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INJECTIVITY OF LINEAR COMBINATIONS IN $\mathcal{B}(\mathcal{H})$*

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Abstract. The aim of this paper is to provide sufficient and necessary conditions under which the linear combination $\alpha A + \beta B$, for given operators $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, is injective. Using these results, necessary and sufficient conditions for left(right) invertibility are given. Some special cases will be studied as well.

Key words. Hilbert space, Invertibility, Injectivity, Operator matrix.

AMS subject classifications. 47A05, 47A99.

1. Notations, motivations, and preliminaries. If $\mathcal{M}$ is a closed subspace of a Hilbert space $\mathcal{H}$, we use the symbol $P_{\mathcal{M}}$ to denote the orthogonal projection onto $\mathcal{M}$. For a given operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. For a projection $P_{\mathcal{M},\mathcal{N}} : \mathcal{H} \to \mathcal{H}$ onto $\mathcal{M}$ parallel to $\mathcal{N}$, we introduce the operator $P'_{\mathcal{M},\mathcal{N}} : \mathcal{H} \to \mathcal{M}$ defined as $P'_{\mathcal{M},\mathcal{N}}x = P_{\mathcal{M},\mathcal{N}}x$, for $x \in \mathcal{H}$ (as all spaces we use in this paper are separable Hilbert spaces, there is no danger of confusing this notation with the adjoint operator).

If $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $\mathcal{M}$ is a subspace of $\mathcal{K}$, then the restriction of the operator A to the subspace $\mathcal{M}$ will be denoted by $A|_{\mathcal{M}}$. The inverse image of a set $S \subseteq \mathcal{H}$ will be denoted by $A^{-1}(S)$.

The motivation behind this paper are the papers [15] and [11] where the invertibility of the linear combination $\alpha A + \beta B$ was considered in the case when $A, B \in \mathcal{B}(\mathcal{H})$ are regular operators and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, and also some recently published papers (see [17, 12, 18, 19, 22, 20]) which considered the independence of the invertibility of the linear combination $\alpha A + \beta B$ in the cases when $A, B \in \mathcal{B}(\mathcal{H})$ are projectors or orthogonal projectors (see [12, 17, 18, 19, 20, 22] for results concerning specifically projections and orthogonal projections). The aim of this paper is to investigate injectivity of the linear combination $\alpha A + \beta B$ where $A, B \in \mathcal{B}(\mathcal{H})$ are given operators and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Expanding on that analysis, sufficient and necessary conditions for left invertibility (right invertibility) will be given and some special cases will be considered.

One of the basic ideas of this paper is to utilize results concerning the injectivity and left (right) invertibility of the operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \mathcal{X}_1 \oplus \mathcal{Y}_1 \to \mathcal{X}_2 \oplus \mathcal{Y}_2,$$

where $\mathcal{X}_i, \mathcal{Y}_i$, $i = 1, 2$ are Banach (Hilbert) spaces and $A \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_1)$, $B \in \mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_2)$, $C \in \mathcal{B}(\mathcal{Y}_1, \mathcal{X}_2)$, in order to obtain analogous results for the linear combination $\alpha A + \beta B$. Completions of operator matrices have been studied extensively (for some examples, see [1, 4, 5, 6, 7, 8, 9, 10]) and have found applications; some examples include [2, 3].

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To this end, we will use the following Theorems.

