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THE MAXIMAL $\alpha-$INDEX OF TREES WITH K PENDENT VERTICES AND ITS COMPUTATION*

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Abstract. Let G be a graph with adjacency matrix $A(G)$ and let $D(G)$ be the diagonal matrix of the degrees of G. The $\alpha-$index of G is the spectral radius $\rho_\alpha(G)$ of the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $\alpha \in [0,1]$. Let $T_{n,k}$ be the tree of order n and k pendent vertices obtained from a star $K_{1,k}$ and k pendent paths of almost equal lengths attached to different pendent vertices of $K_{1,k}$. It is shown that if $\alpha \in [0,1)$ and T is a tree of order n with k pendent vertices then

$$\rho_\alpha(T) \leq \rho_\alpha(T_{n,k}),$$

with equality holding if and only if $T = T_{n,k}$. This result generalizes a theorem of Wu, Xiao and Hong [6] in which the result is proved for the adjacency matrix ($\alpha = 0$). Let $q = [\frac{n-1}{k}]$ and $n-1 = kq + r$, $0 \leq r \leq k-1$. It is also obtained that the spectrum of $A_\alpha(T_{n,k})$ is the union of the spectra of two special symmetric tridiagonal matrices of order q and $q+1$ when $r = 0$ or the union of the spectra of three special symmetric tridiagonal matrices of order q, $q+1$ and $2q+2$ when $r \neq 0$. Thus, the $\alpha-$index of $T_{n,k}$ can be computed as the largest eigenvalue of the special symmetric tridiagonal matrix of order $q+1$ if $r = 0$ or order $2q+2$ if $r \neq 0$.

Key words. Convex combination of matrices, Signless Laplacian, Adjacency matrix, Tree, Pendent vertices, Spectral radius.

AMS subject classifications. 05C50, 15A48.

1. Introduction.

Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. Let $D(G)$ be the diagonal matrix of order n whose $(i,i)-$entry is the degree of the $i-th$ vertex of G and let $A(G)$ be the adjacency matrix of G.

As usual, $K_{1,s}$ denotes the star on $s+1$ vertices, K_n and P_n are the complete graph and the path, both on n vertices, respectively.

In [2], Nikiforov introduces the matrix $A_\alpha(G)$,

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G),$$

with $\alpha \in [0,1]$ together with basic results and several open problems. Observe that $A_\alpha(G)$ is a symmetric nonnegative matrix for all $\alpha \in [0,1]$ and that $A_0(G) = A(G)$ and $A_{1/2}(G) = \frac{1}{2}(D(G) + A(G)) = \frac{1}{2}Q(G)$. Since $A_1(G) = D(G)$, from now on, we take $\alpha \in [0,1)$.

Let $\rho_\alpha(G)$ be the $\alpha-$index of G, that is, the spectral radius of $A_\alpha(G)$. From the Perron - Frobenius Theory for nonnegative matrices, it follows that for a connected graph G, $\rho_\alpha(G)$ (Perron root) is a simple eigenvalue of $A_\alpha(G)$ having a positive eigenvector (Perron vector).

Let n and k given positive integers with $2 \leq k \leq n-1$. Let $T_{n,k}$ be the tree of order n and k pendent vertices obtained from a star $K_{1,k}$ and k pendent paths of almost equal lengths attached to different

