le \widetilde{x}^T L(\Gamma^*)\widetilde{x} = \lambda(\Gamma).$$

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The equality $\lambda(\Gamma^*) = \lambda(\Gamma)$ holds if and only if $\widetilde{x}^T\widetilde{x} = 1$ and $\widetilde{x}$ is an eigenvector corresponding to $\lambda(\Gamma^*)$. The later means that $x_{v_1} = x_{v_2}$ and $d_{\Gamma_2}(u)x_u = \sum_{v\in N_{\Gamma_2}(u)} x_v$ by the eigenvector equations of $x$ and $\widetilde{x}$ at vertex $v_2$

$$\lambda(\Gamma)x_{v_2} = d_{\Gamma_1}(v_2)x_{v_2} + d_{\Gamma_2}(u)x_u - \sum_{v\in N_{\Gamma_1}(v_2)} \sigma(vv_2)x_v - \sum_{v\in N_{\Gamma_2}(u)} x_v$$

and

$$\lambda(\Gamma^*)\widetilde{x}_{v_2} = d_{\Gamma_1}(v_2)\widetilde{x}_{v_2} - \sum_{v\in N_{\Gamma_1}(v_2)} \sigma(vv_2)\widetilde{x}_v. \qquad \square$$

Let $\bar{C}_g$ be the cycle with length $g$ in the unbalanced unicyclic signed graph $\Gamma$ and $T_i$ be the attaching tree with root $v_i \in V(\bar{C}_g)$. We call a subgraph of $T_i$ a *main path* if it is a path from the root $v_i$ to vertex $v_{i_t}$, where $v_{i_t} \in V(T_i)$ is at the largest distance from $v_i$. Note that the main path of $T_i$ is not unique.

Lemma 3.4. *Let $\hat{\Gamma} \in \mathcal{U}(n, g, k, \bar{\sigma})$ be a signed graph minimizing the least Laplacian eigenvalue and let $\bar{C}_g$ be the cycle in $\hat{\Gamma}$, and the vertices of $\bar{C}_g$ be written as $v_1, v_2, \ldots, v_g$ $(3 \le g \le n-k)$. Then there is just one tree attaching on the cycle $\bar{C}_g$ in $\hat{\Gamma}$, with root $v_i$ on the cycle. Moreover, $d_{v_i} = 3$ if $3 \le g < n-k$.*

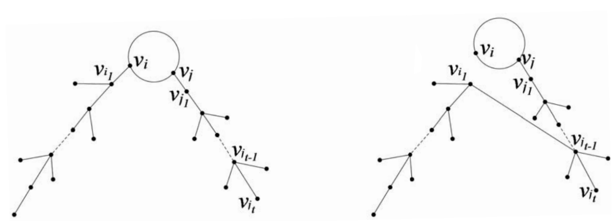


Figure 2. $\hat{\Gamma}$ *and* $\tilde{\Gamma} = \hat{\Gamma} - v_iv_{i_1} + v_{i_1}v_{j_{t-1}}$.

*Proof.* For the sake of contradiction, suppose that there exist trees $T_i$ and $T_j$ attaching on the cycle $\bar{C}_g$ of $\hat{\Gamma}$ with roots $v_i$ and $v_j$, respectively such that $v_i \sim v_{i_1}$ and $v_j \sim v_{j_1}$(see the left graph of Figure 2). Let $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^\top$ be a nonnegative unit eigenvector corresponding to the least Laplacian eigenvalue $\hat{\lambda}$ of $\hat{\Gamma}$. By Lemma 2.6, there is at most one vertex $v \in V(\bar{C}_g)$ such that $x_v = 0$. Thus, without loss of generality, we can assume that $x_{v_j} \ge x_{v_i} > 0$ or $x_{v_j} > x_{v_i} = 0$. Let $P_{T_j}$ denote a main path between vertices $v_j$ and $v_{j_t}$, and $v_{j_{t-1}} \sim v_{j_t}$, where $v_{j_{t-1}} \in V(P_{T_j})$. Note that it is possible that $v_j = v_{j_{t-1}}$. We construct a new signed graph $\tilde{\Gamma}$ as follows: $\tilde{\Gamma} = \hat{\Gamma} - v_iv_{i_1} + v_{i_1}v_{j_{t-1}}$, and the edge $v_{i_1}v_{j_{t-1}}$ is positive (see the right graph of Figure 2). The number of pendant vertices in $\tilde{\Gamma}$ is also $k$, and hence, $\tilde{\Gamma} \in \mathcal{U}(n, g, k, \bar{\sigma})$.

