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SPECTRAL PROPERTIES OF SIGN PATTERNS*

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Abstract. In this paper, an infinite family of irreducible sign patterns that are spectrally arbitrary, for which the nilpotent-Jacobian method does not apply, is given. It is demonstrated that it is possible for an irreducible sign pattern to be refined inertially arbitrary and not spectrally arbitrary. It is observed that not every nonzero spectrally arbitrary pattern has a signing which is spectrally arbitrary. It is also shown that every superpattern of the reducible pattern $\mathcal{T}_2 \oplus \mathcal{T}_2$ is spectrally arbitrary.

Key words. Sign pattern, Nonzero pattern, Spectrally arbitrary pattern, Inertially arbitrary pattern, Nilpotent-Jacobian method.

AMS subject classifications. 15A18, 15A29, 05C50.

1. Introduction. The concept of a sign pattern being spectrally arbitrary was introduced in [12]. Since then there has been a lot of activity (see, for example, [15] and [22]) around recognizing when a sign pattern is spectrally arbitrary and relating it to other concepts, such as a pattern being potentially nilpotent, potentially stable, inertially arbitrary, or refined inertially arbitrary. In this paper we answer a number of open questions about spectrally arbitrary patterns, over the real numbers. In particular, we demonstrate the following items:

(a) There exists an irreducible sign pattern that is refined inertially arbitrary, but not spectrally arbitrary (Theorems 3.2 and 4.7 (iii)).
(b) There exists irreducible sign patterns $\mathcal{S}_1$ and $\mathcal{S}_2$ both of which are not spectrally arbitrary, but $\mathcal{S}_1 \oplus \mathcal{S}_2$ is spectrally arbitrary (Theorem 3.3).
(c) Not every nonzero spectrally arbitrary pattern has a signing which is spectrally arbitrary (Theorem 4.7 (iii)).
(d) Some irreducible spectrally arbitrary patterns do not allow a nonderogatory nilpotent realization (Corollary 4.8 (i)).
(e) There exists irreducible spectrally arbitrary sign patterns for which the nilpotent-Jacobian method does not apply (Corollary 4.8 (ii)).
(f) Given $\mathcal{T}_2 = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$, then every superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$ allows a nilpotent matrix (Theorem 5.1).
(g) Every superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$ is spectrally arbitrary (Theorem 5.1).

Previous work has explored the relationship between classes of refined inertially arbitrary patterns and spectrally arbitrary patterns. It was shown in [11] that an irreducible nonzero pattern exists which is refined

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inertially arbitrary, but not spectrally arbitrary. In [21], it was demonstrated that there exists a reducible sign pattern that is refined inertially arbitary but not spectrally arbitray. In Sections 3 and 4, we illustrate item (a) by providing irreducible sign patterns $\mathcal{S}$ and $\mathcal{C}_{n,k}$, with n and k both odd, that are refined inertially arbitrary but not spectrally arbitrary. This gives a negative answer to an open question raised in [22], [11], and [16]. A pattern equivalent to $\mathcal{C}_{4,2}$ was shown to be inertially arbitrary but not spectrally arbitrary in [7] and the nonzero patterns of order 4 which are inertially arbitrary but not spectrally arbitrary were characterized in [8]. The question of characterizing the (nonzero) patterns with this property for all orders n was raised in [1]. In Theorem 4.7, we demonstrate that $\mathcal{C}_{n,k}$ (and its nonzero pattern $\mathcal{C}^*_{n,k}$), with n and k even, provide a class of patterns that are inertially arbitrary but not refined inertially arbitrary (and hence not spectrally arbitrary) for all even $n \geq 4$.

The idea of constructing reducible spectrally arbitrary patterns from irreducible spectrally arbitrary patterns was first considered in [12]. In [7], it was demonstrated that one can construct a reducible sign pattern that is inertially arbitrary, using irreducible blocks, not all of which were inertially arbitrary. An example of a reducible inertially arbitrary nonzero pattern with none of the blocks inertially arbitrary was given in [17]. A corresponding example of a sign pattern was provided in [5]. In [10], it was shown that a sign pattern $\mathcal{M}_4 \oplus \mathcal{T}_2$ is spectrally arbitrary, even though $\mathcal{M}_4$ is not spectrally arbitrary. Work in [21] demonstrates that is possible to construct a direct sum of two reducible blocks which is spectrally arbitrary, but for which neither of the reducible blocks are spectrally arbitrary. In Section 3, we illustrate item (b) by providing the pattern $\mathcal{S} \oplus \mathcal{M}_4$ which is spectrally arbitrary, but both $\mathcal{S}$ and $\mathcal{M}_4$ are irreducible sign patterns that are not spectrally arbitrary. This answers a question raised in [4], [11], and recently in [22].

The spectrally arbitrary nonzero patterns of order four were characterized in [9]. Each of these patterns were shown to have a corresponding sign pattern that is spectrally arbitrary. In [8], an order four nonzero pattern was described that was inertially arbitrary, but had no signing which is is inertially arbitrary. The question was raised in [9] if every spectrally arbitrary nonzero pattern has a spectrally arbitrary signing. In Section 4, we illustrate item (c) by showing that for any odd order larger than four, there is a nonzero spectrally arbitrary pattern that has no corresponding sign pattern which is spectrally arbitrary.

The nilpotent-Jacobian method was developed in [12] as a tool to demonstrate an irreducible sign pattern is spectrally arbitrary, but the question was raised (see e.g. [18]) if this method would work on every irreducible spectrally arbitrary pattern. The authors of [18] demonstrated that there are spectrally arbitrary patterns over the complex numbers for which the nilpotent-Jacobian method does not apply. It is also known that there are *reducible* sign patterns that are spectrally arbitrary, for which the nilpotent-Jacobian method does not apply (see e.g. $\mathcal{T}_2 \oplus \mathcal{T}_2$ in [5]; see also [21]), since the nilpotent-Jacobian method will not apply to reducible patterns (see e.g. [19, Theorem 1.1]). In Section 4, we show that there are *irreducible* spectrally arbitrary patterns for which the nilpotent-Jacobian method does not apply. We use an observation developed in [2]: If a nilpotent matrix is successfully employed with the nilpotent-Jacobian method, then that nilpotent matrix must have been nonderogatory. This raised the question if every spectrally arbitrary sign pattern allows a nonderogatory nilpotent realization. We note that full sign patterns that are spectrally arbitrary do allow a nonderogatory nilpotent realization [20]. We demonstrate item (e) by first demonstrating item (d) and answer the question raised in [2]. The technique we use to show a pattern is spectrally arbitrary involves analyzing an associated characteristic polynomial and showing that by taking a certain variable to be sufficiently large, we can obtain all possible characteristic polynomials. This technique has been used previously for spectrally arbitrary patterns over the complex numbers [18] and for reducible spectrally arbitrary sign patterns [21].

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In [5], it was shown that $\mathcal{T} = \mathcal{T}_2 \oplus \mathcal{T}_2$ is spectrally arbitrary. This result also follows from the result noted in [10] that if the direct sum of spectrally arbitrary sign patterns has at most one odd order summand, then the direct sum is spectrally arbitrary. In [19], it was shown that every superpattern of $\mathcal{T}$ is inertially arbitrary, and it was left as an open problem whether every superpattern is spectrally arbitrary. In Section 5 we demonstrate item (g) by showing every superpattern of $\mathcal{T}$ is spectrally arbitrary using the nilpotent-Jacobian method. This also answers a question raised in [16] by demonstrating item (f), that every superpattern of $\mathcal{T}$ is in fact potentially nilpotent.