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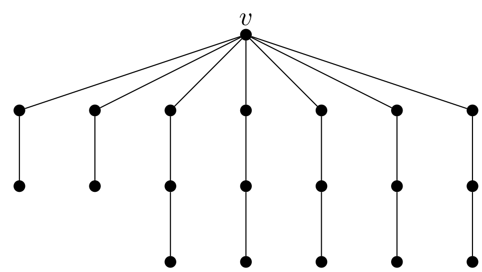
pendent vertices of $K_{1,k}$. More precisely, if $q = [\frac{n-1}{k}]$ and $n-1 = kq+r$, $0 \leq r \leq k-1$, then $T_{n,k}$ is the tree obtained from the star $K_{1,k}$ together with $k-r$ pendent paths P_q and r pendent paths P_{q+1} attached to different pendent vertices of $K_{1,k}$ whenever $r \neq 0$ (see Example 1.1). If $r=0$, then $T_{n,k}$ is the tree obtained from the star $K_{1,k}$ and k pendent paths P_q attached to different vertices of $K_{1,k}$ (see Example 1.2). Clearly, $T_{n,k}$ is a tree having exactly k pendents vertices and the number of vertices of $T_{n,k}$ is $(k-r)q + r(q+1) + 1 = kq + r + 1 = n$.

Example 1.1. Let $n = 20$ and $k = 7$. Then $q = 2$ and $r = 5$. The tree $T_{20,7}$ is displayed below:



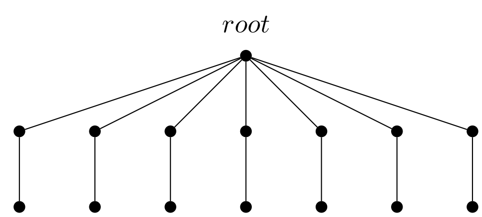
In Section 2, we prove that if $\alpha \in [0,1)$ and T is a tree of order n with k pendent vertices, then

$$\rho_\alpha(T) \leq \rho_\alpha(T_{n,k}),$$

with equality holding if and only if $T = T_{n,k}$. This result generalizes a theorem of Wu, Xiao and Hong [6] in which the result is proved for the adjacency matrix ($\alpha = 0$).

A rooted graph is a graph in which one vertex has been designated as a special vertex called the root. Given a rooted graph the level of a vertex is one more than its distance to the root vertex. A generalized Bethe tree is a rooted tree in which vertices at the same level have the same degree. For instance, if $r = 0$, then $T_{n,k}$ is a generalized Bethe tree. In Example 1.2, we illustrate this case.

Example 1.2. Let $n = 15$ and $k = 7$. Then $q = 2$ and $r = 0$. The tree $T_{15,7}$ is displayed below:



If $r \neq 0$, then $T_{n,k}$ is a tree defined by the coalescence of two generalized Bethe trees at their roots (see Example 1.1).

Let $\{B_i : 1 \leq i \leq m\}$ be a set of trees such that, for $i = 1, 2, \ldots, m$. Then,

(1) B_i is a generalized Bethe tree of k_i levels,

(2) the vertices of B_i at the level j have degree d_{i,k_i-j+1} for $j = 1, 2, \ldots, k_i$, and

(3) the edges of B_i joining the vertices at the level j with the vertices at the level $(j+1)$ have weight w_{i,k_i-j} for $j = 1, 2, \ldots, k_i - 1$.

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Let $v\{B_i : 1 \leq i \leq m\}$ be the tree obtained from the coalescence of the trees B_i at their roots in a common vertex v.

The Laplacian matrix of G is $L(G) = D(G) - A(G)$. In [5], we give a complete characterization of the eigenvalues of the Laplacian matrix and adjacency matrix of $v\{B_i : 1 \leq i \leq m\}$ including results on their multiplicities. In Section 3, we extend these results to $A_\alpha(v\{B_i : 1 \leq i \leq m\})$. Finally, in Section 4, we apply the results of Section 3 to deduce that the spectrum of $A_\alpha(T_{n,k})$ is the union of the spectra of two special symmetric tridiagonal matrices of order q and $q+1$ when $r = 0$ or the union of the spectra of three special symmetric tridiagonal matrices of order q, $q+1$ and $2q+2$ when $r \neq 0$. Thus, the $\alpha-$index of $T_{n,k}$ can be computed as the largest eigenvalue of the special symmetric tridiagonal matrix of order $q+1$ if $r = 0$ or order $2q+2$ if $r \neq 0$.