THEOREM 1.1 ([1]). *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$, and $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ be given operators, where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. The operator matrix $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is injective if and only if A is injective and $\mathcal{R}(C|_{\mathcal{N}(B)}) \cap \mathcal{R}(A) = \{0\}$.*

REMARK 1.2. Analysing the proof of Theorem 1.1 given in [1] we see that without any modification, it holds for the operator matrices of the type:

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \mathcal{X}_1 \oplus \mathcal{Y}_1 \to \mathcal{X}_2 \oplus \mathcal{Y}_2,$$

where $\mathcal{X}_i, \mathcal{Y}_i$, $i = 1, 2$ are Banach spaces, that is:

THEOREM 1.3. *Let $A \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$, $B \in \mathcal{B}(\mathcal{Y}_1, \mathcal{Y}_2)$, and $C \in \mathcal{B}(\mathcal{Y}_1, \mathcal{X}_2)$ be given operators, where $\mathcal{X}_i, \mathcal{Y}_i$, $i = 1, 2$ are Banach spaces. The operator matrix $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is injective if and only if A is injective and $\mathcal{R}(C|_{\mathcal{N}(B)}) \cap \mathcal{R}(A) = \{0\}$.*

THEOREM 1.4 ([1]). *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$, and $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ be given operators. The operator matrix M_C is left invertible if and only if:*

(i) A is left invertible;
(ii) $\begin{bmatrix} (I - P_{\mathcal{R}(A)})C \\ B \end{bmatrix}$ is left invertible.

REMARK 1.5. The notation $P_{\mathcal{R}(A)}$ denotes an arbitrary, but fixed, oblique projection onto $\mathcal{R}(A)$. Similarly, bellow we will use the notation $P_{\mathcal{N}(B)}$ for an arbitrary, but fixed oblique projection onto $\mathcal{N}(B)$.

THEOREM 1.6 ([1]). *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$ and $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ be given operators. The operator matrix M_C is right invertible if and only if:*

(i) B is right invertible;
(ii) $\begin{bmatrix} A & CP_{\mathcal{N}(B)} \end{bmatrix}$ is right invertible.

REMARK 1.7. As noted in the paper, in the case of Hilbert spaces $\mathcal{H}, \mathcal{K}$ some of these conditions have the following form. Condition (ii) from Theorem 1.4 is equivalent to that $\begin{bmatrix} C^* P_{\mathcal{R}(A)^\perp} & B^* \end{bmatrix}$ is right invertible, that is

$$\mathcal{R}(C^* P_{\mathcal{R}(A)^\perp}) + \mathcal{R}(B^*) = \mathcal{K},$$

and condition (ii) from Theorem 1.6 is equivalent to

$$\mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)}) = \mathcal{H}.$$

Furthermore, as in the case of Theorem 1.1, Theorems 1.4 and 1.6 can be reformulated in the case when $A \in B(\mathcal{H}_1, \mathcal{H}_2), B \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ and $C \in \mathcal{B}(\mathcal{K}_1, \mathcal{H}_2)$, where $\mathcal{H}_i, \mathcal{K}_i$, $i = 1, 2$ are Hilbert spaces.

2. Injectivity.

THEOREM 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then the operator $\alpha A + \beta B$ is injective if and only if the following conditions hold:*

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(i) $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$ *;*
(ii) $\mathcal{N}(\alpha A + \beta B) \cap \mathcal{N}(B)^{\perp} = \{0\}$*;*
(ii) $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}}) \cap \mathcal{R}(AP_{\mathcal{N}(B)}) = \{0\}$,

where $\mathcal{T} = B^{-1}(\mathcal{R}(A)) \cap \mathcal{N}(B)^{\perp}$.

Proof. First, observe that the operator A and B have the following representations:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}, \tag{2.1}$$

$$B = \begin{bmatrix} 0 & B_1 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}, \tag{2.2}$$

where $\mathcal{H} = \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp} = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^{\perp}$ are decompositions of the space $\mathcal{H}$. So, the linear combinations $\alpha A + \beta B$ is injective if and only if the operator matrix

$$\begin{bmatrix} \alpha A_1 & \beta B_1 + \alpha A_2 \\ 0 & \beta B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}, \tag{2.3}$$

is injective. According to Theorem 1.3, the operator matrix in (2.3) is injective if and only if the following holds:

(i) A_1 is injective;
(ii) $(\alpha A_2 + \beta B_1)|_{\mathcal{N}(B_2)}$ is injective and $\mathcal{R}((\alpha A_1 + \beta B_2)|_{\mathcal{N}(B_2)}) \cap \mathcal{R}(A_1) = \{0\}$.