2. Technical terms and background results. Throughout, we will assume matrices are $n \times n$ unless specified otherwise. A *sign pattern* is a matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ with entries in $\{+, -, 0\}$. A real matrix A is said to *have* sign pattern $\mathcal{A}$ (or is called a *realization* of $\mathcal{A}$) if $A_{ij} > 0$ when $\mathcal{A}_{ij} = +$, $A_{ij} < 0$ when $\mathcal{A}_{ij} = -$, and $A_{ij} = 0$ when $\mathcal{A}_{ij} = 0$. A *nonzero pattern* is a matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ with entries in $\{*, 0\}$. A real matrix A is said to *have* nonzero pattern $\mathcal{A}$ (or is called a *realization* of $\mathcal{A}$) if $A_{ij} \neq 0$ if and only if $\mathcal{A}_{ij} \neq 0$. We use the term *pattern* when statements hold for both sign patterns and nonzero patterns. A *signing* of a nonzero pattern $\mathcal{A}$ is a sign pattern $\mathcal{B}$ such that $\mathcal{B}_{ij} = 0$ whenever $\mathcal{A}_{ij} = 0$ and $\mathcal{B}_{ij} \in \{+, -\}$ whenever $\mathcal{A}_{ij} = *$. We say $\mathcal{A}$ is a *subpattern* of $\mathcal{B}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by replacing some (or possibly none) of the nonzero symbols in $\mathcal{B}$ with 0. If $\mathcal{A}$ is a subpattern of $\mathcal{B}$, then we say $\mathcal{B}$ is a *superpattern* of $\mathcal{A}$. A sign pattern $\mathcal{A}$ is *signature similar* to sign pattern $\mathcal{B}$ if $\mathcal{A} = \mathcal{D}\mathcal{B}\mathcal{D}^T$ where $\mathcal{D}$ is a diagonal matrix with diagonal entries from $\{+, -\}$. Since we will be focusing on the eigenvalues that a pattern allows, we say a sign pattern $\mathcal{A}$ is *equivalent* to $\mathcal{B}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by a combination of signature similarity, negation, transposition and permutation similarity. Likewise a nonzero pattern $\mathcal{A}$ is *equivalent* to $\mathcal{B}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ via transposition and/or permutation similarity. A pattern $\mathcal{A}$ is *irreducible* if there is no permutation matrix P such that $P\mathcal{A}P^T$ is a nontrivial block triangular matrix.

A pattern $\mathcal{A}$ *allows* a particular property if there is some real matrix A with pattern $\mathcal{A}$ that has the specified property. A pattern $\mathcal{A}$ is a *spectrally arbitrary pattern (SAP)*, if $\mathcal{A}$ allows every characteristic polynomial, that is, for every polynomial $f(x) = x^n + r_1 x^{n-1} + \cdots + r_{n-1}x + r_n$ with real coefficients, there is some real matrix A with pattern $\mathcal{A}$ such that $f(x)$ is the characteristic polynomial of A. The *inertia* of a matrix A is an ordered triple $i(A) = (i_+, i_-, i_0)$ with i_+ (resp., i_- and i_0) the number of eigenvalues of A with positive (resp., negative and zero) real part. A pattern $\mathcal{A}$ is an *inertially arbitrary pattern (IAP)* if $\mathcal{A}$ allows every possible inertia, that is, if for each triple of nonnegative integers a, b, c with $a + b + c = n$, there is some matrix A with pattern $\mathcal{A}$ and inertia $i(A) = (a, b, c)$. The *refined inertia* of a matrix A is $ri(A) = (i_+, i_-, i_z, 2i_p)$ with i_z being the number of zero eigenvalues and $2i_p$ is the number of purely imaginary eigenvalues of A. A pattern $\mathcal{A}$ is a *refined inertially arbitrary pattern (rIAP)* if it allows every refined inertia.

We next describe a method for determining that a pattern is spectrally arbitrary. Given a matrix A with m nonzero entries, let X be the matrix obtained from A by replacing the nonzero entries of A with variables $x_1, x_2, \ldots, x_m$. Suppose that X has characteristic polynomial $x^n + f_1 x^{n-1} + f_2 x^{n-2} + \cdots + f_{n-1}x + f_n$. The Jacobian matrix J_X has entry (i, j) equal to the partial derivative $\frac{\partial f_i}{\partial x_j}$. If $\operatorname{rank}(J_X|_{X=A}) = n$, then we say that A has a *full-rank Jacobian.*

Theorem 2.1. [12] *If a nilpotent matrix A with sign pattern $\mathcal{A}$ has a full-rank Jacobian, then every superpattern of $\mathcal{A}$ is spectrally arbitrary.*

Theorem 2.1 essentially provides a method, called the *nilpotent-Jacobian method*, to demonstrate a pattern is spectrally arbitrary: Find a nilpotent matrix that has a full-rank Jacobian. This nilpotent-

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Jacobian method has often been the key tool for demonstrating a pattern is spectrally arbitrary (see, for example, [12] and [22]).

A matrix is *nonderogatory* if its characteristic polynomial is equal to its minimum polynomial. In particular, if A is a nilpotent matrix, then A is nonderogatory if $k = n$ is the smallest positive integer such that $A^k = 0$. It was observed in [2] that being nonderogatory is a necessary condition for a nilpotent matrix to have a full-rank Jacobian. That is, it is necessary that a pattern $\mathcal{A}$ allows a nonderogatory nilpotent matrix if the nilpotent-Jacobian method applies to $\mathcal{A}$.

A variation of the nilpotent-Jacobian method has been used to help show a pattern is refined inertially arbitrary when a pattern fails to be spectrally arbitrary. The following lemma is an abbreviated version of Lemma 3.4 in [6] that can be used recursively to obtain many refined inertias.

Lemma 2.2. [6] *Let $\mathcal{A}$ be an $n \times n$ sign pattern. Suppose there is a matrix A, with sign pattern $\mathcal{A}$, that has a full-rank Jacobian. Suppose $ri(A) = (a_p, a_n, a_z, 2a_i)$. If $a_i \geq 1$, then $(a_p + 2, a_n, a_z, 2a_i - 2)$, $(a_p, a_n + 2, a_z, 2a_i - 2) \in ri(\mathcal{A})$ and if $a_z \geq 1$ then $(a_p + 1, a_n, a_z - 1, 2a_i)$, $(a_p, a_n + 1, a_z - 1, 2a_i) \in ri(\mathcal{A})$. Furthermore, for each modified refined inertia, there is a realization of $\mathcal{A}$ with this refined inertia that has a full-rank Jacobian.*

3. A refined inertially arbitrary sign pattern.

While every irreducible SAP is an rIAP, it has been an open question, raised in [11] and [22], if the converse is true. In this section, we demonstrate that there is an irreducible sign pattern that is refined inertially arbitrary but not spectrally arbitrary. In Section 4, we determine other patterns with this property. We end this section by demonstrating the existence of a reducible spectrally arbitrary pattern which is a direct sum of two irreducible patterns, both of which are not spectrally arbitrary.

First we define a nonzero pattern $\mathcal{L}^*$, introduced in [11], along with a particular signing $\mathcal{S}$ of $\mathcal{L}^*$:

$$\mathcal{L}^* = \begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & * & 0 & 0 & * \end{bmatrix} \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} + & + & 0 & 0 & + \\ 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & + & 0 \\ 0 & 0 & + & 0 & + \\ - & - & 0 & 0 & - \end{bmatrix}.$$

Note that in [11], the notation $\mathcal{L}$ is used for the nonzero pattern $\mathcal{L}^*$. In [11, Theorem 2.9], it is shown that $\mathcal{L}^*$ is refined inertially arbitrary but not spectrally arbitrary. We tweak the proofs given in [11] to apply to the *sign* pattern $\mathcal{S}$.