2. The maximal $\alpha-$index of trees with k pendent vertices. In [6], the authors proved the following:

THEOREM 2.1. (Wu, Xiao, and Hong [6]) *Among all trees on n vertices and k pendent vertices, the maximal spectral radius of the adjacency matrix is obtained uniquely at $T_{n,k}$.*

In this section, we extend Theorem 2.1 to all $\alpha \in [0, 1)$. We begin recalling the following lemma that generalizes results known for the adjacency matrix and the signless Laplacian matrix of graphs.

LEMMA 2.2. (Nikiforov and Rojo [4]) *Let $\alpha \in [0, 1)$ and let G be a graph of order n. Suppose that $u, v \in V(G)$ and $S \subset V(G)$ satisfy $u, v \notin S$ and for every $w \in S$, $\{u, w\} \in E(G)$ and $\{v, w\} \notin E(G)$. Let H be the graph obtained by deleting the edges $\{u, w\}$ and adding the edges $\{v, w\}$ for all $w \in S$. If S is nonempty and there is a positive eigenvector $(x_1, \ldots, x_n)$ to $\rho_\alpha(G)$ such that $x_v \geq x_u$, then*

$$\rho_\alpha(H) > \rho_\alpha(G).$$

For any vertex u of a connected graph G, let $G_{p,q}(u)$ be the graph obtained by attaching the paths P_p and P_q to u. This is done by identifying one end vertex of P_p and one end vertex of P_q with u. The following theorem was proposed as a Conjecture 18 in [4].

THEOREM 2.3. (Lin, Huang, and Xue [1]) *Let $\alpha \in [0, 1)$. If G is a connected graph and $p \geq q + 2 \geq 3$, then*

$$\rho_\alpha(G_{p,q}(u)) < \rho_\alpha(G_{p-1,q+1}(u)).$$

Given a graph G and a vertex $u \in V(G)$, let $\Gamma_G(u)$ be the set of neighbors of u.

We are ready to extend Theorem 2.1 to all $\alpha \in [0, 1)$.

THEOREM 2.4. *Let $\alpha \in [0, 1)$ and T be a tree of order n and k pendent vertices. Then*

$$\rho_\alpha(T) \leq \rho_\alpha(T_{n,k}),$$

with equality if and only if $T = T_{n,k}$.

Proof. Let T be a tree on n vertices and k pendent vertices. Let d_v be the degree of $v \in V(T)$. Let t be the number of vertices of T with a degree greater than or equal to 3. The following cases can occur:

Case 1: $t = 0$. In this case, $T = P_n = T_{n,2}$. Then $\rho_\alpha(T) = \rho_\alpha(T_{n,2})$.

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Case 2: $t = 1$. Repeated application of Theorem 2.3 enables to conclude that $\rho_\alpha(T) \le \rho_\alpha(T_{n,k})$ with equality if only if $T = T_{n,k}$.

Case 3: $t > 1$. Let $\mathbf{x}$ be a positive unit eigenvector corresponding to $\rho_\alpha(T)$ in which x_v is the component of $\mathbf{x}$ corresponding to $v \in V(T)$. Let $u, v \in V(T)$ such that $d_u \ge 3$ and $d_v \ge 3$. There is no loss of generality in assuming $x_u \ge x_v$. There is a unique path P connecting u and v and let $z \in P$ be unique neighbour of v. Let $v_1, \ldots, v_{d_v-2} \in \Gamma_T(v) \setminus z$. Let T_1 be the tree obtained from T by deleting the edges $\{v, v_1\}, \ldots, \{v, v_{d_v-2}\}$ and adding the edges $\{u, v_1\}, \ldots, \{u, v_{d_v-2}\}$. Clearly T_1 is a tree of order n with k pendent vertices having $t-1$ vertices with a degree greater than or equal to 3. Since $x_u \ge x_v$, by Lemma 2.2, it follows that $\rho_\alpha(T) < \rho_\alpha(T_1)$. If $t - 1 = 1$, we stop and if $t - 1 > 1$, we continue in this fashion to obtain a sequence of trees $T_1, T_2, \ldots, T_{t-1}$ of order n with k pendent vertices such that $\rho_\alpha(T) < \rho_\alpha(T_1) < \rho_\alpha(T_2) < \cdots < \rho_\alpha(T_{t-1})$, in which T_{t-1} has a unique vertex with a degree greater than or equal to 3. Finally, we apply Case 2 to conclude that $\rho_\alpha(T) < \rho_\alpha(T_{n,k})$. ☐

