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ON INEQUALITIES FOR A-NUMERICAL RADIUS OF OPERATORS*

PINTU BHUNIA†, KALLOL PAUL†, AND RAJ KUMAR NAYAK†

Abstract. Let A be a positive operator on a complex Hilbert space $\mathcal{H}$. Inequalities are presented concerning upper and lower bounds for A-numerical radius of operators, which improve on and generalize the existing ones, studied recently in [A. Zamani. A-Numerical radius inequalities for semi-Hilbertian space operators. *Linear Algebra Appl.*, 578:159–183, 2019.]. Also, some inequalities are obtained for B-numerical radius of 2×2 operator matrices, where B is the 2×2 diagonal operator matrix whose diagonal entries are A. Further, upper bounds are obtained for A-numerical radius for product of operators, which improve on the existing bounds.

Key words. A-numerical radius, A-adjoint operator, A-selfadjoint operator, Positive operator.

AMS subject classifications. 47A12, 47A30, 47A63.

1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space with usual inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the norm induced from $\langle\cdot,\cdot\rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Throughout this article, we assume I and O are the identity operator and the zero operator on $\mathcal{H}$, respectively. A selfadjoint operator $A\in\mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax,x\rangle\geq 0$ for all $x\in\mathcal{H}$, and is called strictly positive if $\langle Ax,x\rangle>0$ for all $(0\neq)x\in\mathcal{H}$. For a positive (strictly positive) operator A, we write $A\geq 0$ $(A>0)$. Let $B=\begin{pmatrix} A & O\\ O & A\end{pmatrix}$. Then $B\in\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ is positive or strictly positive if A is positive or strictly positive, respectively. Let us fix the alphabets A and B for positive operator on $\mathcal{H}$ and $\mathcal{H}\oplus\mathcal{H}$, respectively. Clearly, A induces a positive semidefinite sesquilinear form $\langle\cdot,\cdot\rangle_A:\mathcal{H}\times\mathcal{H}\to\mathbb{C}$ defined as $\langle x,y\rangle_A=\langle Ax,y\rangle$ for $x,y\in\mathcal{H}$. Let $\|\cdot\|_A$ denote the seminorm on $\mathcal{H}$ induced from the sesquilinear form $\langle\cdot,\cdot\rangle_A$, that is, $\|x\|_A=\sqrt{\langle x,x\rangle_A}$ for all $x\in\mathcal{H}$. It is easy to verify that $\|\cdot\|_A$ is a norm if and only if A is a strictly positive operator. Also, $(\mathcal{H},\|\cdot\|_A)$ is complete if and only if the range $\mathcal{R}(A)$ of A is closed in $\mathcal{H}$. By $\overline{\mathcal{R}(T)}$ we denote the norm closure of $\mathcal{R}(T)$ in $\mathcal{H}$. For $T\in\mathcal{B}(\mathcal{H})$, A-operator seminorm of T, denoted as $\|T\|_A$, is defined as

$$\|T\|_A=\sup_{x\in\overline{\mathcal{R}(A)},x\neq 0}\frac{\|Tx\|_A}{\|x\|_A}.$$

Here, we note that for a given $T\in\mathcal{B}(\mathcal{H})$, if there exists $c>0$ such that $\|Tx\|_A\leq c\|x\|_A$ for all $x\in\overline{\mathcal{R}(A)}$ then $\|T\|_A<+\infty$. Again A-minimum modulus of T, denoted as $m_A(T)$ (see [26]), is defined as

$$m_A(T)=\inf_{x\in\overline{\mathcal{R}(A)},x\neq 0}\frac{\|Tx\|_A}{\|x\|_A}.$$

We set $\mathcal{B}^A(\mathcal{H})=\{T\in\mathcal{B}(\mathcal{H}):\|T\|_A<+\infty\}$. It is easy to verify that $\mathcal{B}^A(\mathcal{H})$ is not generally a subalgebra of $\mathcal{B}(\mathcal{H})$ and $\|T\|_A=0$ if and only if $ATA=0$. For $T\in\mathcal{B}(\mathcal{H})$, an operator $R\in\mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if for every $x,y\in\mathcal{H}$ such that $\langle Tx,y\rangle_A=\langle x,Ry\rangle_A$, that is, $AR=T^*A$, where T^* is the adjoint of T.

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†Department of Mathematics, Jadavpur University, Kolkata 700032, West Bengal, India (pintubhunia5206@gmail.com, kalloldada@gmail.com, rajkumarju51@gmail.com).

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For any operator $T \in \mathcal{B}(\mathcal{H})$, A-adjoint of T may or may not exist. In fact, an operator $T \in \mathcal{B}(\mathcal{H})$ may have one or more than one A-adjoint operators, also it may have none. By Douglas Theorem [12], we have that an operator $T \in \mathcal{B}(\mathcal{H})$ admits A-adjoint if and only if

$$\mathcal{R}(T^*A) \subseteq \mathcal{R}(A).$$

Now we consider an example that $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. Then we see that $\mathcal{R}(T^*A) = \{(x,0) : x \in \mathbb{C}\}$ and $\mathcal{R}(A) = \{(0,x) : x \in \mathbb{C}\}$. So, by Douglas Theorem [12], we conclude that T have no A-adjoint.

Let $\mathcal{B}_A(\mathcal{H})$ be the collection of all operators in $\mathcal{B}^A(\mathcal{H})$ which admits A-adjoint. Note that $\mathcal{B}_A(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$, A-adjoint operator of T is written as $T^{\sharp_A}$. It is well known that $T^{\sharp_A} = A^{\dagger}T^*A$ where $A^{\dagger}$ is the Moore-Penrose inverse of A, (see [20]). It is useful that if $T \in \mathcal{B}_A(\mathcal{H})$ then $AT^{\sharp_A} = T^*A$. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be A-selfadjoint operator if AT is selfadjoint, that is, $AT = T^*A$ and it is called A-positive if $AT \geq 0$. For A-positive operator T we have

$$\|T\|_A = \sup\{\langle Tx, x\rangle_A : x \in \mathcal{H}, \|x\|_A = 1\}.$$

An operator $U \in \mathcal{B}_A(\mathcal{H})$ is said to be A-unitary if $U^{\sharp_A}U = (U^{\sharp_A})^{\sharp_A}U^{\sharp_A} = P_A$, P_A is the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Here we note that if $T \in \mathcal{B}_A(\mathcal{H})$ then $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_ATP_A$. Also $T^{\sharp_A}T$, $TT^{\sharp_A}$ are A-selfadjoint and A-positive operators and so

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2.$$

Also, for $T, S \in \mathcal{B}_A(\mathcal{H})$, $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$, $\|TS\|_A \leq \|T\|_A\|S\|_A$ and $\|Tx\|_A \leq \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$. For further details we refer the reader to [1, 2, 3]. For an operator $T \in \mathcal{B}_A(\mathcal{H})$, we write $Re_A(T) = \frac{1}{2}(T + T^{\sharp_A})$ and $Im_A(T) = \frac{1}{2i}(T - T^{\sharp_A})$.