Proposition 3.1. *Let $f(x) = x^5 + r_1x^4 + r_2x^3 + r_3x^2 + r_4x + r_5$. Then $\mathcal{S}$ does not allow characteristic polynomial $f(x)$ if and only if $r_1 = r_3 = 0$ while $r_5 \neq 0$.*

Proof. Follow the proof of [11, Proposition 2.7] and consider appropriate choices for the variables: In Case 1, take a and c to be sufficiently large and positive. This guarantees that the variables a, b, c, h are positive and the variables d, f, g are negative. In Case 2, take c to be sufficiently large and positive and let $h = 1/c^2$. Then $1 + c + p_3 - hq_4(c)$ is asymptotically equivalent to c as $c \to \infty$. Thus, for sufficiently large c, we are guaranteed that a, b, c, h are positive while d, f, g are negative. ∎

Similarly, the proofs of [11, Corollary 2.8] and [11, Theorem 2.9] also hold for $\mathcal{S}$. This gives the following result:

Theorem 3.2. *The irreducible sign pattern $\mathcal{S}$ of order 5 is an rIAP but not a SAP.*

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In [21], a spectrally arbitrary reducible sign pattern is constructed with none of its summands spectrally arbitrary. In [11], an example is provided in the case of nonzero patterns of order 9. This example also extends to an example for sign patterns. Consider the sign pattern $\mathcal{M}_4$ introduced in [10] with nonzero pattern $\mathcal{M}^*$:

$$\mathcal{M}_4 = \begin{bmatrix} + & + & - & 0 \\ - & - & + & 0 \\ 0 & 0 & 0 & - \\ + & + & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{M}^* = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix}.$$

Properties of the characteristic polynomials realized by $\mathcal{M}_4$ are given in [10, Proposition 2.2] and in [11, Lemma 2.2] (see [11, p. 465], where it is mentioned that [11, Lemma 2.2] also remains true for the sign pattern $\mathcal{M}_4$). Since [11, Lemma 4.1] also holds for the sign pattern $\mathcal{S}$, and the proof of [11, Theorem 4.2] only relies on [11, Lemma 2.2 and Lemma 4.1], we have the following result for sign patterns.

Theorem 3.3. *The reducible sign pattern $\mathcal{S} \oplus \mathcal{M}_4$ of order 9 is a SAP with neither of its summands a SAP.*

4. A class of inertially arbitrary patterns. Given $2 \leq k \leq n$, and $n \geq 4$, consider the $n \times n$ sign pattern $\mathcal{C}_{n,k}$ and its corresponding nonzero pattern $\mathcal{C}^*_{n,k}$ defined as

$$\mathcal{C}_{n,k} = \begin{bmatrix} + & + & \cdots & + & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & + & \cdots & + \\ - & - & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & - & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & - & - \end{bmatrix} \quad \text{and} \quad \mathcal{C}^*_{n,k} = \begin{bmatrix} * & * & \cdots & * & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & * & \cdots & * \\ * & * & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & * & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & * & * \end{bmatrix},$$

each with k nonzero entries in the first row. Note that $C_{n,n}$ (resp., $C^*_{n,n}$) has a row of zeros, and hence is not an IAP. In this section, we will observe that $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) is inertially arbitrary if $k \neq n$, and we will characterize when it is also refined inertially arbitary and spectrally arbitrary. Further, we show in Lemma 4.3 that $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) does not allow a nilpotent matrix that is nonderogatory, unless $k = n-1$. Thus, when $k < n-1$ with values of n and k for which $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) is a SAP, the nilpotent-Jacobian method does not apply and an alternative argument is needed.

By a positive diagonal similarity (resp., diagonal similarity), every matrix having sign pattern $\mathcal{C}_{n,k}$ (resp., nonzero pattern $\mathcal{C}^*_{n,k}$) is equivalent to a matrix having all nonzero entries below the main diagonal equal to -1. Then, without loss of generality, any $B_{n,k} \in Q(\mathcal{C}_{n,k})$ (resp., $B_{n,k} \in Q(\mathcal{C}^*_{n,k})$) is equal to

$$B_{n,k} = \begin{bmatrix} c_1 & c_2 & \cdots & c_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & c_{k+1} & \cdots & c_n \\ -1 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & -d \end{bmatrix} \tag{4.1}$$

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for some $d, c_1, c_2, \dots, c_n \in \mathbb{R}_{>0}$ (resp., $d, c_1, c_2, \dots, c_n \in \mathbb{R} \setminus \{0\}$).

LEMMA 4.1. *Let $f_{n,k}(x)$ be the characteristic polynomial of $B_{n,k}$ and $w_i = (-1)^i(c_{i+1} - c_i d)$ for $i = 3, 4, \dots, n-1$. Then,*

$$\begin{aligned} f_{n,n}(x) &= x^n + (d-c_1)x^{n-1} + (c_3 - c_1 d)x^{n-2} + \sum_{i=3}^{n-1} w_i x^{n-i}, \\ f_{n,n-1}(x) &= f_{n,n}(x) + c_n(c_2 - c_1)(-1)^{n-1}, \end{aligned}$$

and for $2 \le k < n-1$,

$$\begin{aligned} f_{n,k}(x) = x^n &+ (d-c_1)x^{n-1} + (c_3 - c_1 d)x^{n-2} + \sum_{i=3}^{k} w_i x^{n-i} \\ &+ \left[w_{k+1} + (-1)^k (c_2 - c_1)c_{k+1}\right] x^{n-k-1} \\ &+ \sum_{i=k+2}^{n-1} \left[w_i + (c_2 - c_1)w_{i-1}\right] x^{n-i} + (c_2 - c_1)w_{n-1}. \end{aligned}$$

Proof. We start by showing $f_{n,n}(x) = x^n + (d-c_1)x^{n-1} + (c_3 - c_1 d)x^{n-2} + \sum_{i=3}^{n-1} w_i x^{n-i}$. We calculate the characteristic polynomial by first applying cofactor expansion along row two, then applying the linearity of a determinant to the last row, and then observing the remaining two determinants come from characteristic polynomials of companion matrices:

$$\begin{aligned} f_{n,n}(x) &= \det \begin{bmatrix} x-c_1 & -c_2 & -c_3 & \cdots & \cdots & -c_n \\ 0 & x & 0 & \cdots & \cdots & 0 \\ 1 & 1 & x & \ddots & & \vdots \\ 0 & 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & x+d \end{bmatrix} \\ &= x \left(\det \left[\begin{array}{ccccc} x-c_1 & -c_3 & \cdots & \cdots & -c_n \\ 1 & x & 0 & \cdots & 0 \\ 0 & 1 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & x & 0 \\ \hline 0 & \cdots & 0 & 1 & x \end{array}\right] + \det \left[\begin{array}{cccc|c} x-c_1 & -c_3 & \cdots & \cdots & -c_n \\ 1 & x & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & x & 0 \\ \hline 0 & \cdots & 0 & 0 & d \end{array}\right] \right) \\ &= x \left(x^{n-1} - c_1 x^{n-2} + \sum_{i=3}^{n} c_i x^{n-i} (-1)^{i+1} \right) + xd \left(x^{n-2} - c_1 x^{n-3} + \sum_{i=3}^{n-1} c_i x^{n-i-1} (-1)^{i+1} \right) \\ &= x^n + (d-c_1)x^{n-1} + (c_3 - c_1 d)x^{n-2} + \sum_{i=3}^{n-1} w_i x^{n-i}. \end{aligned}$$

Next we consider $f_{n,k}$ for $k < n$, noting its relationship with $f_{n,k+1}$. In particular, observe that every nonzero transversal of $xI - B_{n,k+1}$ containing c_{k+1} also contains the $(2,2)$ entry; each such transversal corresponds to a nonzero transversal of $xI - B_{n,k}$ corresponding c_{k+1} in row two and using an x from the