Obviously, (i) holds if and only if $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$. Now we examine the conditions in (ii). Since $\alpha A_2 + \beta B_1 = P'_{\overline{\mathcal{R}(A)}}(\alpha A + \beta B)|_{\mathcal{N}(B)^{\perp}}$, and $\mathcal{T}' = \mathcal{N}(B_2) \subseteq \mathcal{N}(B)^{\perp}$, we have that $(\alpha A_2 + \beta B_1)|_{\mathcal{T}'}$ is injective if and only if $P'_{\overline{\mathcal{R}(A)}}(\alpha A + \beta B)|_{\mathcal{T}'}$ is injective and its range intersected with $\mathcal{R}(A_1) = \mathcal{R}(A|_{\mathcal{N}(B)}) = \mathcal{R}(AP_{\mathcal{N}(B)})$ contains only the null vector. Since $\mathcal{T}' = B^{-1}(\overline{\mathcal{R}(A)}) \cap \mathcal{N}(B)^{\perp}$ we have that the second condition in (ii) is equivalent to $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}'}) \cap \mathcal{R}(AP_{\mathcal{N}(B)}) = \{0\}$ (here we utilize the fact that $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}'}) \subseteq \overline{\mathcal{R}(A)}$ implies that $P'_{\overline{\mathcal{R}(A)}}(\alpha A + \beta B)_{\mathcal{T}'} = (\alpha A + \beta B)_{\mathcal{T}'}$). Again, since $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}'}) \subseteq \overline{\mathcal{R}(A)}$, we have that $P_{\overline{\mathcal{R}(A)}}(\alpha A + \beta B)|_{\mathcal{T}'}$ is injective if and only if $\mathcal{N}(P_{\overline{\mathcal{R}(A)}}(\alpha A + \beta B)|_{\mathcal{N}(B)^{\perp}}) \cap \mathcal{T}' = \{0\}$, which is clearly equivalent to $\mathcal{N}(\alpha A + \beta B) \cap \mathcal{N}(B)^{\perp} = \{0\}$. To complete the proof let us show that $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}'}) \cap \mathcal{R}(AP_{\mathcal{N}(B)}) = \{0\}$ if and only of $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}}) \cap \mathcal{R}(AP_{\mathcal{N}(B)}) = \{0\}$ where $\mathcal{T} = B^{-1}(\mathcal{R}(A)) \cap \mathcal{N}(B)^{\perp}$. We only need to consider the case $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}$. We can write $\mathcal{T}'$ as $\mathcal{T}' = \mathcal{T} \cup \mathcal{T}_1$ where

$$\mathcal{T}_1 = B^{-1}(\overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)) \cap \mathcal{N}(B)^{\perp}.$$

Notice that $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}_1}) \cap \mathcal{R}(AP_{\mathcal{N}(B)}) = \{0\}$ always holds. Indeed, if we assumed that there exists a non-zero vector in $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}_1}) \cap \mathcal{R}(AP_{\mathcal{N}(B)})$, there would exist vectors $x \in \mathcal{T}_1$ and $y \in N(B)$ such that $\alpha Ax + \beta Bx = Ay$. But then we would get

$$\beta Bx = A(\alpha x + y),$$

which implies that $x \in \mathcal{T}$, which is a contradiction since $\mathcal{T} \cap \mathcal{T}_1 = \emptyset$. Because

$$\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}'}) = \mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}}) \cup \mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}_1}),$$

we see from the previous conclusion that $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}}) \cap \mathcal{R}(AP_{\mathcal{N}(B)}) = \{0\}$ if and only if $\mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}'}) \cap \mathcal{R}(AP_{\mathcal{N}(B)}) = \{0\}$. This completes the proof. ∎