Let $y$ be a vector defined on the vertices of $\tilde{\Gamma}$ such that

$$\begin{cases} y_v = x_v + x_{v_{j_{t-1}}} - x_{v_i}, & \text{for } v \in V(T_i); \\ y_v = x_v, & \text{for } v \in V(\hat{\Gamma})\backslash V(T_i). \end{cases}$$

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By (2.2), we have $y^T L(\tilde{\Gamma})y = x^T L(\hat{\Gamma})x$, and

$$\begin{aligned} y^T y &= \sum_{v\in V(\hat{\Gamma})} y_v^2 \\ &= \sum_{v\in V(\hat{\Gamma}-T_i)} x_v^2 + \sum_{v\in V(T_i)} (x_v + x_{v_{j_{t-1}}} - x_{v_i})^2 \\ &= \sum_{v\in V(\hat{\Gamma}-T_i)} x_v^2 + \sum_{v\in V(T_i)} x_v^2 + 2(x_{v_{j_{t-1}}} - x_{v_i}) \sum_{v\in V(T_i)} x_{v_{i_t}} + \sum_{v\in V(T_i)} (x_{v_{j_{t-1}}} - x_{v_i})^2 \\ &= 1 + 2(x_{v_{j_{t-1}}} - x_{v_i}) \sum_{v\in V(T_i)} x_v + |T_i|(x_{v_{j_{t-1}}} - x_{v_i})^2. \end{aligned}$$

By Lemma 2.4, we have $0 < x_{v_j} < \cdots < x_{v_{j_{t-1}}} < x_{v_{j_t}}$ as $x_{v_j} > 0$. So whether $x_{v_j} \geq x_{v_i} > 0$, or $x_{v_j} > x_{v_i} = 0$, we have $x_{v_{j_{t-1}}} > x_{v_i}$. Thus, $y^\top y > 1$, and

$$\hat{\lambda} = x^T L(\hat{\Gamma})x = y^T L(\tilde{\Gamma})y \geq y^T y\lambda(\tilde{\Gamma}) > \lambda(\tilde{\Gamma}).$$

This contradicts with the minimality of $\hat{\lambda}$.

Next, we will show that $d_{v_i} = 3$ if $g < n-k$. Suppose to the contrary that there are trees $T_1, T_2$ attaching at vertex $v_i \in V(\bar{C}_g)$. Let $v_i \sim v_{i_{11}}$ and $v_{i_{21}} \sim v_i$, where $v_{i_{11}} \in V(T_1)$ and $v_{i_{21}} \in V(T_2)$. Moreover, without loss of generality, assume that $v_{i_{11}}$ is not a pendant vertex since $g < n-k$. By Lemma 3.2 we have $x_{v_i} > 0$. Then by Lemma 2.4, it follows $x_{v_{i_{11}}} > x_{v_i}$. So, we can see the signed graph $\bar{C}_g(v_i) \diamond T_1(v_i, v_{i_{11}}) \diamond T_2(v_i) \in \mathcal{U}(n, g, k, \bar{\sigma})$ has smaller least Laplacian eigenvalue than $\hat{\Gamma}$ by using Lemma 3.3. ∎

Lemma 3.5. *Let $\hat{\Gamma} \in \mathcal{U}(n, g, k, \bar{\sigma})$ be a signed graph minimizing the least Laplacian eigenvalue and $3 \leq g \leq n-k$. Then $\hat{\Gamma} = \bar{C}_g(v_g) \diamond P_{(v_g, v_{n-k})}(v_g, v_{n-k}) \diamond S_{(k+1, v_{n-k})}(v_{n-k})$ (see Figure 3).*

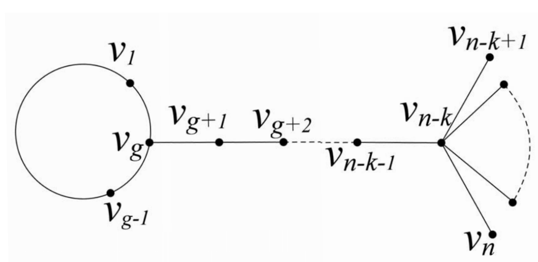


Figure 3. *The signed graph $\hat{\Gamma}(g) = \overline{C_g}(v_g) \diamond P_{(v_g, v_{n-k})}(v_g, v_{n-k}) \diamond S_{(k+1, v_{n-k})}(v_{n-k})$.*