For $T \in \mathcal{B}_A(\mathcal{H})$, A-numerical radius of T, denoted as $w_A(T)$, is defined as (see [4])

$$w_A(T) = \sup\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}.$$

Also, for $T \in \mathcal{B}_A(\mathcal{H})$, A-Crawford number of T, denoted as $c_A(T)$ (see [26]), is defined as

$$c_A(T) = \inf\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}.$$

For $T \in \mathcal{B}_A(\mathcal{H})$, it is well-known that A-numerical radius of T is equivalent to A-operator seminorm of T, (see [25]), satisfying the following inequality:

$$\frac{1}{2}\|T\|_A \leq w_A(T) \leq \|T\|_A.$$

Over the years many mathematicians have studied numerical radius inequalities in [5, 7, 8, 9, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24]. Recently, Zamani [25] have studied A-numerical radius and computed some inequalities for A-numerical radius. In this paper, we compute some inequalities for B-numerical radius of 2×2 operator matrices which generalize and improve on the existing inequalities. Also, we obtain some inequalities for A-numerical radius of operators in $\mathcal{B}_A(\mathcal{H})$ which improve on the existing inequalities in [25]. Further, we obtain A-numerical radius bounds for sum of product of operators in $\mathcal{B}_A(\mathcal{H})$ which improve on the existing bounds.

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2. A-numerical radius inequalities for operators in $\mathcal{B}_A(\mathcal{H})$. We begin this section with the following three results proved by Zamani [25].

LEMMA 2.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be an A-selfadjoint operator. Then*

$$w_A(T) = \|T\|_A.$$

LEMMA 2.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$. For every $\theta \in \mathbb{R}$,*

$$w_A\left(Re_A(e^{i\theta}T)\right) = \left\|Re_A(e^{i\theta}T)\right\|_A.$$

LEMMA 2.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_A(T) = \sup_{\theta\in\mathbb{R}} \left\|Re_A(e^{i\theta}T)\right\|_A \quad \textit{and} \quad w_A(T) = \sup_{\theta\in\mathbb{R}} \left\|Im_A(e^{i\theta}T)\right\|_A.$$

Next we compute B-numerical radius for some 2×2 operator matrices. First we note that the operator $T = (T_{ij})_{2\times 2}$ is in $\mathcal{B}_B(\mathcal{H}\oplus\mathcal{H})$ if the operator T_{ij} (for $i,j = 1,2$) are in $\mathcal{B}_A(\mathcal{H})$, and in this case (see [10, Lemma 3.1]), $T^{\sharp_B} = (T_{ji}^{\sharp_A})_{2\times 2}$. We now prove the following lemma.

LEMMA 2.4. *Let $X, Y \in \mathcal{B}_A(\mathcal{H})$. Then the following results hold:*

$$(i) \quad w_B\begin{pmatrix} X & O \\ O & Y \end{pmatrix} = \max\{w_A(X), w_A(Y)\}.$$

$$(ii) \quad \textit{If } A > 0, \textit{ then } w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix} = w_B\begin{pmatrix} O & Y \\ X & O \end{pmatrix}.$$

$$(iii) \quad \textit{If } A > 0, \textit{ then for any } \theta \in \mathbb{R}, \; w_B\begin{pmatrix} O & X \\ e^{i\theta}Y & O \end{pmatrix} = w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}.$$

$$(iv) \quad \textit{If } A > 0, \textit{ then } w_B\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} = \max\{w_A(X+Y), w_A(X-Y)\}.$$

$$\textit{In particular, } w_B\begin{pmatrix} O & Y \\ Y & O \end{pmatrix} = w_A(Y).$$

Proof. (i) Let $T = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$ and $u = (x,y) \in \mathcal{H}\oplus\mathcal{H}$ with $\|u\|_B = 1$, i.e., $\|x\|_A^2 + \|y\|_A^2 = 1$. Now,

$$\begin{aligned} |\langle Tu, u\rangle_B| &\le |\langle Xx, x\rangle_A| + |\langle Yy, y\rangle_A| \\ &\le w_A(X)\|x\|_A^2 + w_A(Y)\|y\|_A^2 \\ &\le \max\{w_A(X), w_A(Y)\}. \end{aligned}$$

Taking supremum over $\|u\|_B = 1$, we get

$$w_B(T) \le \max\{w_A(X), w_A(Y)\}.$$

Suppose $u = (x, 0) \in \mathcal{H}\oplus\mathcal{H}$ where $\|x\|_A = 1$. Then

$$|\langle Tu, u\rangle_B| = |\langle AXx, x\rangle| = |\langle Xx, x\rangle_A|.$$

Taking supremum over $\|x\|_A = 1$, we get

$$\sup_{\|x\|_A=1} |\langle Tu, u\rangle_B| = w_A(X),$$

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and so, we have $w_B(T) \geq w_A(X)$. Similarly, if we take $v = (0, y) \in \mathcal{H} \oplus \mathcal{H}$ with $\|y\|_A = 1$, then we can show that $w_B(T) \geq w_A(Y)$. Therefore, $w_B(T) \geq \max\{w_A(X), w_A(Y)\}$. This completes the proof of Lemma 2.4 (i).

(ii) The proof follows from the observation that $w_B(U^{\sharp_B}TU) = w_B(T)$ (see [10, Lemma 3.8]) if U is an B-unitary operator on $\mathcal{H} \oplus \mathcal{H}$, here we take $U = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$.