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It is important to note that due to the nature of Theorem 1.3 we do not need to utilize orthogonal decompositions of $\mathcal{H}$ and we can use

$$\mathcal{H} = \mathcal{N}(B) \oplus \mathcal{N} = \overline{\mathcal{R}(A)} \oplus \mathcal{M},$$

where $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces complementary to $\overline{\mathcal{R}(A)}$ and $\mathcal{N}(B)$, respectively. So condition (ii) from Theorem 2.1 becomes: $\mathcal{N}(\alpha A + \beta B) \cap \mathcal{N} = \{0\}$; and the subspace $\mathcal{T}$ is now $B^{-1}(\mathcal{R}(A)) \cap \mathcal{N}$.

Before we continue our investigations, in the following elementary examples, we will see that the injectivity of a linear combination of operators can depend on choice of constants, which will serve as motivation to find special cases where injectivity of a linear combination of operators is independent on the choice of the constants.

Example 1: Let $A, B \in \mathcal{B}(l_2)$ be operators defined as block diagonal operators whose block-diagonal entries are the matrices

$$M_A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \; M_B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

for A, and B, respectively. It is easy to see that $A+B$ is injective, whereas $A-B$ is not. If we analyze this situation using Theorem 2.1 (and its conditions) we see that:

$$\mathcal{N}(A) = \{(x_n)_{n\in\mathbb{N}} \mid x_{3k-1} = -x_{3k-2}, x_{3k} = 0,\; k \in \mathbb{N}\},$$

$$\mathcal{N}(B) = \{(x_n)_{n\in\mathbb{N}} \mid x_{3k-1} = x_{3k-2}, x_{3k} = 0,\; k \in \mathbb{N}\},$$

$$\mathcal{N}(B)^{\perp} = \{(x_n)_{n\in\mathbb{N}} \mid x_{3k-1} = -x_{3k-2},\; k \in \mathbb{N}\},$$

$$\mathcal{R}(A) = \{(x_n)_{n\in\mathbb{N}} \mid x_{3k-1} = x_{3k-2},\; k \in \mathbb{N}\},$$

$$\mathcal{T} = \{0\}.$$

We see that $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$, and since $\mathcal{T} = \{0\}$, condition (i) from Theorem 2.1 is satisfied, and condition (iii) trivially holds. It is easy to check that $A+B$ is injective on $\mathcal{N}(B)^{\perp}$, whereas $A-B$ is not, so by Theorem 2.1 we have that $A+B$ is injective, and $A-B$ is not.

Example 2: Let $A, B \in \mathcal{B}(l_2)$ be block-diagonal operators whose diagonal blocks are the matrices

$$N_A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \; N_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $A+B$ is injective, but $A-B$ is not. Furthermore, since $N_A^3 = N_B^3 = 0$ we see that A and B are nilpotent operators.

This example shows that even elementary classes of operators do not have the property that the injectivity of the linear combination is independent of the choice of scalars.

Some special cases in which the injectivity of the linear combination $\alpha A + \beta B$ is independent of the choice of scalars $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ are investigated now:

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators such that $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha A + \beta B$ is injective of and only if $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$.*

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Proof. From the assumption that $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}$ we have that $\mathcal{T}=\{0\}$ so conditions (ii) and (iii) from Theorem 2.1 trivially hold. So, we can conclude that the linear combination $\alpha A+\beta B$ is injective if and only if $\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}$. $\square$

Similarly, we get

THEOREM 2.3. *Let $A,B\in\mathcal{B}(\mathcal{H})$ be given operators such that $\mathcal{R}(A)=\mathcal{R}(AP_{\mathcal{N}(B)})$, and $\alpha,\beta\in\mathbb{C}\setminus\{0\}$. Then $\alpha A+\beta B$ is injective if and only if $\mathcal{N}(A)\cap\mathcal{N}(B)=\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}$.*