3. The $A_\alpha-$spectrum of the coalescence of generalized Bethe trees at their roots. Let $\sigma(M)$ be the spectrum of the matrix M. From now on, let $\beta = 1 - \alpha$.

The $A_\alpha-$spectrum of a generalized Bethe tree was studied in [3] and the results are presented in Theorem 3.2 below.

Let B_k be a generalized Bethe tree on k levels. For $j = 1, \ldots, k$, let n_{k-j+1} be the number of vertices at level j and let d_{k-j+1} be their degree. In particular, $d_1 = 1$ and $n_k = 1$. Let

$$\Omega = \{j : 1 \le j \le k-1, n_j > n_{j+1}\}. \tag{3.1}$$

DEFINITION 3.1. For $j = 1, 2, \ldots, k-1$, let T_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$T = \begin{bmatrix} \alpha & \beta\sqrt{d_2 - 1} & 0 & & 0 \\ \beta\sqrt{d_2 - 1} & \alpha d_2 & \ddots & & \\ & \ddots & \ddots & \beta\sqrt{d_{k-1} - 1} & \\ & & \beta\sqrt{d_{k-1} - 1} & \alpha d_{k-1} & \beta\sqrt{d_k} \\ 0 & & 0 & \beta\sqrt{d_k} & \alpha d_k \end{bmatrix}.$$

THEOREM 3.2. (Nikiforov and Rojo [3, Theorem 8]) *Let B_k be a generalized Bethe tree, and $\alpha \in [0, 1)$. If the matrices $T_1, \ldots, T_{k-1}, T$ are defined as in Definition 3.1, then:*

(*a*)
$$\sigma(A_\alpha(B_k)) = (\cup_{j\in\Omega}\sigma(T_j)) \cup \sigma(T).$$

(*b*) *The multiplicity of each eigenvalue of T_j as an eigenvalue of $A_\alpha(B_k)$ is $n_j - n_{j+1}$ if $j \in \Omega$ and the eigenvalues of T as eigenvalues of $A_\alpha(B_k)$ are simple. If some eigenvalues obtained in different matrices are equal, their multiplicities are added together.*

(*c*) *The largest eigenvalue of T is the largest eigenvalue of $A_\alpha(B_k)$.*

We now search for $A_\alpha-$spectrum of $v\{B_i : 1 \le i \le m\}$. We recall that $\{B_i : 1 \le i \le m\}$ is a set of trees such that, for $i = 1, 2, \ldots, m$,

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(1) B_i is a generalized Bethe tree of k_i levels,

(2) the vertices of B_i at the level j have degree d_{i,k_i-j+1} for $j = 1, 2, \ldots, k_i$, and

(3) the edges of B_i joining the vertices at the level j with the vertices at the level $(j+1)$ have weight w_{i,k_i-j} for $j = 1, 2, \ldots, k_i - 1$.