(iii) As in (ii), we now take $U = \begin{pmatrix} I & O \\ O & e^{\frac{i\theta}{2}} I \end{pmatrix}$.

(iv) Let $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix}$ and $T = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$. Then an easy calculation we have

$$U^{\sharp_B}TU = \begin{pmatrix} X - Y & O \\ O & X + Y \end{pmatrix}.$$

Using Lemma 2.4 (i) and $w_B(U^{\sharp_B}TU) = w_B(T)$, we get

$$w_B(T) = \max\{w_A(X + Y), w_A(X - Y)\}.$$

Taking $X = O$, we get

$$w_B \begin{pmatrix} O & Y \\ Y & O \end{pmatrix} = w_A(Y).$$

This completes the proof of Lemma 2.4 (iv). □

Next we prove the following important lemma for A-positive operators.

Lemma 2.5. *Let $X, Y \in \mathcal{B}_A(\mathcal{H})$ be A-positive. If $X - Y$ is A-positive, then*

$$\|X\|_A \geq \|Y\|_A.$$

Proof. From the definition of A-positive operator we have , for all $x \in \mathcal{H}$

$$\langle (X - Y)x, x\rangle_A \geq 0$$
$$\Rightarrow \langle Xx, x\rangle_A \geq \langle Yx, x\rangle_A$$
$$\Rightarrow w_A(X) \geq \langle Yx, x\rangle_A.$$

Taking supremum over $\|x\|_A = 1$, we get

$$w_A(X) \geq w_A(Y).$$

Since X, Y are A-selfadjoint operators, so $\|X\|_A \geq \|Y\|_A$. □

We are now in a position to prove the following theorem.

Theorem 2.6. *Let $X, Y \in \mathcal{B}_A(\mathcal{H})$. Then*

$$w_B^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} \geq \frac{1}{4} \max\left\{\|XX^{\sharp_A} + Y^{\sharp_A}Y\|_A, \|X^{\sharp_A}X + YY^{\sharp_A}\|_A\right\},$$
$$w_B^2 \begin{pmatrix} O & X \\ Y & O \end{pmatrix} \leq \frac{1}{2} \max\left\{\|XX^{\sharp_A} + Y^{\sharp_A}Y\|_A, \|X^{\sharp_A}X + YY^{\sharp_A}\|_A\right\}.$$

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Proof. Let $T = \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$, $H_\theta = Re_A(e^{i\theta}T)$ and $K_\theta = Im_A(e^{i\theta}T)$. Then, from an easy calculation, we have

$$H_\theta^2 + K_\theta^2 = \frac{1}{2}\begin{pmatrix} M & O \\ O & N \end{pmatrix},$$

where $M = XX^{\sharp_A} + Y^{\sharp_A}Y$, $N = X^{\sharp_A}X + YY^{\sharp_A}$.

Taking norm on both sides and then using Lemma 2.3, we get

$$\frac{1}{2}\left\|\begin{pmatrix} M & O \\ O & N \end{pmatrix}\right\|_B = \|H_\theta^2 + K_\theta^2\|_B \le \|H_\theta\|_B^2 + \|K_\theta\|_B^2 \le 2w_B^2(T).$$

Therefore, we get

$$\frac{1}{2}\max\left\{\|M\|_A, \|N\|_A\right\} \le 2w_B^2(T).$$

This completes the proof of the first inequality.

Again, from $H_\theta^2 + K_\theta^2 = \frac{1}{2}\begin{pmatrix} M & O \\ O & N \end{pmatrix}$, we have $H_\theta^2 - \frac{1}{2}\begin{pmatrix} M & O \\ O & N \end{pmatrix} = -K_\theta^2 \le 0$. Therefore, $H_\theta^2 \le \frac{1}{2}\begin{pmatrix} M & O \\ O & N \end{pmatrix}$. Using Lemma 2.5, we get

$$\|H_\theta\|_B^2 \le \frac{1}{2}\left\|\begin{pmatrix} M & O \\ O & N \end{pmatrix}\right\|_B = \frac{1}{2}\max\left\{\|M\|_A, \|N\|_A\right\}.$$

Taking supremum over $\theta \in \mathbb{R}$, we get

$$w_B^2(T) \le \frac{1}{2}\max\left\{\|M\|_A, \|N\|_A\right\}.$$

This completes the proof of the second inequality of the theorem. ∎

Next we state the corollary, the proof of which follows easily by considering $X = Y = T$ and $A > 0$ in Theorem 2.6.

Corollary 2.7. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $A > 0$. Then*

$$\frac{1}{4}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A \le w_A^2(T) \le \frac{1}{2}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A.$$

Remark 2.8. (i) Kittaneh [18, Theorem 1] proved that if $T \in \mathcal{B}(\mathcal{H})$, then

$$\frac{1}{4}\|TT^* + T^*T\| \le w^2(T) \le \frac{1}{2}\|TT^* + T^*T\|,$$

which follows easily from Corollary 2.7 by taking $A = I$.

(ii) Zamani [25, Theorem 2.10] proved that

$$w_A^2(T) \le \frac{1}{2}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A,$$

which clearly follows from the inequality obtained in Corollary 2.7.

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Next we prove the following theorem.

THEOREM 2.9. *Let* $X, Y \in \mathcal{B}_A(\mathcal{H})$*. Then* $w_B^4\begin{pmatrix} O & X \\ Y & O \end{pmatrix} \geq \frac{1}{16}\max\left\{\|P\|_A, \|Q\|_A\right\}$ *and*

$$w_B^4\begin{pmatrix} O & X \\ Y & O \end{pmatrix} \leq \frac{1}{8}\max\left\{\|XX^{\sharp_A}+Y^{\sharp_A}Y\|_A^2+4w_A^2(XY), \|X^{\sharp_A}X+YY^{\sharp_A}\|_A^2+4w_A^2(YX)\right\},$$

where $P=(XX^{\sharp_A}+Y^{\sharp_A}Y)^2+4(Re_A(XY))^2, Q=(X^{\sharp_A}X+YY^{\sharp_A})^2+4(Re_A(YX))^2$.