*Proof.* From Lemma 3.4, the result is obvious if $g = n-k$. Now, we will prove the result in the case of $g < n-k$. If $\hat{\Gamma} = \bar{C}_g(v_g) \diamond P_{(v_g, v_{n-k})}(v_g, v_{n-k}) \diamond S_{(k+1, v_{n-k})}(v_{n-k})$, the result is obvious. Otherwise, let $x$ be a nonnegative eigenvector of $\hat{\Gamma}$ corresponding $\lambda(\hat{\Gamma})$ and $T$ be the attaching tree on the cycle in $\hat{\Gamma}$, with root $v_g$. Suppose that $T_i$ is a main path with consecutive vertices $v_i, v_{i+1}, \ldots v_j$ and $T_{v_k}$ is the subtree of $T$ on $T_i$, with root $v_k$ for some $i < k < j-1$. By Lemma 2.4, it follows $x_{v_{j-1}} > x_{v_k}$. So, $(\hat{\Gamma} - T_{v_k})(v_{j-1}) \diamond T_{v_k}(v_k)$ has a smaller least Laplacian eigenvalue than $\hat{\Gamma}$ by using Lemma 3.4. Repeating this process until $\hat{\Gamma} = \bar{C}_g(v_g) \diamond P_{(v_g, v_{n-k})}(v_g, v_{n-k}) \diamond S_{(k+1, v_{n-k})}(v_{n-k})$. ∎

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For fixed integers $n$ and $k$, let $\hat{\Gamma}(g) = \bar{C}_g(v_g) \diamond P_{(v_g, v_{n-k})}(v_g, v_{n-k}) \diamond S_{(k+1, v_{n-k})}(v_{n-k})$ for $g = 3, 4, \ldots, n-k$.

LEMMA 3.6. *The least Laplacian eigenvalue of $\hat{\Gamma}(g)$ has multiplicity* 1.

*Proof.* Let $v$ be the unique vertex lying on $\bar{C}_g$ with degree 3. Assume, to the contrary, $x$ and $y$ are two linear independent eigenvectors of $\hat{\Gamma}(g)$ corresponding to $\lambda(\hat{\Gamma}(g))$. There exists a nonzero linear combination of $x$ and $y$ such that its value at $v$ equals zero, which contradict Lemma 3.2. ∎

LEMMA 3.7. *Let $3 \leq g \leq n-k$ and $\hat{\Gamma}(g) \in U(n, g, k, \bar{\sigma})$ be defined as above (see Figure 3). Then*

$$\lambda(\hat{\Gamma}(n-k)) > \lambda(\hat{\Gamma}(n-k-1)) > \cdots > \lambda(\hat{\Gamma}(4)) > \lambda(\hat{\Gamma}(3)).$$

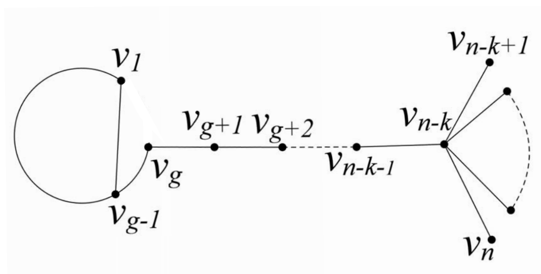


FIGURE 4. *The signed graph $\hat{\Gamma}(g-1)$.*

*Proof.* Let $\bar{C}_g$ be the cycle of the signed graph $\hat{\Gamma}(g)$. Up to switching equivalence, suppose that $v_1 v_g$ is the negative edge of $\hat{\Gamma}(g)$. Suppose $4 \leq g \leq n-k$. Deleting $v_1 v_g$ of $\hat{\Gamma}(g)$ and adding a negative edge $v_1 v_{g-1}$ in $\hat{\Gamma}(g)$, we obtain an unbalanced unicyclic signed graph $\hat{\Gamma}(g-1)$, which belongs to $\mathcal{U}(n, g-1, k, \bar{\sigma})$ (see Figure 4).

Let $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T$ be any unit eigenvector of $\hat{\Gamma}(g)$ corresponding to $\lambda(\hat{\Gamma}(g))$.