We recall the results obtained in [5] on the spectrum of $L(v\{B_i : 1 \leq i \leq m\})$. Assume that the common root v is at the level 1. For $j = 1, \ldots, k_i$, let n_{i,k_i-j+1} be the number of vertices at the level j of B_i. Let

$$\delta_{i,1} = w_{i,1},$$

$$\delta_{i,j} = (d_{i,j} - 1)\, w_{i,j-1} + w_{i,j}$$

for $j = 2, \ldots, k_i - 1$, and

$$\delta = \sum_{i=1}^{m} d_{i,k_i} w_{i,k_i-1}.$$

Definition 3.3. For $i = 1, \ldots, m$ and for $j = 1, \ldots, k_i - 1$, let $T_{i,j}$ be the $j \times j$ leading principal submatrix of the $(k_i - 1) \times (k_i - 1)$ symmetric tridiagonal matrix

$$T_{i,k_i-1} = \begin{bmatrix} \delta_{i,1} & w_{i,1}\sqrt{d_{i,2}-1} & & \\ w_{i,1}\sqrt{d_{i,2}-1} & \delta_{i,2} & \ddots & \\ & \ddots & \ddots & w_{i,k_i-2}\sqrt{d_{i,k_i-1}-1} \\ & & w_{i,k_i-2}\sqrt{d_{i,k_i-1}-1} & \delta_{i,k_i-1} \end{bmatrix}.$$

Definition 3.4. Let $r = \sum_{i=1}^{m} k_i - m + 1$. Let T be the symmetric matrix of order $r \times r$ defined by

$$T = \begin{bmatrix} T_{1,k_1-1} & 0 & \cdots & 0 & w_{1,k_1-1}\mathbf{p}_1 \\ 0 & T_{2,k_2-1} & \ddots & & w_{2,k_2-1}\mathbf{p}_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & & 0 & T_{m,k_m-1} & w_{m,k_m-1}\mathbf{p}_m \\ w_{1,k_1}\mathbf{p}_1^T & w_{2,k_2-1}\mathbf{p}_2^T & \cdots & w_{m,k_m-1}\mathbf{p}_m^T & \delta \end{bmatrix},$$

where $T_{1,k_1-1}, T_{2,k_2-1}, \ldots, T_{m,k_m-1}$ are the symmetric tridiagonal matrices defined in Definition 3.3 and

$$\mathbf{p}_i^T = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \sqrt{n_{i,k_i-1}} \end{bmatrix}$$

for $i = 1, \ldots, m$.

For $i = 1, \ldots, m$, let

$$\Omega_i = \{j : 1 \leq j \leq k_i - 1, n_{i,j} > n_{i,j+1}\}. \tag{3.2}$$

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THEOREM 3.5. (Rojo [5, Theorem 2]) *(a)* $\sigma\left(L\left(v\left\{B_i : 1 \le i \le m\right\}\right)\right) = \left(\cup_{i=1}^{m} \cup_{j\in\Omega_i} \sigma\left(T_{i,j}\right)\right) \cup \sigma\left(T\right),$ *where the matrices $T_{i,j}$ and T are as in Definitions 3.3 and 3.4.*

(b) The multiplicity of each eigenvalue of the matrix $T_{i,j}$, as an eigenvalue of $L\left(v\left\{B_i : 1 \le i \le m\right\}\right)$, is at least $\left(n_{i,j} - n_{i,j+1}\right)$ for $j \in \Omega_i$, and the eigenvalues of T as eigenvalues of $L\left(v\left\{B_i : 1 \le i \le m\right\}\right)$ are simple.

Taking into consideration that the diagonal entries $\delta_{i,j}$ and δ defined above become

$$\delta_{i,1} = \alpha,$$

$$\delta_{i,j} = \alpha d_{i,j},$$

for $j = 1, \dots, k_i - 1$, and

$$\delta = \alpha \sum_{i=1}^{m} d_{i,k_i}$$

in case of the matrix $A_\alpha(v\{B_i : 1 \le i \le m\})$ and using the fact that $A_\alpha(G)$ can be viewed as a matrix on a weighted graph G in which all its edges have a weight $\beta = 1 - \alpha$, the technique and the same steps used in [5] to obtain Theorem 3.5 can be applied to find the spectrum of $A_\alpha(v\{B_i : 1 \le i \le m\})$ getting that :

THEOREM 3.6. *(a)*

$$\sigma\left(A_\alpha\left(v\left\{B_i : 1 \le i \le m\right\}\right)\right) = \left(\cup_{i=1}^{m} \cup_{j\in\Omega_i} \sigma\left(T_{i,j}(\alpha)\right)\right) \cup \sigma\left(T(\alpha)\right),$$

where the matrices $T_{i,j}(\alpha)$ and $T(\alpha)$ are as in Definitions 3.7 and 3.8.