Proof. Let $T=\begin{pmatrix} O & X \\ Y & O \end{pmatrix}$, $H_\theta = Re_A(e^{i\theta}T)$ and $K_\theta = Im_A(e^{i\theta}T)$. Then, we get

$$H_\theta^4+K_\theta^4=\frac{1}{8}\begin{pmatrix} P_0 & O \\ O & Q_0 \end{pmatrix},$$

where $P_0=(XX^{\sharp_A}+Y^{\sharp_A}Y)^2+4(Re_A(e^{2i\theta}XY))^2$, $Q_0=(X^{\sharp_A}X+YY^{\sharp_A})^2+4(Re_A(e^{2i\theta}YX))^2$. Taking norm on both sides and using Lemma 2.3, we get

$$\frac{1}{8}\left\|\begin{pmatrix} P_0 & O \\ O & Q_0 \end{pmatrix}\right\|_B = \|H_\theta^4+K_\theta^4\|_B \leq \|H_\theta\|_B^4+\|K_\theta\|_B^4 \leq 2w_B^4(T).$$

Therefore, we get

$$\frac{1}{8}\max\left\{\|P_0\|_A, \|Q_0\|_A\right\} \leq 2w_B^4(T).$$

This holds for all $\theta \in \mathbb{R}$, so taking $\theta = 0$, we get

$$\frac{1}{8}\max\left\{\|P\|_A, \|Q\|_A\right\} \leq 2w_B^4(T).$$

This completes the proof of the first inequality of the theorem.

Again, from $H_\theta^4+K_\theta^4=\frac{1}{8}\begin{pmatrix} P_0 & O \\ O & Q_0 \end{pmatrix}$, we have $H_\theta^4-\frac{1}{8}\begin{pmatrix} P_0 & O \\ O & Q_0 \end{pmatrix}=-K_\theta^4 \leq 0$. Therefore, $H_\theta^4 \leq \frac{1}{8}\begin{pmatrix} P_0 & O \\ O & Q_0 \end{pmatrix}$. Using Lemma 2.5, we get

$$\|H_\theta\|_B^4 \leq \frac{1}{8}\left\|\begin{pmatrix} P_0 & O \\ O & Q_0 \end{pmatrix}\right\|_B = \frac{1}{8}\max\left\{\|P_0\|_A, \|Q_0\|_A\right\}.$$

Therefore, using Lemma 2.3, we get

$$\|H_\theta\|_B^4 \leq \frac{1}{8}\max\left\{\|XX^{\sharp_A}+Y^{\sharp_A}Y\|_A^2+4w_A^2(XY), \|X^{\sharp_A}X+YY^{\sharp_A}\|_A^2+4w_A^2(YX)\right\}.$$

Taking supremum over $\theta \in \mathbb{R}$ and using Lemma 2.3, we get

$$w_B^4(T) \leq \frac{1}{8}\max\left\{\|XX^{\sharp_A}+Y^{\sharp_A}Y\|_A^2+4w_A^2(XY), \|X^{\sharp_A}X+YY^{\sharp_A}\|_A^2+4w_A^2(YX)\right\}.$$

This completes the proof of the second inequality of the theorem. □

Now, taking $X=Y=T$ (say) and $A>0$ in the above Theorem 2.9, we get the following inequality.

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Corollary 2.10. *Let $T \in B_A(H)$ where $A > 0$. Then*

$$\frac{1}{16}\|(TT^{\sharp_A}+T^{\sharp_A}T)^2+4(Re_A(T^2))^2\|_A \leq w_A^4(T) \leq \frac{1}{8}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A^2+\frac{1}{2}w_A^2(T^2).$$

Remark 2.11. (i) In [5, Theorem 2.11] we proved that if $T \in \mathcal{B}(\mathcal{H})$ then

$$\frac{1}{16}\|TT^*+T^*T\|^2+\frac{1}{4}m\left((Re(T^2))^2\right) \leq w^4(T) \leq \frac{1}{8}\|TT^*+T^*T\|^2+\frac{1}{2}w^2(T^2),$$

which follows easily from Corollary 2.10 by taking $A = I$.

(ii) Zamani [25, Theorem 2.10] proved that

$$w_A^2(T) \leq \frac{1}{2}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A.$$

Since $w_A(T^2) \leq w_A^2(T)$ (see [19, Proposition 3.10]), so $w_A(T^2) \leq \frac{1}{2}\|TT^{\sharp_A}+T^{\sharp_A}T\|_A$. Therefore, the right hand inequality obtained in Corollary 2.10 improves on the inequality obtained by Zamani [25, Theorem 2.10].

We next prove the following theorem.

Theorem 2.12. *Let $T \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then*

$$w_A^4(T) \leq \frac{1}{4}w_A^2(T^2)+\frac{1}{8}w_A(T^2P+PT^2)+\frac{1}{16}\|P\|_A^2,$$

where $P = T^{\sharp_A}T + TT^{\sharp_A}$.

Proof. From Lemma 2.3, we have $w_A(T) = \sup_{\theta\in\mathbb{R}}\|H_\theta\|_A$ where $H_\theta = Re_A(e^{i\theta}T)$. Then

$$\begin{aligned}
H_\theta &= \frac{1}{2}(e^{i\theta}T+e^{-i\theta}T^{\sharp_A}) \\
\Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2+e^{-2i\theta}{T^{\sharp_A}}^2+P \\
\Rightarrow 16{H_\theta}^4 &= \left(e^{2i\theta}T^2+e^{-2i\theta}{T^{\sharp_A}}^2+P\right)\left(e^{2i\theta}T^2+e^{-2i\theta}{T^{\sharp_A}}^2+P\right) \\
&= \left(e^{2i\theta}T^2+e^{-2i\theta}{T^{\sharp_A}}^2\right)^2+\left(e^{2i\theta}T^2+e^{-2i\theta}{T^{\sharp_A}}^2\right)P \\
&\quad +P\left(e^{2i\theta}T^2+e^{-2i\theta}{T^{\sharp_A}}^2\right)+P^2 \\
&= 4\left(Re_A(e^{2i\theta}T^2)\right)^2+2Re_A(e^{2i\theta}(T^2P+PT^2))+P^2 \\
\Rightarrow \|{H_\theta}^4\|_A &\leq \frac{1}{4}\left\|Re_A(e^{2i\theta}T^2)\right\|_A^2+\frac{1}{8}\left\|Re_A(e^{2i\theta}(T^2P+PT^2))\right\|_A+\frac{1}{16}\|P\|_A^2 \\
&\leq \frac{1}{4}w_A^2(T^2)+\frac{1}{8}w_A(T^2P+PT^2)+\frac{1}{16}\|P\|_A^2.
\end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$, we get

$$w_A^4(T) \leq \frac{1}{4}w_A^2(T^2)+\frac{1}{8}w_A(T^2P+PT^2)+\frac{1}{16}\|P\|_A^2.$$