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$(1,1)$ position. The remaining transversals are indicated in the following:

$$f_{n,n-1}(x) = f_{n,n}(x) + c_n(c_2 - c_1)(-1)^{n-1}$$

and for $2 \le k < n-1$,

$$f_{n,k}(x) = f_{n,k+1}(x) + (x+d)c_{k+1}(c_2 - c_1)x^{n-k-2}(-1)^k.$$

It follows that for $2 \le k < n-1$,

$$f_{n,k}(x) = f_{n,n}(x) + c_n(c_2 - c_1)(-1)^{n-1} + \sum_{i=k+1}^{n-1}(x+d)c_i(c_2 - c_1)(-1)^{i+1},$$

and hence,

$$f_{n,k}(x) = f_{n,n}(x) + (-1)^k c_{k+1}(c_2 - c_1)x^{n-k-1} + \sum_{i=k+1}^{n-1} w_i(c_2 - c_1)x^{n-i-1}. \qquad \square$$

Remark 4.2. The nilpotent-Jacobian method can be used to show that the sign pattern $\mathcal{C}_{n,n-1}$ (resp., nonzero pattern $\mathcal{C}^*_{n,n-1}$) is a SAP. In particular, consider $B_{n,n-1}$ with $d=1$ and variables c_i, $i = 1, 2, \ldots, n$. From Lemma 4.1, suppose that $B_{n,n-1}$ has characteristic polynomial $x^n + f_1x^{n-1} + f_2x^{n-2} + \cdots + f_{n-1}x + f_n$, and note that the matrix is nilpotent when $c_1 = \cdots = c_n = 1$. Then the Jacobian matrix $J = \frac{\partial(f_1, f_n, f_2, f_3, \ldots, f_{n-1})}{\partial(c_1, c_2, \ldots, c_n)}$, after rearranging the f_i as specified, is equal to

$$J = \left[\begin{array}{ccc|cccc} -1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ -a & a & 0 & 0 & \cdots & 0 & b \\ -1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & (-1)^4 & (-1)^3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & (-1)^5 & (-1)^4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & (-1)^n & (-1)^{n-1} \end{array}\right],$$

where $a = (-1)^{n-1}c_n$ and $b = (-1)^{n-1}(c_2 - c_1)$. When evaluated at $(c_1, c_2, \ldots, c_n) = (1, 1, \ldots, 1)$, we have $b = 0$ and $a \neq 0$, and thus, J is a lower triangular matrix with nonzero determinant. Hence, by Theorem 2.1, every superpattern of $\mathcal{C}_{n,n-1}$ (resp., $\mathcal{C}^*_{n,n-1}$) is a SAP.

Lemma 4.3. *Let $n \ge 4$ and $2 \le k < n-1$. Then $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) does not allow a nonderogatory nilpotent matrix.*

Proof. Let B be a nilpotent matrix having sign pattern $\mathcal{C}_{n,k}$ (resp., nonzero pattern $\mathcal{C}^*_{n,k}$). Since nilpotence is invariant under matrix scaling, we may assume the (n,n)-entry of B is equal to -1. Thus, by a positive diagonal similarity (resp., diagonal similarity), we may further assume $B = B_{n,k}$ for some $c_1, c_2, \ldots, c_n \in \mathbb{R}_{>0}$ (resp., $c_1, c_2, \ldots, c_n \in \mathbb{R} \setminus \{0\}$) with $d = 1$. By Lemma 4.1, since $d = 1$ we must have $c_1 = 1$ and $c_i = 1$ for $i = 3, \ldots, k+1$. If $c_2 \neq c_1$, then for $\det B = 0$ to hold we must have $w_{n-1} = 0$. Then $w_i + (c_2 - c_1)w_{i-1} = 0$ for $i = k+2, \ldots, n-1$ implies that $w_{n-2} = w_{n-3} = \cdots = w_{k+1} = 0$ giving a nonzero coefficient of x^{n-k-1}, contradicting that B is a nilpotent matrix. Thus, we must have $c_2 = c_1 = 1$. Hence, $w_i = 0$ for $i = k+1, \ldots, n-1$ implying $c_1 = c_2 = \cdots = c_n = 1$. Now, the first two columns of B and the last two columns of B form two pairs of dependent columns as $n - k < 1$. [Note if $n - k = 1$, the last two columns do not form a pair of dependent columns.] This implies $\operatorname{rank}(B) \le n-2$, hence, B must be a derogatory matrix. $\square$

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Since it was shown in [2] that it is necessary that a nilpotent matrix is nonderogatory if it has a full-rank Jacobian, by Lemma 4.3, the nilpotent-Jacobian method does not apply to $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) with $n \geq 4$ and $2 \leq k < n-1$. To analyze characteristic polynomials realizable by $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) we introduce the following polynomial.

Notation 4.4. Let $n \geq 4$ and $2 \leq k \leq n-1$. For a fixed $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ and $c \neq 0$, define $P_{\mathbf{r},c}(y)$ to be the polynomial

$$P_{\mathbf{r},c}(y) = y^{n-k} + \frac{(-1)^{k+1} r_{k+1}}{c} y^{n-k-1} + \frac{(-1)^{k+2} r_{k+2}}{c} y^{n-k-2} + \cdots + \frac{(-1)^n r_n}{c}.$$

The next lemma is the main tool used in this paper. It shows a relationship between characteristic polynomials realizable by $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) and polynomials $P_{\mathbf{r},c}(y)$ with real roots.

Lemma 4.5. *Let $n \geq 4$, $2 \leq k \leq n-1$ and fix $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$.*

(a) If $B_{n,k}$ has characteristic polynomial $x^n + r_1 x^{n-1} + \cdots + r_n$, then $P_{\mathbf{r},c_{k+1}}(y)$ has a real root and $c_{k+1} = d^k - r_1 d^{k-1} + r_2 d^{k-2} + \cdots + (-1)^k r_k$.
(b) If $P_{\mathbf{r},c}(y)$ has a real root for every $c > 0$, then $\mathcal{C}_{n,k}$ allows the characteristic polynomial $x^n + r_1 x^{n-1} + \cdots + r_n$.

Proof. Assume $2 \leq k < n-1$ for some $n \geq 4$ and fix $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$. Consider the matrix $B_{n,k} \in Q(\mathcal{C}_{n,k})$ (resp., $B_{n,k} \in Q(\mathcal{C}^*_{n,k})$) and let $f(x) = x^n + r_1 x^{n-1} + \cdots + r_n$. For the characteristic polynomial of $B_{n,k}$ to be equal to $f(x)$, from Lemma 4.1, we require that

$$d - c_1 - r_1 = 0 \tag{4.2}$$

$$c_3 - c_1 d - r_2 = 0 \tag{4.3}$$

$$\left\{ \begin{aligned} w_3 - r_3 &= 0 \\ w_4 - r_4 &= 0 \\ &\vdots \\ w_k - r_k &= 0 \end{aligned} \right. \tag{4.4}$$

$$w_{k+1} + (-1)^k (c_2 - c_1) c_{k+1} - r_{k+1} = 0 \tag{4.5}$$

$$\left\{ \begin{aligned} w_{k+2} + (c_2 - c_1) w_{k+1} - r_{k+2} &= 0 \\ w_{k+3} + (c_2 - c_1) w_{k+2} - r_{k+3} &= 0 \\ &\vdots \\ w_{n-1} + (c_2 - c_1) w_{n-2} - r_{n-1} &= 0 \end{aligned} \right. \tag{4.6}$$

$$(c_2 - c_1) w_{n-1} - r_n = 0 \tag{4.7}$$

for some choice of $d, c_1, c_2, \ldots, c_n \in \mathbb{R}_{>0}$ (resp., $d, c_1, c_2, \ldots, c_n \in \mathbb{R} \setminus \{0\}$), where $w_i = (-1)^i (c_{i+1} - c_i d)$ for $i = 3, 4, \ldots, n-1$. Making the substitution $y = c_2 - c_1$ and applying forward substitution to the system of equations (4.5) and (4.6) gives

$$w_i = (-1)^i y^{i-k} c_{k+1} + \sum_{t=0}^{i-(k+1)} (-1)^t r_{i-t} y^t$$

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for $i = k+1, \ldots, n-1$, and hence, by equation (4.7), we must have

$$\begin{aligned} r_n &= (-1)^{n-1} y^{n-k} c_{k+1} + \sum_{t=0}^{n-(k+1)} (-1)^t r_{n-t} y^t \\ &= (-1)^{n-1} c_{k+1} y^{n-k} + r_{n-1} y - r_{n-2} y^2 + \cdots + (-1)^{n-k-2} r_{k+1} y^{n-k-1}. \end{aligned} \tag{4.8}$$