Proof. Assume that the linear combination $\alpha A+\beta B$ is injective. It is evident that in that case $\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}$. It follows that the conditions from Theorem 2.1 hold. Assume that $\mathcal{R}(A)\cap\mathcal{R}(B)\neq\{0\}$. This means that there exists a non-zero $z\in\mathcal{H}$ such that $z=Ax=By$, where $x\in\mathcal{N}(B)$ (here we used the fact that $\mathcal{R}(A)=\mathcal{R}(AP_{\mathcal{N}(B)})$) and $y\in\mathcal{N}(B)^{\perp}$ (this also means that $y\in\mathcal{T}=B^{-1}(\mathcal{R}(A))\cap\mathcal{N}(B)^{\perp}$). Now, since $\alpha A+\beta B$ is injective we have

$$(\alpha A+\beta B)(\frac{1}{\alpha}x-\frac{1}{\beta}y)=-\frac{\alpha}{\beta}Ay\neq 0.$$

Using the condition $\mathcal{R}(A)=\mathcal{R}(AP_{\mathcal{N}(B)})$, exists a $w\in\mathcal{N}(B)$ such that $\frac{\alpha}{\beta}Ay=Aw$. It follows that :

$$0\neq(\alpha A+\beta B)(\frac{1}{\beta}y)=A(w+x),$$

where we again use the fact that $\alpha A+\beta B$ is injective, and that $\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}$. This implies that $\mathcal{R}((\alpha A+\beta B)|_{\mathcal{T}})\cap\mathcal{R}(AP_{\mathcal{N}(B)})\neq\{0\}$ which is in contradiction with condition (iii) from Theorem 2.1. Hence, $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}$.

Conversely, if $\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}$ and $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}$ conditions (i) and (iii) of Theorem 2.1 are satisfied. Assume that condition (ii) does not hold. That means that there exists a nonzero vector $x\in\mathcal{N}(B)^{\perp}$ such that $Bx=A(-\frac{\alpha}{\beta}x)$. This is in contradictions with $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}$, so condition (ii) is satisfied as well. $\square$

From Theorems 2.2 and 2.3, we get the following corollary:

COROLLARY 2.4. *Let $A,B\in\mathcal{B}(\mathcal{H})$ be given operators and $\alpha,\beta\in\mathbb{C}\setminus\{0\}$. If one of the conditions, $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}$ or $\mathcal{R}(A)=\mathcal{R}(AP_{\mathcal{N}(B)})$ holds, then the injectivity of the linear combination $\alpha A+\beta B$ is independent of the choice of the scalars $\alpha,\beta\in\mathbb{C}\setminus\{0\}$.*

Injectivity of linear combinations of projections is investigated next (in a manner which will cover both cases of sums and differences).

THEOREM 2.5. *Let $P,Q\in\mathcal{B}(\mathcal{H})$ be given projections and $\alpha,\beta\in\mathbb{C}\setminus\{0\}$. Then the operator $\alpha P+\beta Q$ is injective if and only if:*

$$\begin{cases}\mathcal{N}(P)\cap\mathcal{N}(Q)=\mathcal{R}(Q)\cap\mathcal{R}(P(I-Q))=\{0\}, & \alpha+\beta\neq 0;\\ \mathcal{N}(P)\cap\mathcal{N}(Q)=\mathcal{R}(P)\cap\mathcal{R}(Q)=\{0\}, & \alpha+\beta=0.\end{cases}$$

Proof. First let us note that since $\mathcal{H}$ is the direct sum $\mathcal{N}(Q)\oplus\mathcal{R}(Q)$ and $\mathcal{N}(P)\oplus\mathcal{R}(P)$. We have that the linear combination $\alpha P+\beta Q$ has the following representation in this case:

$$\alpha P+\beta Q=\begin{bmatrix}\alpha P_1 & \alpha P_2+\beta Q_1\\ 0 & \beta Q_2\end{bmatrix}:\begin{bmatrix}\mathcal{N}(Q)\\ \mathcal{R}(Q)\end{bmatrix}\to\begin{bmatrix}\mathcal{R}(P)\\ \mathcal{N}(P)\end{bmatrix}. \tag{2.4}$$