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REMARK 2.13. Using the inequality in Corollary 3.3, it is easy to see that if $A > 0$ then $w_A(T^2P+PT^2) \leq 2w_A(T^2)\|P\|_A$. In case $A > 0$, we would like to remark that the inequality obtained in Theorem 2.12 improves on the inequality [25, Theorem 2.11] obtained by Zamani. As for numerical example, if we consider $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $\mathbb{C}^3$, then by simple computation we have

$$\frac{1}{4}w_A^2(T^2) + \frac{1}{8}w_A(T^2P + PT^2) + \frac{1}{16}\|P\|_A^2 = \frac{39}{16} < \frac{1}{16}\left(\|P\|_A + 2w_A(T^2)\right)^2 = \frac{49}{16}.$$

Now we prove the following theorem.

THEOREM 2.14. *Let $T \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then*

$$w_A^3(T) \leq \frac{1}{4}w_A(T^3) + \frac{1}{4}w_A(T^2T^{\sharp_A} + T^{\sharp_A}T^2 + TT^{\sharp_A}T).$$

Moreover, if $T^2 = 0$, then $w_A(T) = \frac{1}{2}\sqrt{\|TT^{\sharp_A} + T^{\sharp_A}T\|_A}$, and if $T^3 = 0$, then $w_A^3(T) = \frac{1}{4}w_A(T^2T^{\sharp_A} + T^{\sharp_A}T^2 + TT^{\sharp_A}T)$.

Proof. From Lemma 2.3, we have $w_A(T) = \sup_{\theta\in\mathbb{R}} \|H_\theta\|_A$ where $H_\theta = Re_A(e^{i\theta}T)$. Then,

$$\begin{aligned}
H_\theta &= \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^{\sharp_A}) \\
\Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2 + T^{\sharp_A}T + TT^{\sharp_A} \\
\Rightarrow 8H_\theta^3 &= \left(e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2 + T^{\sharp_A}T + TT^{\sharp_A}\right)(e^{i\theta}T + e^{-i\theta}T^{\sharp_A}) \\
\Rightarrow H_\theta^3 &= \frac{1}{4}Re_A(e^{3i\theta}T^3) + \frac{1}{4}Re_A(e^{i\theta}(T^2T^{\sharp_A} + T^{\sharp_A}T^2 + TT^{\sharp_A}T) \\
\Rightarrow \|H_\theta^3\|_A &\leq \frac{1}{4}\|Re_A(e^{3i\theta}T^3)\|_A + \frac{1}{4}\|Re_A(e^{i\theta}(T^2T^{\sharp_A} + T^{\sharp_A}T^2 + TT^{\sharp_A}T))\|_A \\
&\leq \frac{1}{4}w_A(T^3) + \frac{1}{4}w_A(T^2T^{\sharp_A} + T^{\sharp_A}T^2 + TT^{\sharp_A}T).
\end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$, we get the desired inequality.

If $T^2 = 0$, then $4{H_\theta}^2 = T^{\sharp_A}T + TT^{\sharp_A}$, and so, $w_A(T) = \frac{1}{2}\sqrt{\|TT^{\sharp_A} + T^{\sharp_A}T\|_A}$.

If $T^3 = 0$, then $H_\theta^3 = \frac{1}{4}Re_A(e^{i\theta}(T^2T^{\sharp_A} + T^{\sharp_A}T^2 + TT^{\sharp_A}T))$, and so, $w_A^3(T) = \frac{1}{4}w_A(T^2T^{\sharp_A} + T^{\sharp_A}T^2 + TT^{\sharp_A}T)$. ∎

REMARK 2.15. Here we would like to remark that the bound obtained in Theorem 2.14 improves on the existing upper bound in [25, Corollary 2.8] when $A > 0$. Note that if $T^2 = 0$ then $w_A(T) = \frac{1}{2}\sqrt{\|TT^{\sharp_A} + T^{\sharp_A}T\|_A}$. But the converse is not true, that is, $w_A(T) = \frac{1}{2}\sqrt{\|TT^{\sharp_A} + T^{\sharp_A}T\|_A}$ does not always imply $T^2 = O$. As for example, we consider $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $\mathbb{C}^3$. Then we see that $w_A(T) = \frac{1}{2}\sqrt{\|TT^{\sharp_A} + T^{\sharp_A}T\|_A} = 1$ but $T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq O$.

Next we prove the following inequality.

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THEOREM 2.16. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then for each $r \geq 1$,*

$$w_A^{2r}(T) \leq \frac{1}{2} w_A^r(T^2) + \frac{1}{4} \left\| (T^{\sharp_A} T)^r + (T T^{\sharp_A})^r \right\|_A.$$

Proof. From Lemma 2.3, we get $w_A(T) = \sup_{\theta \in \mathbb{R}} \|H_\theta\|_A$, where $H_\theta = Re_A(e^{i\theta}T)$. Now,

$$\begin{aligned} H_\theta &= \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^{\sharp_A}) \\ \Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2 + T^{\sharp_A}T + TT^{\sharp_A} \\ \Rightarrow {H_\theta}^2 &= \frac{1}{2} Re_A(e^{2i\theta}T^2) + \frac{1}{4}(T^{\sharp_A}T + TT^{\sharp_A}) \\ \Rightarrow \|{H_\theta}^2\|_A &\leq \frac{1}{2}\left\| Re_A(e^{2i\theta}T^2)\right\|_A + \frac{1}{4}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A \end{aligned}$$