Note that equation (4.8) is equivalent to $P_{\mathbf{r},c_{k+1}}(y) = 0$ since we require $c_{k+1} > 0$ (resp., $c_{k+1} \neq 0$). Then the following statements are equivalent with $w_i = (-1)^i (c_{i+1} - c_i d)$:

(i) the characteristic polynomial of $B_{n,k} \in Q(\mathcal{C}_{n,k})$ (resp., $B_{n,k} \in Q(\mathcal{C}^*_{n,k})$) is equal to $f(x)$,
(ii) the system of n equations (4.2)–(4.7) has a solution for some $d, c_1, c_2, \ldots, c_n \in \mathbb{R}_{>0}$ (resp., $d, c_1, c_2, \ldots, c_n \in \mathbb{R} \setminus \{0\}$),
(iii) the system of $n+1$ equations (4.2)–(4.6), $P_{\mathbf{r},c_{k+1}}(y) = 0$ and $y = c_2 - c_1$ has a solution for some $d, c_1, c_2, \ldots, c_n \in \mathbb{R}_{>0}$ (resp., $d, c_1, c_2, \ldots, c_n \in \mathbb{R} \setminus \{0\}$) and $y \in \mathbb{R}$.

We note that we also have the equivalence of (i) and (iii) when $k = n-1$ by omitting equations (4.5)–(4.7) in (iii): For the characteristic polynomial of $B_{n,k}$ to be equal to $f(x)$, by Lemma 4.1, we require equations (4.2)–(4.4) along with $c_n(c_2 - c_1)(-1)^{n-1} - r_n = 0$ to hold, and this last equation is equivalent to both equation (4.5) by defining $w_n = 0$ and also the equation $P_{\mathbf{r},c_{k+1}}(y) = 0$.

By equations (4.2)–(4.4) any solution in (iii) must satisfy $c_1 = d - r_1$, $c_3 = d^2 - r_1 d + r_2$ and $c_{i+1} = c_i d + (-1)^i r_i$ for $i = 3, 4, \ldots, k$. That is, $c_1 = d - r_1$ and $c_{i+1} = d^i - r_1 d^{i-1} + r_2 d^{i-2} + \cdots + (-1)^i r_i$ for $i = 2, 3, \ldots, k$.

Part (a) follows from the equivalence between (i) and (iii).

For part (b), assume $P_{\mathbf{r},c}(y)$ has a real root for every $c > 0$. We show (iii) holds which then implies (i) holds, completing the proof since $B_{n,k} \in Q(\mathcal{C}_{n,k})$. For every sufficiently large d, choose any real root y_d of $P_{\mathbf{r},\hat{c}}(y)$ where $\hat{c} = d^k - r_1 d^{k-1} + r_2 d^{k-2} + \cdots + (-1)^k r_k$; y_d exists since $\hat{c} > 0$ for every sufficiently large d. Note that as $d \to \infty$ we have $y_d \to 0$ since $P_{\mathbf{r},\hat{c}}(y)$ is a minor perturbation of y^{n-k}. For equations (4.2)–(4.4) to hold, let $c_1 = d - r_1$ and $c_{i+1} = d^i - r_1 d^{i-1} + r_2 d^{i-2} + \cdots + (-1)^i r_i$ for $i = 2, 3, \ldots, k$. Note $c_{k+1} = \hat{c}$. For equation $y = c_2 - c_1$ to hold, let $c_2 = y_d + d - r_1$. Thus, as $d \to \infty$, then c_1 is asymptotically equivalent to d and c_i is asymptotically equivalent to d^{i-1} for $i = 2, 3, \ldots, k+1$. The system of equations (4.5) and (4.6) then yields real solutions c_i with c_i asymptotically equivalent to d^{i-1} as $d \to \infty$ for $i = k+2, \ldots, n$. Thus, $d, c_1, c_2, \ldots, c_n$ are positive for sufficiently large d. Hence, (iii) holds. ∎

The proof of Lemma 4.5 (b) shows that for sufficiently large d, if $P_{\mathbf{r},c}(y)$ has a root, then $\mathcal{C}_{n,k}$ is spectrally arbitrary. This implies that over the complex numbers, $\mathcal{C}^*_{n,k}$ will be spectrally arbitrary. Thus, we have the following theorem.

Theorem 4.6. *Let $n \geq 4$ and $2 \leq k \leq n-1$. Then $\mathcal{C}^*_{n,k}$ is a spectrally arbitrary pattern over the complex numbers.*

A pattern equivalent to $\mathcal{C}^*_{4,2}$ appears in [18, Lemma 3.1]: It is spectrally arbitrary over the complex numbers, though it is not spectrally arbitrary over the real numbers. A pattern equivalent to $\mathcal{C}_{4,2}$ appears in [8, Proposition 3.1]: It is shown to be inertially arbitrary, but not spectrally arbitrary, in [8]. The next result characterizes the values of n and k for which $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) is spectrally arbitrary (over the real numbers), refined inertially arbitrary or inertially arbitrary.

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Theorem 4.7. *Let $n \geq 4$ and $2 \leq k \leq n-1$.*

(i) If $n-k$ is odd, then $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) is a SAP.
(ii) If n and k are both even, then $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) is an IAP but not an rIAP.
(iii) If n and k are both odd, then $\mathcal{C}_{n,k}$ is an rIAP but not a SAP, and $\mathcal{C}^*_{n,k}$ is a SAP but has no signing that is a SAP.

Proof. (i) Assume $n-k$ is odd and arbitrarily fix any $r_1, r_2, \ldots, r_n \in \mathbb{R}$. For every $c > 0$, the polynomial $P_{\mathbf{r},c}(y)$ has a real root since it has odd degree. By Lemma 4.5 (b), it follows that $\mathcal{C}_{n,k}$ allows the characteristic polynomial $x^n + r_1 x^{n-1} + \cdots + r_n$. Since $r_1, r_2, \ldots, r_n \in \mathbb{R}$ were chosen to be arbitrary, $\mathcal{C}_{n,k}$ is a SAP. Hence, $\mathcal{C}^*_{n,k}$ is also a SAP.