Let $y$ be a vector defined on the vertices of $\hat{\Gamma}(g)$ such that

$$\begin{cases} y_{v_j} = -x_{v_{g-j}}, & j = 1, 2, \ldots, g-1; \\ y_{v_j} = x_{v_j}, & j = g, g+1, \ldots, n. \end{cases}$$

It is easy to check that $y^\top L(\hat{\Gamma}(g-1))y = x^\top L(\hat{\Gamma}(g))x = \lambda(\hat{\Gamma}(g))$. By the Rayleigh-Ritz theorem, this implies that $y$ is an eigenvector of $L(\hat{\Gamma}(g-1))$ corresponding to $\lambda(\hat{\Gamma}(g))$. As the multiplicity of $\lambda(\hat{\Gamma}(g))$ is one and $x_{v_g} \neq 0$, without loss generality, we assume that $x_{v_g} > 0$. Thus, $y = x$, and hence,

$$x_{v_j} = -x_{v_{g-j}}, \quad \text{for } j = 1, 2, \ldots, g-1. \tag{3.3}$$

By Lemma 2.4 and (3.3), $x_{v_{g+1}} > x_{v_g}$ and $x_{v_{g-1}} = -x_{v_1}$. We also find that $x_{v_g} \neq x_{v_{g-1}}$. Otherwise, $3x_{v_g} > (3 - \lambda(\hat{\Gamma}(g)))x_{v_g} = -x_{v_1} + x_{v_{g-1}} + x_{v_{g+1}} = 2x_{v_{g-1}} + x_{v_{g+1}} > 3x_{v_g}$ if $g < n-k$, and also if $g = n-k$

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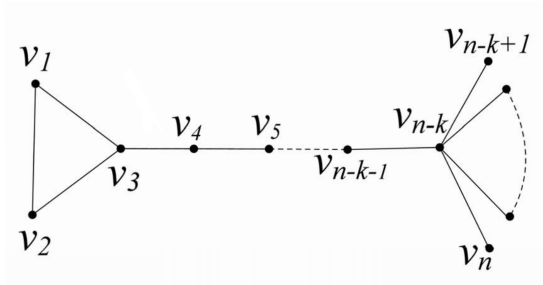


FIGURE 5. $\hat{\Gamma}(3) = \overline{C}_3(v_3) \diamond P_{(v_3,v_{n-k})}(v_3, v_{n-k}) \diamond S_{k+1,v_{n-k}}(v_{n-k})$.

then

$$\begin{aligned}
(k+2)x_{v_g} > [(k+2) - \lambda(\hat{\Gamma}(g))]x_{v_g} &= -x_{v_1} + x_{v_{g-1}} + \sum_{k=g+1}^{n} x_{v_k} \\
&= 2x_{v_{g-1}} + \sum_{k=g+1}^{n} x_{v_k} \\
&> (k+2)x_{v_g}.
\end{aligned}$$

In either case, we reach a contradiction. Hence,

$$(x_{v_1} + x_{v_g})^2 = (x_{v_g} - x_{v_{g-1}})^2 > 0 = (x_{v_{g-1}} + x_{v_1})^2. \tag{3.4}$$

By (2.2) and (3.4),

$$\begin{aligned}
\lambda(\hat{\Gamma}(g)) &= \sum_{v_iv_j \in E(\hat{\Gamma}(g))} (x_{v_i} - \sigma(v_iv_j)x_{v_j})^2 \\
&= \sum_{v_iv_j \in E(\hat{\Gamma}(g)) - v_1v_g} (x_{v_i} - \sigma(v_iv_j)x_{v_j})^2 + (x_{v_1} + x_{v_g})^2 \\
&> \sum_{v_iv_j \in E(\hat{\Gamma}(g)) - v_1v_g} (x_{v_i} - \sigma(v_iv_j)x_{v_j})^2 + (x_{v_1} + x_{v_{g-1}})^2 \\
&= \sum_{v_iv_j \in E(\hat{\Gamma}(g-1))} (x_{v_i} - \sigma(v_iv_j)x_{v_j})^2 \\
&\geq \lambda(\hat{\Gamma}(g-1)).
\end{aligned}$$

Therefore,

$$\lambda(\hat{\Gamma}(n-k)) > \lambda(\hat{\Gamma}(n-k-1)) > \cdots > \lambda(\hat{\Gamma}(4)) > \lambda(\hat{\Gamma}(3)). \qquad \square$$

The main result of this paper now follows readily from Lemma 3.7.

THEOREM 3.8. *The unique, up to switching equivalence, unbalanced unicyclic signed graph* $\Gamma \in U(n, g, k, \bar{\sigma})$ *which minimizes the least Laplacian eigenvalue is* $\hat{\Gamma}(3)$*, namely,* $\bar{C}_3(v_3) \diamond P_{(v_3,v_{n-k})}(v_3, v_{n-k}) \diamond S_{k+1,v_{n-k}}(v_{n-k})$ *(see Figure 5).*

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