For $r \geq 1$, t^r and $t^{\frac{1}{r}}$ are convex and concave functions, respectively, and using that, we get

$$\begin{aligned} \|{H_\theta}^2\|_A^r &\leq \left\{ \frac{1}{2}\left\| Re_A(e^{2i\theta}T^2)\right\|_A + \frac{1}{2}\left\| \frac{T^{\sharp_A}T + TT^{\sharp_A}}{2}\right\|_A \right\}^r \\ &\leq \frac{1}{2}\left\| Re_A(e^{2i\theta}T^2)\right\|_A^r + \frac{1}{2}\left\| \frac{T^{\sharp_A}T + TT^{\sharp_A}}{2}\right\|_A^r \\ &\leq \frac{1}{2}\left\| Re_A(e^{2i\theta}T^2)\right\|_A^r + \frac{1}{2}\left\| \left(\frac{(T^{\sharp_A}T)^r + (TT^{\sharp_A})^r}{2}\right)^{\frac{1}{r}}\right\|_A^r \\ &= \frac{1}{2}\left\| Re_A(e^{2i\theta}T^2)\right\|_A^r + \frac{1}{2}\left\| \frac{(T^{\sharp_A}T)^r + (TT^{\sharp_A})^r}{2}\right\|_A \\ &\leq \frac{1}{2} w_A^r(T^2) + \frac{1}{4}\left\| (T^{\sharp_A}T)^r + (TT^{\sharp_A})^r\right\|_A. \end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$, we get

$$w_A^{2r}(T) \leq \frac{1}{2} w_A^r(T^2) + \frac{1}{4} \left\| (T^{\sharp_A} T)^r + (T T^{\sharp_A})^r \right\|_A.$$

REMARK 2.17. Here, we would like to remark that if we take $r = 1$ in the above Theorem 2.16, we get the inequality [25, Theorem 2.11] proved by Zamani.

Now we obtain a lower bound for A-numerical radius.

THEOREM 2.18. *Let $T \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then*

$$w_A^4(T) \geq \frac{1}{4} C_A^2(T^2) + \frac{1}{8} c_A(T^2P + PT^2) + \frac{1}{16}\|P\|_A^2,$$

where $P = T^{\sharp_A}T + TT^{\sharp_A}$, $C_A(T) = \inf_{\|x\|_A = 1} \inf_{\phi \in \mathbb{R}} \|Re_A(e^{i\phi}T)x\|_A$.

Proof. We know that $w_A(T) = \sup_{\phi \in \mathbb{R}} \|H_\phi\|_A$, where $H_\phi = Re_A(e^{i\phi}T)$. Let x be a unit vector in H and θ be a real number such that

$$e^{2i\theta}\langle (T^2P + PT^2)x, x\rangle_A = |\langle (T^2P + PT^2)x, x\rangle_A|.$$

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Then,

$$\begin{aligned}
H_\theta &= \frac{1}{2}(e^{i\theta}T + e^{-i\theta}T^{\sharp_A}) \\
\Rightarrow 4{H_\theta}^2 &= e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2 + P \\
\Rightarrow 16{H_\theta}^4 &= \big(e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2 + P\big)\big(e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2 + P\big) \\
&= \big(e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2\big)^2 + \big(e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2\big)P \\
&\quad +P\big(e^{2i\theta}T^2 + e^{-2i\theta}{T^{\sharp_A}}^2\big) + P^2 \\
&= 4\big(Re_A(e^{2i\theta}T^2)\big)^2 + 2Re_A(e^{2i\theta}(T^2P + PT^2)) + P^2 \\
\Rightarrow 16w_A^4(T) &\geq \|4\big(Re_A(e^{2i\theta}T^2)\big)^2 + 2Re_A(e^{2i\theta}(T^2P + PT^2)) + P^2\|_A \\
&\geq |\langle\big(4\big(Re_A(e^{2i\theta}T^2)\big)^2 + 2Re_A(e^{2i\theta}(T^2P + PT^2)) + P^2\big)x, x\rangle_A| \\
&= |4\langle\big(Re_A(e^{2i\theta}T^2)\big)^2x, x\rangle_A + 2Re_A(e^{2i\theta}\langle(T^2P + PT^2)x, x\rangle_A) + \langle P^2x, x\rangle_A| \\
&= 4\|\big(Re_A(e^{2i\theta}T^2)\big)x\|_A^2 + 2|\langle(T^2P + PT^2)x, x\rangle_A| + \|Px\|_A^2 \\
&\geq 4\|\big(Re_A(e^{2i\theta}T^2)\big)x\|_A^2 + 2c_A(T^2P + PT^2) + \|Px\|_A^2 \\
\Rightarrow 16w_A^4(T) &\geq 4C_A^2(T^2) + 2c_A(T^2P + PT^2) + \sup_{\|x\|_A=1}\|Px\|_A^2 \\
&= 4C_A^2(T^2) + 2c_A(T^2P + PT^2) + \|P\|_A^2 \\
\Rightarrow w_A^4(T) &\geq \frac{1}{4}C_A^2(T^2) + \frac{1}{8}c_A(T^2P + PT^2) + \frac{1}{16}\|P\|_A^2.
\end{aligned}$$

This completes the proof. ∎

Remark 2.19. It is clear that $\frac{1}{4}C_A^2(T^2) + \frac{1}{8}c_A(T^2P + PT^2) + \frac{1}{16}\|P\|_A^2 \geq \frac{1}{16}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A^2 \geq \frac{1}{16}\|T\|_A^4$. So, if $A > 0$, then the inequality obtained in Theorem 2.18 is better than the first inequality in [25, Corollary 2.8], obtained by Zamani.

3. A-numerical radius inequalities for product of operators in $\mathcal{B}_A(\mathcal{H})$.

We begin this section with the following A-numerical radius inequality for sum of product of operators.