(ii) Suppose that n and k are even and that $(0,0,0,n) \in ri(\mathcal{C}^*_{n,k})$. Then there is a $B \in Q(\mathcal{C}^*_{n,k})$ of the form in equation (4.1) for some $d, c_1, c_2, \ldots, c_n \in \mathbb{R} \setminus \{0\}$ with $ri(B) = (0,0,0,n)$ and characteristic polynomial $x^n + r_1 x^{n-1} + \cdots + r_n$ with $r_i = 0$ for all odd i and $r_i > 0$ for all even i. By Lemma 4.5 (a), $P_{\mathbf{r},c_{k+1}}(y)$ has a real root and $c_{k+1} = d^k + r_2 d^{k-2} + \cdots + r_k$. Note $c_{k+1} > 0$ since it is a polynomial in even powers of d and $r_i > 0$ for all even i. Thus, since n and k are even, the polynomial

$$P_{\mathbf{r},c_{k+1}}(y) = y^{n-k} + \frac{(-1)^{k+2} r_{k+2}}{c_{k+1}} y^{n-k-2} + \cdots + \frac{(-1)^{n-2} r_{n-2}}{c_{k+1}} y^2 + \frac{r_n}{c_{k+1}}$$

is strictly positive for all y. Hence, $P_{\mathbf{r},c_{k+1}}(y)$ cannot have any real roots, a contradiction. Thus, $(0,0,0,n) \notin ri(\mathcal{C}^*_{n,k})$ implying $\mathcal{C}^*_{n,k}$ is not an rIAP. This also implies $\mathcal{C}_{n,k}$ is not an rIAP.

We next show $\mathcal{C}_{n,k}$ is an IAP when n and k are even. Arbitrarily fix any $r_1, r_2, \ldots, r_n \in \mathbb{R}$ with $r_n \leq 0$. For every $c > 0$, $P_{\mathbf{r},c}(0) = \frac{(-1)^n r_n}{c} \leq 0$ and $P_{\mathbf{r},c}(y) \to \infty$ as $y \to \infty$. Thus, for every $c > 0$, $P_{\mathbf{r},c}(y)$ has a real root by the Intermediate Value Theorem. By Lemma 4.5 (b), $\mathcal{C}_{n,k}$ can realize all characteristic polynomials of the form $x^n + r_1 x^{n-1} + \cdots + r_n$ with $r_n \leq 0$. This implies $(n_1, n_2, n_3) \in i(\mathcal{C}_{n,k})$ for all $n_3 \geq 1$, $n_1 + n_2 + n_3 = n$.

Now consider $(n_1, n_2, 0)$ with $n_1 + n_2 = n$. Choose $B \in Q(\mathcal{C}_{n,k})$ of the form in equation (4.1) with inertia $i(B) = (p, q, 1)$, $p + q = n-1$, and characteristic polynomial $x^n + r_1 x^{n-1} + \cdots + r_n$ for some $d, c_1, c_2, \ldots, c_n \in \mathbb{R}_{>0}$; such a B exists since $r_n = 0$ in this case. Consider the perturbation $\hat{B}$ of B, also of the form in equation (4.1), by letting $\hat{c}_1 = c_1 + \epsilon_1$, $\hat{c}_n = c_n + \epsilon_2$ and $\hat{d} = d$, $\hat{c}_i = c_i$ for $i = 2, 3, \ldots, n-1$, where $\epsilon_1, \epsilon_2 \in \mathbb{R}$ remain to be chosen. For ϵ_1, ϵ_2 sufficiently small, $\hat{c}_1, \hat{c}_n > 0$. Let $x^n + \hat{r}_1 x^{n-1} + \cdots + \hat{r}_n$ denote the characteristic polynomial of $\hat{B}$. Then

$$\begin{aligned}\hat{r}_n &= (-1)^{n-1}(\hat{c}_2 - \hat{c}_1)(\hat{c}_n - \hat{c}_{n-1}d) = (-1)^{n-1}(c_2 - c_1 - \epsilon_1)(c_n - c_{n-1}d + \epsilon_2)\\ &= r_n - (-1)^{n-1}\epsilon_1(c_n - c_{n-1}d) + (-1)^{n-1}\epsilon_2(c_2 - c_1) - (-1)^{n-1}\epsilon_1\epsilon_2.\end{aligned}$$

Since $r_n = 0$, either $c_2 - c_1 = 0$ or $c_n - c_{n-1}d = 0$ (or both are zero); if $c_2 - c_1 \neq 0$ take $\epsilon_1 = 0$ and if $c_n - c_{n-1}d \neq 0$ take $\epsilon_2 = 0$. Then we may choose the signs of ϵ_1 and ϵ_2 so that $\hat{r}_n > 0$ (resp., $\hat{r}_n < 0$). For ϵ_1, ϵ_2 sufficiently small of appropriate sign, $\hat{r}_i - r_i$ is sufficiently close to zero. Since the roots of a polynomial are continuous functions of its coefficients, ϵ_1 and ϵ_2 may be chosen so that $\hat{B}$ has inertia $(p+1, q, 0)$ or has inertia $(p, q+1, 0)$. This implies $(n_1, n_2, 0) \in i(C_{n,k})$ for all $n_1, n_2 \geq 0$, $n_1 + n_2 = n$. Hence, $\mathcal{C}_{n,k}$ is an IAP, implying $\mathcal{C}^*_{n,k}$ is also an IAP.

(iii) We first show no signings of $\mathcal{C}^*_{n,k}$ are a SAP when n and k are both odd. Consider a signing $\mathcal{C}$ of $\mathcal{C}^*_{n,k}$ and suppose it is a SAP. By applying a signature similarity, we may assume all entries below the main diagonal of $\mathcal{C}$ are negative. Given $B \in Q(\mathcal{C})$, using a positive diagonal similarity, we may assume $B = B_{n,k}$ for

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some $d, c_1, c_2, \ldots, c_n \in \mathbb{R}$ of appropriate sign. Since $\mathcal{C}$ is a SAP, we may choose such a B with characteristic polynomial $x^n + r_1 x^{n-1} + \cdots + r_n$ satisfying $r_{k+1} = r_{k+2} = \cdots = r_{n-1} = 0$ and $r_n = -sgn(\mathcal{C}(2, k+1))$. Then by Lemma 4.5 (a), $P_{\mathbf{r},c_{k+1}}(y) = y^{n-k} - \frac{r_n}{c_{k+1}}$ has a real root, a contradiction since $-\frac{r_n}{c_{k+1}} > 0$ and $n-k$ is even. Thus, no signings of $\mathcal{C}^*_{n,k}$ are a SAP. Note this implies that $\mathcal{C}_{n,k}$ is not a SAP.

We now show $\mathcal{C}_{n,k}$ is an rIAP and that $\mathcal{C}^*_{n,k}$ is a SAP when n and k are odd. Arbitrarily fix any $r_1, r_2, \ldots, r_n \in \mathbb{R}$ with $r_n \geq 0$. For every $c > 0$, $P_{\mathbf{r},c}(0) = \frac{(-1)^n r_n}{c} \leq 0$ and $P_{\mathbf{r},c}(y) \to \infty$ as $y \to \infty$. Thus, for every $c > 0$, $P_{\mathbf{r},c}(y)$ has a real root by the Intermediate Value Theorem. By Lemma 4.5(b), $\mathcal{C}_{n,k}$ can realize all characteristic polynomials of the form $x^n + r_1 x^{n-1} + \cdots + r_n$ with $r_n \geq 0$. This implies that $\mathcal{C}^*_{n,k}$ is a SAP since $-\mathcal{C}_{n,k}$ can realize all characteristic polynomials of the form $x^n + r_1 x^{n-1} + \cdots + r_n$ with $r_n \leq 0$ as n is odd. Furthermore, $\mathcal{C}_{n,k}$ can realize all characteristic polynomials which have a root at $x = 0$ with multiplicity at least 1. Therefore, there exists a matrix with sign pattern $\mathcal{C}_{n,k}$ and refined inertia $(a_p, a_n, a_z, 2a_i)$ for all $a_p + a_n + a_z + 2a_i = n$ and $a_z \geq 1$. To show $(a_p, a_n, 0, 2a_i) \in ri(\mathcal{C}_{n,k})$ with $a_p + a_n + 2a_i = n$ (note $a_p + a_n$ is odd in this case) we apply Lemma 2.2 recursively to a realization having a full-rank Jacobian with refined inertia $(0, 0, 1, n-1)$. Consider $B_{n,k}$ with $d = 1$ and variables c_i, $i = 1, 2, \ldots, n$. Define functions $f_1, f_2, \ldots, f_n$ of $c_1, c_2, \ldots, c_n, r_1, r_2, \ldots, r_n$ to be the left sides of the equations (4.2)–(4.7), in order. Note that setting $c_1 = 1$ and $c_{2i} = c_{2i-1}$ for $i = 1, 2, \ldots, \frac{n-1}{2}$ gives the characteristic polynomial of $B_{n,k}$ to be $x^n + \sum_{i=1}^{(n-1)/2} (c_{2i+1} - c_{2i-1}) x^{n-2i}$. This implies we can choose c_i so that the characteristic polynomial is equal to $x(x^2+1)^{(n-1)/2}$, in particular, let $\hat{c}_1 = 1$, $\hat{c}_{2i+1} = \hat{c}_{2i-1} + \binom{(n-1)/2}{i}$ and $\hat{c}_{2i} = \hat{c}_{2i-1}$ for $i = 1, 2, \ldots, \frac{n-1}{2}$. Note that $\hat{c}_i > 0$, $i = 1, 2, \ldots, n$, and that $\hat{c}_1 = \hat{c}_2$, $\hat{c}_n \neq \hat{c}_{n-1}$. Then the Jacobian matrix $J = \frac{\partial(f_1, f_n, f_2, f_3 \ldots, f_{n-1})}{\partial(c_1, c_2, \ldots, c_n)}$, after rearranging the f_i as specified, has the form