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Applying Theorem 1.3 to (2.4) $\alpha P + \beta Q$ is injective if and only if $P_1 : \mathcal{N}(Q) \to \mathcal{R}(P)$ is injective and $(\alpha P_2 + \beta Q_1)|_{\mathcal{N}(Q_2)}$ is injective with range disjoint from $\mathcal{R}(P_1)$. P_1 is injective if and only if $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$. We have $\mathcal{N}(Q_2) = \mathcal{R}(P) \cap \mathcal{R}(Q)$. So for $x \in \mathcal{R}(P) \cap \mathcal{R}(Q)$, we have $(\alpha P_2 + \beta Q_1)x = (\alpha + \beta)x$. Because of this, we continue this proof analyzing two separate cases, $\alpha + \beta \neq 0$ and $\alpha + \beta = 0$.

If $\alpha + \beta \neq 0$, then $(\alpha P_2 + \beta Q_1)|_{\mathcal{N}(Q_2)}$ is obviously injective and $\mathcal{R}((\alpha P_2 + \beta Q_1)|_{\mathcal{N}(Q_2)}) = \mathcal{R}(P) \cap \mathcal{R}(Q)$ so the second condition of Theorem 1.3 is equivalent to $\{0\} = \mathcal{R}(P) \cap \mathcal{R}(Q) \cap \mathcal{R}(P|_{\mathcal{N}(Q)}) = \mathcal{R}(Q) \cap \mathcal{R}(P(I-Q))$. In conclusion, $\alpha P + \beta Q$, where $\alpha + \beta \neq 0$ will be injective if and only if $\mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{R}(Q) \cap \mathcal{R}(P(I-Q)) = \{0\}$.

If $\alpha + \beta = 0$, we have that for each $x \in \mathcal{R}(P) \cap \mathcal{R}(Q)$ $(\alpha P_2 + \beta Q_1)x = 0$, so Theorem 1.3 will hold if and only if $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$. □

Conditions for injectivity in the case of orthogonal projections are:

THEOREM 2.6. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be given orthogonal projections and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then the operator $\alpha P + \beta Q$ is injective if and only if*

$$\begin{cases} \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}, & \alpha + \beta \neq 0; \\ \mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}, & \alpha + \beta = 0. \end{cases}$$

Proof. This theorem immediately follows from Theorem 2.5 and from

$$\begin{aligned} \mathcal{R}(Q) \cap \mathcal{R}(P(I-Q)) &= \mathcal{R}(Q) \cap \mathcal{N}((I-Q)P)^{\perp} = \\ &= \mathcal{R}(Q) \cap (\mathcal{N}(P) \oplus (\mathcal{R}(P) \cap \mathcal{R}(Q)))^{\perp} = \\ &= \mathcal{R}(Q) \cap N(P)^{\perp} \cap (\mathcal{R}(P) \cap \mathcal{R}(Q))^{\perp} = \\ &= \mathcal{R}(Q) \cap \mathcal{R}(P) \cap (\mathcal{R}(P) \cap \mathcal{R}(Q))^{\perp} = \{0\}, \end{aligned}$$

where we simply utilize the fact that P and Q are orthogonal projections. □

3. Left (right) invertibility. Using Theorems 1.4 and 1.6, we now give necessary and sufficient conditions for left (right) invertibility of the linear combination $\alpha A + \beta B$, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

THEOREM 3.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha A + \beta B$ is left invertible if and only if the following conditions hold:*

(i) $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$;
(ii) $\mathcal{R}(AP_{\mathcal{N}(B)})$ *is closed;*
(iii) $\mathcal{N}(B)^{\perp} = \mathcal{R}(B^* P_{\mathcal{R}(A)^{\perp}}) + \mathcal{R}((\alpha A + \beta B)^*|_{\mathcal{S}})$;

where $\mathcal{S} = \mathcal{R}(AP_{\mathcal{N}(B)})^{\perp} \cap \overline{\mathcal{R}(A)}$.