Theorem 3.1. *Let $P, Q, X, Y \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then*

$$w_A(PXQ^{\sharp_A} \pm QYP^{\sharp_A}) \leq 2\|P\|_A\|Q\|_A w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}.$$

In particular,

$$w_A(PXQ^{\sharp_A} \pm QXP^{\sharp_A}) \leq 2\|P\|_A\|Q\|_A w_A(X).$$

Proof. Let $C = \begin{pmatrix} P & Q \\ O & O \end{pmatrix}$ and $Z = \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$. Then, from an easy calculation, we get

$$CZC^{\sharp_B} = \begin{pmatrix} PXQ^{\sharp_A} + QYP^{\sharp_A} & O \\ O & O \end{pmatrix}.$$

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Therefore,

$$\begin{aligned} w_A(PXQ^{\sharp_A}+QYP^{\sharp_A}) &= w_B\begin{pmatrix} PXQ^{\sharp_A}+QYP^{\sharp_A} & O \\ O & O \end{pmatrix} \\ &= w_B(CZC^{\sharp_B}), \quad \text{using Lemma 2.4 } (i) \\ &\le \|C\|_B^2 w_B(Z), \quad \text{using [25, Lemma 4.4]} \\ &= \|PP^{\sharp_A}+QQ^{\sharp_A}\|_A w_B(Z) \\ &\le (\|P\|_A^2+\|Q\|_A^2) w_B(Z). \end{aligned}$$

Replacing P and Q by tP and $\frac{1}{t}Q$, respectively, with $t>0$ in the above inequality, we get

$$w_A(PXQ^{\sharp_A}+QYP^{\sharp_A}) \le \left(\frac{t^4\|P\|_A^2+\|Q\|_A^2}{t^2}\right) w_B(Z).$$

Note that

$$\min_{t>0} \frac{t^4\|P\|_A^2+\|Q\|_A^2}{t^2} = 2\|P\|_A\|Q\|_A,$$

and so,

$$w_A(PXQ^{\sharp_A}+QYP^{\sharp_A}) \le 2\|P\|_A\|Q\|_A w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}.$$

Replacing Y by $-Y$ in the above inequality and using Lemma 2.4 (iii), we get

$$w_A(PXQ^{\sharp_A}-QYP^{\sharp_A}) \le 2\|P\|_A\|Q\|_A w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}.$$

Taking $X=Y$ and using Lemma 2.4 (iv), we get

$$w_A(PXQ^{\sharp_A}\pm QXP^{\sharp_A}) \le 2\|P\|_A\|Q\|_A w_A(X).$$

This completes the proof of the theorem. □

Remark 3.2. Here, we note that the inequality

$$w_A(PXQ^{\sharp_A}+QYP^{\sharp_A}) \le 2\|P\|_A\|Q\|_A w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}$$

in Theorem 3.1 holds also when $A \ge 0$.

Considering $X=Y=T$ (say), $P=I$ in Theorem 3.1, we get the following inequality.

Corollary 3.3. *Let $T, Q \in \mathcal{B}_A(\mathcal{H})$, where $A>0$. Then*

$$w_A(TQ^{\sharp_A}\pm QT) \le 2w_A(T)\|Q\|_A.$$

Next we prove the following lemma, the idea of which is based on the result [6, Lemma 3] proved by Bernau and Smithes.

Lemma 3.4. *Let $X, T, Y \in \mathcal{B}_A(\mathcal{H})$ where $A>0$. Then, for all $x \in \mathcal{H}$,*

$$|\langle X^{\sharp_A}TYx, x\rangle_A| + |\langle Y^{\sharp_A}TXx, x\rangle_A| \le 2w_A(T)\|Xx\|_A\|Yx\|_A.$$

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Proof. Let $x \in \mathcal{H}$ and θ, ϕ be real numbers such that $e^{i\phi}\langle Y^{\sharp_A}TXx, x\rangle_A = |\langle Y^{\sharp_A}TXx, x\rangle_A|$, $e^{2i\theta}\langle e^{-i\phi}X^{\sharp_A}TYx, x\rangle_A = |\langle e^{-i\phi}X^{\sharp_A}TYx, x\rangle_A| = |\langle X^{\sharp_A}TYx, x\rangle_A|$. Then, for a non-zero real number λ, we have

$$\begin{aligned}
2e^{2i\theta}\langle TYx, e^{i\phi}Xx\rangle_A + 2e^{i\phi}\langle TXx, Yx\rangle_A &= \langle e^{i\theta}T\left(\lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\rangle_A \\
&\quad -\langle e^{i\theta}T\left(\lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\rangle_A \\
\Rightarrow 2e^{2i\theta}\langle e^{-i\phi}X^{\sharp_A}TYx, x\rangle_A + 2e^{i\phi}\langle Y^{\sharp_A}TXx, x\rangle_A &= \langle e^{i\theta}T\left(\lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\rangle_A \\
&\quad -\langle e^{i\theta}T\left(\lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\rangle_A \\
\Rightarrow 2\left|\langle X^{\sharp_A}TYx, x\rangle_A\right| + 2\left|\langle Y^{\sharp_A}TXx, x\rangle_A\right| &= \langle e^{i\theta}T\left(\lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\rangle_A \\
&\quad -\langle e^{i\theta}T\left(\lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\rangle_A \\
\Rightarrow 2\left|\langle X^{\sharp_A}TYx, x\rangle_A\right| + 2\left|\langle Y^{\sharp_A}TXx, x\rangle_A\right| &\leq \left|\langle e^{i\theta}T\left(\lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\rangle_A\right| \\
&\quad +\left|\langle e^{i\theta}T\left(\lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\right), \lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\rangle_A\right| \\
\Rightarrow 2\left|\langle X^{\sharp_A}TYx, x\rangle_A\right| + 2\left|\langle Y^{\sharp_A}TXx, x\rangle_A\right| &\leq w_A(T)\left(\left\|\lambda e^{i\theta}Yx + \frac{1}{\lambda}e^{i\phi}Xx\right\|_A^2 + \left\|\lambda e^{i\theta}Yx - \frac{1}{\lambda}e^{i\phi}Xx\right\|_A^2\right) \\
\Rightarrow \left|\langle X^{\sharp_A}TYx, x\rangle_A\right| + \left|\langle Y^{\sharp_A}TXx, x\rangle_A\right| &\leq w_A(T)\left(\lambda^2\|Yx\|_A^2 + \frac{1}{\lambda^2}\|Xx\|_A^2\right).
\end{aligned}$$

This holds for all non-zero real λ. If $\|Yx\|_A \neq 0$, then we choose $\lambda^2 = \frac{\|Xx\|_A}{\|Yx\|_A}$. So, we get

$$|\langle X^{\sharp_A}TYx, x\rangle_A| + |\langle Y^{\sharp_A}TXx, x\rangle_A| \leq 2w_A(T)\|Xx\|_A\|Yx\|_A.$$

Clearly, this inequality also holds when $\|Yx\|_A = 0$, i.e., $Yx = 0$. This completes the proof of the lemma. □

Remark 3.5. In [11], we have already generalized the result obtained by Bernau and Smithes [6, Lemma 3], and proved some important numerical radius inequalities.