$$J = \left[\begin{array}{ccc|cccc} -1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \circledast & a & 0 & \cdots & 0 & -b & b \\ -1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline \circledast & \cdots & \circledast & (-1)^3 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \circledast & (-1)^4 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ \circledast & \cdots & \circledast & \circledast & \cdots & \circledast & (-1)^{n-1} \end{array}\right],$$

where $a = (c_n - c_{n-1})$, $b = (c_2 - c_1)$, and $\circledast$ represents an arbitrary entry. When evaluated at $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n)$, $b = 0$ and $a \neq 0$, and thus, J is a lower triangular matrix with nonzero determinant. Thus, there is a realization of $\mathcal{C}_{n,k}$ with refined inertia $(0, 0, 1, n-1)$ that has a full-rank Jacobian. By recursive application of Lemma 2.2, $\mathcal{C}_{n,k}$ can realize all refined inertias of the form $(a_p, a_n, 0, 2a_i)$ with $a_p + a_n + 2a_i = n$. Therefore, $\mathcal{C}_{n,k}$ is an rIAP. ∎

Lemma 4.3, Theorem 4.6 and Theorem 4.7 give the following corollary.

Corollary 4.8.

(i) *For all $n \geq 5$, there is an irreducible sign (resp., nonzero) pattern of order n that is a SAP which does not allow a nonderogatory nilpotent matrix.*

(ii) *For all $n \geq 5$, there is an irreducible sign (resp., nonzero) pattern of order n that is a SAP for which the nilpotent-Jacobian method does not apply.*

(iii) *For all $n \geq 4$, there is an irreducible nonzero pattern of order n that is a SAP over the complex numbers for which the nilpotent-Jacobian method does not apply.*

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(iv) *For all odd $n \geq 5$, there is an irreducible sign pattern of order n that is an rIAP but not a SAP.*
(v) *For all odd $n \geq 5$, there is an irreducible nonzero pattern of order n that is a SAP with no signing that is a SAP.*

Proof. By considering $\mathcal{C}_{n,n-3}$ and $\mathcal{C}^*_{n,n-3}$ with $n \geq 5$, (i) and (ii) follow from Theorem 4.7 (i) and Lemma 4.3. By considering $\mathcal{C}^*_{n,2}$ with $n \geq 4$, (iii) follows from Theorem 4.6 and noting that the proof of Lemma 4.3 also holds over the complex numbers. By considering $\mathcal{C}_{n,3}$ and $\mathcal{C}^*_{n,3}$ with $n \geq 5$ and n odd, (iv) and (v) follow from Theorem 4.7 (iii). ∎

Note that Corollary 4.8 (iii) for $n = 4$ follows from [18, Lemma 3.1] where it was proven that $\mathcal{C}^*_{4,2}$ is spectrally arbitrary over the complex numbers and has no nilpotent realization that has a full-rank Jacobian.

Remark 4.9. Let $d = 1$ and define functions $f_1, \ldots, f_n$ of $c_1, \ldots, c_n, r_1, \ldots, r_n$ to be the left sides of the equations (4.2)–(4.7), in order. When the nilpotent-Jacobian method is applicable, then the Implicit Function Theorem guarantees for $(r_1, \ldots, r_n)$ sufficiently close to $(0, \ldots, 0)$ there are **unique** continuous functions $c_1, \ldots, c_n$ of $r_1, \ldots, r_n$ that maintain $f_1 = \cdots = f_n = 0$. We remark that for the pattern $\mathcal{C}_{n,k}$, with $n - k > 1$, we do not necessarily have uniqueness for every $(r_1, \ldots, r_n)$ sufficiently close to $(0, \ldots, 0)$.

For example, consider $B_{n,k}$ with $n - k > 1$, $d = 1$, $r_i = 0$ for all $i \neq k+1$ and r_{k+1} sufficiently close to 0. Then both $c_1 = \cdots = c_{k+1} = 1$, $c_{k+2} = \cdots = c_n = 1 + (-1)^{k+1} r_{k+1}$ and $c_1 = 1$, $c_2 = 1 + (-1)^k r_{k+1}$, $c_3 = \cdots = c_n = 1$ maintain $f_1 = \cdots = f_n = 0$, that is, give characteristic polynomial $x^n + r_{k+1} x^{n-k-1}$. These two solutions are obtained from the two roots of $P_{\mathbf{r},c_{k+1}}(y) = y^{n-k} + \frac{(-1)^k r_{k+1}}{c_{k+1}} y^{n-k-1}$ for $y = c_2 - c_1$ and $\mathbf{r} = (0, \ldots, 0, r_{k+1}, 0, \ldots, 0)$.

When $n - k = 1$, the polynomial $P_{\mathbf{r},c}(y)$ is linear and has a unique solution for all $c > 0$. As demonstrated in Remark 4.2, the nilpotent-Jacobian method does indeed apply to $\mathcal{C}_{n,n-1}$ to show it is a SAP.

5. Superpatterns of $\mathcal{T}_2 \oplus \mathcal{T}_2$ are spectrally arbitrary. In [5], it was shown that the reducible sign pattern $\mathcal{T}_2 \oplus \mathcal{T}_2$ is spectrally arbitrary. In this section, we demonstrate that every superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$ is in fact spectrally arbitrary.

Theorem 5.1. *Every superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$ is spectrally arbitrary.*

Proof. If a superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$ is reducible, it is spectrally arbitrary. Let

$$\mathcal{T} = \begin{bmatrix} \mathcal{T}_2 & \mathcal{A} \\ \mathcal{B} & \mathcal{T}_2 \end{bmatrix}$$

be an irreducible superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$. Since $\mathcal{T}$ is irreducible, there must be at least one nonzero entry in each of $\mathcal{A}$ and $\mathcal{B}$. Thus, $\mathcal{T}$ must have at least ten nonzero entries. Hence, by Theorem 2.1, it is sufficient to show that each superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$ with exactly ten nonzero entries, with exactly one nonzero entry in each of $\mathcal{A}$ and $\mathcal{B}$, allows a nilpotent matrix with a full-rank Jacobian.