(b) The multiplicity of each eigenvalue of the matrix $T_{i,j}(\alpha)$, as an eigenvalue of $A_\alpha\left(v\left\{B_i : 1 \le i \le m\right\}\right)$, is at least $\left(n_{i,j} - n_{i,j+1}\right)$ for $j \in \Omega_i$, and the eigenvalues of $T(\alpha)$ as eigenvalues of $A_\alpha\left(v\left\{B_i : 1 \le i \le m\right\}\right)$ are simple.

DEFINITION 3.7. For $i = 1, 2, \dots, m$ and for $j = 1, 2, 3, \dots, k_i - 1$, let $T_{i,j}(\alpha)$ be the $j \times j$ leading principal submatrix of the $(k_i - 1) \times (k_i - 1)$ symmetric tridiagonal matrix

$$T_{i,k_i-1}(\alpha) = \begin{bmatrix} \alpha & \beta\sqrt{d_{i,2}-1} & & \\ \beta\sqrt{d_{i,2}-1} & \alpha d_{i,2} & \ddots & \\ & \ddots & \ddots & \beta\sqrt{d_{i,k_i-1}-1} \\ & & \beta\sqrt{d_{i,k_i-1}-1} & \alpha d_{i,k_i-1} \end{bmatrix}.$$

DEFINITION 3.8. Let $r = \sum_{i=1}^{m} k_i - m + 1$. Let $T(\alpha)$ be the symmetric matrix of order $r \times r$ defined by

$$T(\alpha) = \begin{bmatrix} T_{1,k_1-1}(\alpha) & 0 & \cdots & 0 & \beta\mathbf{p}_1 \\ 0 & T_{2,k_2-1}(\alpha) & \ddots & & \beta\mathbf{p}_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & & 0 & T_{m,k_m-1}(\alpha) & \beta\mathbf{p}_m \\ \beta\mathbf{p}_1^T & \beta\mathbf{p}_2^T & \cdots & \beta\mathbf{p}_m^T & \alpha\sum_{i=1}^{m} d_{i,k_i} \end{bmatrix},$$

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where $T_{1,k_1-1}(\alpha), T_{2,k_2-1}(\alpha), \ldots, T_{m,k_m-1}(\alpha)$ are the symmetric tridiagonal matrices defined in Definition 3.7 and

$$\mathbf{p}_i^T = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \sqrt{n_{i,k_i-1}} \end{bmatrix}$$

for $i = 1, \ldots, m$.

4. The $A_\alpha-$spectrum of $T_{n,k}$. We recall that $n-1 = kq+r$ where $q = [\frac{n-1}{q}]$ and $0 \leq r \leq k-1$. As we will see later, the matrix

$$T(\alpha) = \begin{bmatrix} \alpha & \beta & 0 & & 0 \\ \beta & 2\alpha & \ddots & & \\ & \ddots & \ddots & \beta & \\ & & \beta & 2\alpha & \beta\sqrt{k} \\ 0 & & 0 & \beta\sqrt{k} & k\alpha \end{bmatrix} \tag{4.3}$$

of the appropriate order plays a special role in this section.

We recall that if A is an $m \times m$ symmetric tridiagonal matrix with nonzero codiagonal entries then the eigenvalues of any $(m-1) \times (m-1)$ principal submatrix strictly interlace the eigenvalues of A. Hence, the eigenvalues of any symmetric tridiagonal matrix with nonzero codiagonal entries are simple.