Now using Lemma 3.4, we obtain the following inequalities involving A-numerical radius, A-Crawford number and A-operator norm.

Theorem 3.6. *Let $X, T, Y \in \mathcal{B}_A(\mathcal{H})$, where $A > 0$. Then*

$$c_A(X^{\sharp_A}TY) + w_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A,$$

$$w_A(X^{\sharp_A}TY) + c_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

Proof. Taking $\|x\|_A = 1$ in Lemma 3.4, we have

$$\begin{aligned}
&|\langle X^{\sharp_A}TYx, x\rangle_A| + |\langle Y^{\sharp_A}TXx, x\rangle_A| \leq 2w_A(T)\|X\|_A\|Y\|_A \\
\Rightarrow\, &c_A(X^{\sharp_A}TY) + |\langle Y^{\sharp_A}TXx, x\rangle_A| \leq 2w_A(T)\|X\|_A\|Y\|_A.
\end{aligned}$$

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Taking supremum over $\|x\|_A = 1$, we get

$$c_A(X^{\sharp_A}TY) + w_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

Again taking $\|x\|_A = 1$ in Lemma 3.4, we have

$$\begin{aligned} &|\langle X^{\sharp_A}TYx, x\rangle_A| + |\langle Y^{\sharp_A}TXx, x\rangle_A| \leq 2w_A(T)\|X\|_A\|Y\|_A \\ \Rightarrow &|\langle X^{\sharp_A}TYx, x\rangle_A| + c_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A. \end{aligned}$$

Taking supremum over $\|x\|_A = 1$, we get

$$w_A(X^{\sharp_A}TY) + c_A(Y^{\sharp_A}TX) \leq 2w_A(T)\|X\|_A\|Y\|_A.$$

This completes the proof of the theorem. □

Now taking $Y = I$, $T = X$ and $X = Y$ in the above Theorem 3.6, we get the following upper bounds for the numerical radius of product of two operators, which improve on the existing bounds.

Corollary 3.7. *Let $X, Y \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then the following inequalities hold:*

$$\begin{aligned} w_A(XY) &\leq 2w_A(X)\|Y\|_A - c_A(Y^{\sharp_A}X), \\ w_A(XY) &\leq 2w_A(Y)\|X\|_A - c_A(YX^{\sharp_A}). \end{aligned}$$

Remark 3.8. For $A > 0$, it is clear that the inequalities obtained in Corollary 3.7 improve on the inequalities $w_A(XY) \leq 2w_A(X)\|Y\|_A$ and $w_A(XY) \leq 2w_A(Y)\|X\|_A$ (see [25, Theorem 3.4]).

Finally, using Lemma 3.4, we obtain new inequalities for B-numerical radius of 2×2 operator matrices with zero operators as main diagonal entries.

Theorem 3.9. *Let $X, Y \in \mathcal{B}_A(\mathcal{H})$ where $A > 0$. Then the following inequalities hold:*

$$(i) \quad \|X\|_A^2 + c_A(YX) \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A,$$

$$(ii) \quad m_A^2(X) + w_A(YX) \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A,$$

$$(iii) \quad \|Y\|_A^2 + c_A(XY) \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|Y\|_A,$$

$$(iv) \quad m_A^2(Y) + w_A(XY) \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|Y\|_A.$$

Proof. Taking $X = T$ and $Y = I$ in Lemma 3.4, we get

$$\|Tx\|_A^2 + |\langle T^2x, x\rangle_A| \leq 2w_A(T)\|Tx\|_A\|x\|_A.$$

This also holds if we take $T = \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$ and $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\|_B = 1$, i.e., $\|x_1\|_A^2 + \|x_2\|_A^2 = 1$. Therefore, we get

$$\|Xx_2\|_A^2 + \|Yx_1\|_A^2 + |\langle XYx_1, x_1\rangle_A + \langle YXx_2, x_2\rangle_A| \leq 2w_B(T)\left(\|Xx_2\|_A^2 + \|Yx_1\|_A^2\right)^{\frac{1}{2}}.$$

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Taking $x_1 = 0$, we get

$$\begin{aligned}
&\|Xx_2\|_A^2 + |\langle YXx_2, x_2\rangle|_A \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|Xx_2\|_A \\
\Rightarrow&\|Xx_2\|_A^2 + |\langle YXx_2, x_2\rangle_A| \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A \\
\Rightarrow&\|Xx_2\|_A^2 + c_A(YX) \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A.
\end{aligned}$$

Taking supremum over $\|x_2\|_A = 1$, we get the inequality (i), i.e.,

$$\|X\|_A^2 + c_A(YX) \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A.$$

Again from the inequality

$$\|Xx_2\|_A^2 + |\langle YXx_2, x_2\rangle_A| \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A,$$

we get

$$m_A^2(X) + |\langle YXx_2, x_2\rangle_A| \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A.$$

Taking supremum over $\|x_2\|_A = 1$, we get the inequality (ii), i.e.,

$$m_A^2(X) + w_A(YX) \leq 2w_B\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\|X\|_A.$$

Similarly, taking $x_2 = 0$ and supremum over $\|x_1\|_A = 1$, we can prove the remaining inequalities. □

Next taking $X = Y = T$ in Theorem 3.9 and using Lemma 2.4 (iv), we get the following lower bounds for A-numerical radius.

Theorem 3.10. *Let $T \in \mathcal{B}_A(\mathcal{H})$ with $\|T\|_A \neq 0$ where $A > 0$. Then the following inequalities hold:*

$$w_A(T) \geq \frac{\|T\|_A}{2} + \frac{c_A(T^2)}{2\|T\|_A},$$

$$w_A(T) \geq \frac{m_A^2(T)}{2\|T\|_A} + \frac{w_A(T^2)}{2\|T\|_A}.$$

Remark 3.11. Here, we note that the two inequalities obtain in Theorem 3.10 are incomparable. So, using these bounds we have a new lower bound

$$w_A(T) \geq \frac{1}{2\|T\|_A}\max\left\{\|T\|_A^2 + c_A(T^2), m_A^2(T) + w_A(T^2)\right\},$$

where $T \in \mathcal{B}_A(\mathcal{H})$ with $\|T\|_A \neq 0$. It is clear that this inequality improves on the first inequality in [25, Cor. 2.8].

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