Case 1. Suppose $\mathcal{A}(2,1) \neq 0$. By using a signature similarity $(+,+,-,-)$ if necessary, we may assume $\mathcal{A}(2,1) < 0$. Since $\mathcal{B}$ has exactly one nonzero entry equal to either $+$ or $-$, there are eight subcases to consider. In each case, the resulting sign pattern allows a nilpotent matrix having a full-rank Jacobian, as illustrated with the following nilpotent realizations:

$$\left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ -1 & -2 & -3 & 0 \\ \hline 0 & -3 & 2 & 4 \\ 0 & 0 & -4 & -2 \end{array}\right], \quad \left[\begin{array}{cc|cc} 4 & 2 & 0 & 0 \\ -4 & -1 & -1 & 0 \\ \hline 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right], \quad \left[\begin{array}{cc|cc} 4 & 4 & 0 & 0 \\ -10 & -8 & -4 & 0 \\ \hline 0 & 0 & 8 & 4 \\ -1 & 0 & -10 & -4 \end{array}\right], \quad \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ -4 & -1 & -1 & 0 \\ \hline 0 & 0 & 3 & 3 \\ 3 & 0 & -2 & -3 \end{array}\right],$$

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$$\left[\begin{array}{rr|rr} 1 & 1 & 0 & 0 \\ -7 & -4 & -1 & 0 \\ \hline -9 & 0 & 5 & 1 \\ 0 & 0 & -16 & -2 \end{array}\right], \quad \left[\begin{array}{rr|rr} 1 & 1 & 0 & 0 \\ -22 & -4 & -1 & 0 \\ \hline 81 & 0 & 5 & 1 \\ 0 & 0 & -1 & -2 \end{array}\right], \quad \left[\begin{array}{rr|rr} 8 & 8 & 0 & 0 \\ -2 & -20 & -8 & 0 \\ \hline 0 & 0 & 40 & 8 \\ 0 & -81 & -176 & -28 \end{array}\right], \quad \left[\begin{array}{rr|rr} 8 & 8 & 0 & 0 \\ -32 & -20 & -8 & 0 \\ \hline 0 & 0 & 40 & 8 \\ 0 & 9 & -146 & -28 \end{array}\right].$$

Case 2. Suppose $\mathcal{A}(1,1) \neq 0$. If $\mathcal{B}(1,2) \neq 0$, then $\mathcal{T}$ is equivalent to a spectrally arbitrary pattern considered in Case 1 via transpose and signature similarity $(+,-,+,-)$. Similarly, if $\mathcal{B}(2,1) \neq 0$, then $-\mathcal{T}$ is equivalent to a spectrally arbitrary pattern considered in Case 1 via transpose and permutation similarity $(12)(34)$. By using a signature similarity $(+,+,-,-)$ if necessary, we may assume $\mathcal{A}(1,1) > 0$. For each of the remaining four subcases with $\mathcal{B}(1,1) \in \{+,-\}$ or $\mathcal{B}(2,2) \in \{+,-\}$, the resulting sign pattern $\mathcal{T}$ allows a nilpotent matrix having a full-rank Jacobian, as illustrated with the following nilpotent realizations:

$$\left[\begin{array}{rr|rr} 2 & 2 & 2 & 0 \\ -9 & -6 & 0 & 0 \\ \hline 1 & 0 & 10 & 2 \\ 0 & 0 & -36 & -6 \end{array}\right], \quad \left[\begin{array}{rr|rr} 1 & 1 & 1 & 0 \\ -4 & -2 & 0 & 0 \\ \hline -1 & 0 & 3 & 1 \\ 0 & 0 & -4 & -2 \end{array}\right], \quad \left[\begin{array}{rr|rr} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \end{array}\right], \quad \left[\begin{array}{rr|rr} 10 & 10 & 10 & 0 \\ -25 & -20 & 0 & 0 \\ \hline 0 & 0 & 20 & 10 \\ 0 & -1 & -25 & -10 \end{array}\right].$$

Case 3. Suppose $\mathcal{A}(1,2) \neq 0$ (resp., $\mathcal{A}(2,2) \neq 0$). Then $-\mathcal{T}$ is equivalent to a spectrally arbitrary pattern considered in Case 1 (resp., Case 2), via permutation similarity $(12)(34)$. $\square$

Note that some of the patterns described in the proof of Theorem 5.1 are tridiagonal patterns. Tridiagonal spectrally arbitrary patterns of order 4 were classified in [13] and [1].

A pattern $\mathcal{A}$ is a *minimal* irreducible spectrally arbitrary pattern if $\mathcal{A}$ is not a proper superpattern of any other irreducible spectrally arbitrary pattern. Not every irreducible pattern described in the proof of Theorem 5.1 is a minimal irreducible spectrally arbitrary pattern. For example, let $\mathcal{A}$ be the pattern of the nilpotent matrix

$$A = \left[\begin{array}{rr|rr} 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 1 & -2 & -1 \end{array}\right].$$

Since A has a full-rank Jacobian, $\mathcal{A}$ is spectrally arbitrary via the nilpotent-Jacobian method. Hence, the pattern corresponding to the last matrix in Case 1 is not a minimal irreducible spectrally arbitrary pattern since it is a proper superpattern of $\mathcal{A}$.

6. Concluding comments. In [11, Theorem 2.5], it is shown that for $n \leq 4$, a nonzero pattern of order n is an rIAP if and only if it is a SAP. It was also observed in [11] that this does not hold for $n = 5$. It is known (see e.g. [7]) that for $n \leq 3$, a sign pattern is an IAP if and only if it is a SAP, and hence, a sign pattern is an rIAP if and only if it is a SAP when $n \leq 3$. In Corollary 4.8 we have shown that rIAP does not imply SAP for each odd $n \geq 5$ when working with sign patterns. It would be interesting to know if there exists a sign pattern of order 4 that is an rIAP but not a SAP, or if all sign patterns of order 4 that are an rIAP are also a SAP.

When the nilpotent-Jacobian method is used to show a pattern $\mathcal{A}$ is spectrally arbitrary, then all of the superpatterns of $\mathcal{A}$ must also be spectrally arbitrary [12]. It would be interesting to know if the superpatterns of $\mathcal{C}_{n,k}$ (resp., $\mathcal{C}^*_{n,k}$) are also spectrally arbitrary when $n-k$ is odd (resp., $n-k$ is odd or n,k are both odd).

An outstanding problem is to determine the minimum number of nonzero entries in an irreducible spectrally arbitrary sign pattern. The $2n$-conjecture, introduced in [3], states that an irreducible spectrally

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arbitrary sign pattern of order n requires at least $2n$ nonzero entries. This conjecture has been demonstrated to be true for all patterns up to order five [10]. It is known (see e.g. [18]) that if the nilpotent-Jacobian method is employed to determine a pattern is spectrally arbitrary, then the pattern requires at least $2n$ nonzero entries. Most patterns that have been shown to be spectrally arbitrary to date have used the nilpotent-Jacobian method, or an equivalent method (see e.g. [14]). As demonstrated in Section 4, there exists irreducible spectrally arbitrary patterns for which the nilpotent-Jacobian method does not apply.

If there is an irreducible spectrally arbitrary sign pattern with $2n-1$ nonzero entries, the technique used in this paper to show $\mathcal{C}_{n,k}$ is a SAP (whenever $n-k$ is odd) is unlikely to apply to this pattern. In our arguments, we scaled $n-1$ off-diagonal entries to have magnitude 1, had a free variable d taken to be sufficiently large to guarantee other variables are positive, and we seem to need a variable for each of the n coefficients of $x^{n-1}, \ldots, x, 1$ in the characteristic polynomial so that these coefficient equations hold true for arbitrary $r_1, r_2, \ldots, r_n \in \mathbb{R}$. However, for nonzero patterns we only need to guarantee variables are nonzero, and perhaps this technique may be applied to different signings of a nonzero pattern $\mathcal{A}$ with $2n-1$ nonzero entries to give all possible spectra.

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