There are two cases:

4.1. Case $r = 0$.

THEOREM 4.1. *Let $n = kq+1$. If the matrix $T(\alpha)$ in (4.3) is of order $q+1$ and $T_q(\alpha)$ is its leading principal submatrix of order q, then*

(*a*)

$$\sigma(A_\alpha(T_{n,k})) = \sigma(T_q(\alpha)) \cup \sigma(T(\alpha)); \tag{4.4}$$

(*b*) *the multiplicity of each eigenvalue of $T_q(\alpha)$ as an eigenvalue of $A_\alpha(T_{n,k})$ is exactly $k-1$, and the eigenvalues of $T(\alpha)$ as eigenvalues of $A_\alpha(T_{n,k})$ are simple; and*

(*c*) *the largest eigenvalue of $T(\alpha)$ is the $\alpha-$index of $T_{n,k}$.*

Proof. (a) Assume $r = 0$. Then $n = kq+1$ and $T_{n,k}$ is a generalized Bethe tree of $q+1$ levels in which, from the pendent vertices to the root, the vertex degrees and the number of vertices are

$$d_1 = 1, \quad d_2 = \cdots = d_q = 2, \quad d_{q+1} = k, \quad n_1 = n_2 = \cdots = n_q = k, \quad n_{q+1} = 1.$$

Then the set Ω in (3.1) is $\Omega = \{q\}$ and the matrix T in Definition 3.1 becomes the matrix $T(\alpha)$ in (4.3) of order $(q+1)$. We apply Theorem 3.2, part (a), to obtain that the $A_\alpha-$spectrum of $T_{n,q}$ is given by (4.4).

(b) The eigenvalues of $A_\alpha(T_{n,k})$ are the eigenvalues of $T_q(\alpha)$ and $T(\alpha)$; and, the eigenvalues of $T_q(\alpha)$ strictly interlace the eigenvalues of $T(\alpha)$. These facts and part (b) of Theorem 3.2 imply that the multiplicity of each eigenvalue of $T_q(\alpha)$ as eigenvalue of $A_\alpha(T_{n,k})$ is exactly $k-1$ and each eigenvalue of $T(\alpha)$ as eigenvalue of $A_\alpha(T_{n,k})$ is simple.

(c) It is an immediate consequence of the facts mentioned in the proof of part (b). ∎

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4.2. Case $r \neq 0$. At this point, we introduce the following additional notations: 0 is the all zeros matrix of the appropriate order, I_n is the identity matrix and R_n is the reversal identity matrix, both of order $n \times n$. We recall that R_n is a permutation matrix where the 1 entries reside on the back diagonal and all other entries are zero. If A is a matrix with n rows then R_nA reverses the rows of A and if A is a matrix with n columns then AR_n reverses the columns of A.

THEOREM 4.2. *Let $n = kq + r + 1$ with $0 < r \leq k - 1$. If the matrix $T_q(\alpha)$ and $T_{q+1}(\alpha)$ are the leading principal submatrices of order q and $q + 1$, respectively, of the matrix $T(\alpha)$ as in (4.3), then*

(*a*)

$$\sigma(A_\alpha(T_{n,k})) = \sigma(T_q)(\alpha) \cup \sigma(T_{q+1}(\alpha)) \cup \sigma(R(\alpha)),$$

where $R(\alpha)$ is a symmetric tridiagonal matrix of order $2q + 2$ with diagonal entries

$$\alpha, \overbrace{2\alpha, \ldots, 2\alpha}^{q-1}, k\alpha, \overbrace{2\alpha, \ldots, 2\alpha}^{q}, \alpha \tag{4.5}$$

and codiagonal entries

$$\overbrace{\beta, \ldots, \beta}^{q-1}, \beta\sqrt{k-r}, \beta\sqrt{r}, \overbrace{\beta, \ldots, \beta}^{q}. \tag{4.6}$$

(*b*) *The multiplicity of each eigenvalue of $T_q(\alpha)$ and $T_{q+1}(\alpha)$ as an eigenvalue of $A_\alpha(T_{n,k})$ is $k - r - 1$ and $r - 1$, respectively, and the eigenvalues of $R(\alpha)$ as eigenvalues of $A_\alpha(T_{n,k})$ are simple.*

(*c*) *The largest eigenvalue of $R(\alpha)$ is the $\alpha-$index of $T_{n,k}$.*

Proof. (a) Let now $n = kq + r + 1$, with $r \neq 0$. In this case, $T_{n,k}$ is the tree obtained by the coalescence of $m = 2$ generalized Bethe trees B_1 and B_2 at their roots in a common vertex v, $T_{n,k} = v\{B_1, B_2\}$, in which the number of levels of B_1 is $q + 1$ and the number of levels of B_2 is $q + 2$. Clearly the degree of v is equal to k. From the pendent vertices to the root, the vertex degrees and the number of vertices are

$$d_{1,1} = 1, \quad d_{1,2} = \cdots = d_{1,q} = 2, \quad n_{1,1} = n_{1,2} = \cdots = n_{1,q} = k - r, \quad n_{1,q+1} = 1$$

for the tree B_1, and

$$d_{2,1} = 1, \quad d_{2,2} = \cdots = d_{2,q+1} = 2, \quad n_{2,1} = n_{2,2} = \cdots = n_{2,q+1} = r, \quad n_{2,q+2} = 1$$

for the tree B_2.

The sets Ω_1 and Ω_2 in (3.2) are $\Omega_1 = \{q\}$ and $\Omega_2 = \{q + 1\}$. Then, from Theorem 3.6, part (a), we obtain

$$\sigma(A_\alpha(T_{n,k})) = \sigma(T_q(\alpha)) \cup \sigma(T_{q+1}(\alpha)) \cup \sigma(S(\alpha)),$$

where

$$S(\alpha) = \begin{bmatrix} T_q(\alpha) & 0 & \beta\mathbf{p_1} \\ 0 & T_{q+1}(\alpha) & \beta\mathbf{p_2} \\ \beta\mathbf{p_1}^T & \beta\mathbf{p_2}^T & k\alpha \end{bmatrix}$$

with $\mathbf{p_1}^T = [0, \ldots, 0, \sqrt{k-r}]$ and $\mathbf{p_2}^T = [0, \ldots, 0, \sqrt{r}]$. Let P be the permutation matrix

$$P = \begin{bmatrix} I_q & 0 \\ 0^T & R_{q+2} \end{bmatrix}.$$

Let $R(\alpha) = PS(\alpha)P$. Since $P^2 = I_{2q+2}$, it follows that $S(\alpha)$ and $R(\alpha)$ are similar matrices. We have

$$PS(\alpha) = \begin{bmatrix} T_q(\alpha) & 0 & \beta \mathbf{p_1} \\ \beta \mathbf{p_1}^T & \beta \mathbf{p_2}^T & k\alpha \\ 0 & R_{q+1}T_{q+1}(\alpha) & \beta R_{q+1}\mathbf{p_2} \end{bmatrix}.$$

Hence,

$$R(\alpha) = PS(\alpha)P = \begin{bmatrix} T_q(\alpha) & \beta \mathbf{p_1} & 0 \\ \beta \mathbf{p_1}^T & k\alpha & \beta \mathbf{p_2}^T R_{q+1} \\ 0 & \beta R_{q+1}\mathbf{p_2} & R_{q+1}T_{q+1}(\alpha)R_{q+1} \end{bmatrix}$$

is a symmetric tridiagonal matrix in which its diagonal entries and codiagonal entries are as in (4.5) and (4.6), respectively.

(b) Since $\Omega_1 = \{q\}$, $n_{1,q} = k-r$, $n_{1,q+1} = 1$ and $\Omega_2 = \{q+1\}$, $n_{1,q+1} = r$, $n_{1,q+2} = 1$, the results follow from Theorem 3.6, part (b).

(c) It is an immediate consequence of the interlacing property of the eigenvalues of Hermitian matrices. □

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