Proof. Since $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^{\perp} = \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp}$, $\alpha A + \beta B$ has the following representation

$$\begin{bmatrix} \alpha A_1 & \beta B_1 + \alpha A_2 \\ 0 & \beta B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}. \tag{3.5}$$

Using Theorem 1.4, we see that $\alpha A + \beta B$ is left invertible if and only if:

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(i) A_1 is left invertible,
(ii) $\mathcal{R}((\alpha A_2+\beta B_1)^* P_{\mathcal{S}}) + \mathcal{R}(B_2^*) = \mathcal{N}(B)^{\perp}$,

where $\mathcal{S} = \mathcal{R}(AP_{\mathcal{N}(B)})^{\perp} \cap \overline{\mathcal{R}(A)}$.

Condition (i) is equivalent with $\mathcal{N}(A)\cap\mathcal{N}(B) = \{0\}$ and $\mathcal{R}(AP_{\mathcal{N}(B)})$ is closed. As $\mathcal{S}$ is a subspace of $\overline{\mathcal{R}(A)}$ and $\mathcal{R}(A_1)^{\perp} = \mathcal{N}(A_1^*)$, condition (ii) is equivalent

$$\mathcal{R}((\alpha A+\beta B)^*|_{\mathcal{S}}) + \mathcal{R}(B^* P_{\mathcal{R}(A)^{\perp}}) = \mathcal{N}(B)^{\perp}, \tag{3.6}$$

which completes the proof. ∎

Remark 3.2. Condition (iii) can be reformulated as:

$$\begin{bmatrix} P_{\mathcal{S}}(\alpha A+\beta B)|_{\mathcal{N}(B)^{\perp}} \\ P_{\mathcal{R}(A)^{\perp}} B \end{bmatrix} : \mathcal{N}(B)^{\perp} \to \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \end{bmatrix} \text{ is left invertible.}$$

Similarly, we can give necessary and sufficient conditions for right invertibility:

Theorem 3.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators and $\alpha,\beta \in \mathbb{C}\setminus\{0\}$. Then $\alpha A+\beta B$ is right invertible if and only if the following conditions hold:*

(i) $\overline{\mathcal{R}(A)} + \mathcal{R}(B) = \mathcal{H}$;
(ii) $\overline{\mathcal{R}(A)} = \mathcal{R}(AP_{\mathcal{N}(B)}) + \mathcal{R}((\alpha A+\beta B)|_{\mathcal{T}})$.

where $\mathcal{T} = B^{-1}(\overline{\mathcal{R}(A)}) \cap \mathcal{N}(B)^{\perp}$.

Proof. First, let us assume that $\alpha A+\beta B$ is right invertible, which is equivalent to the operator matrix given by (3.5) being right invertible. Using Theorem 1.6, we see that the following holds:

(i) B_2 is right invertible,
(ii) $\mathcal{R}(A_1) + \mathcal{R}((\alpha A_2+\beta B_1)|_{\mathcal{T}}) = \overline{\mathcal{R}(A)}$,

where $\mathcal{T} = \mathcal{N}(B_2) = B^{-1}(\overline{\mathcal{R}(A)}) \cap \mathcal{N}(B)^{\perp}$. The first condition is equivalent to $\mathcal{R}(A)^{\perp} \subseteq \overline{\mathcal{R}(A)} + \mathcal{R}(B)$, which in turn implies that $\mathcal{H} = \overline{\mathcal{R}(A)} + \mathcal{R}(B)$. Condition (ii) is equivalent to

$$\mathcal{R}(AP_{\mathcal{N}(B)}) + \mathcal{R}((\alpha A+\beta B)|_{\mathcal{T}}) = \overline{\mathcal{R}(A)}.$$

This equality together with $\overline{\mathcal{R}(A)} + \mathcal{R}(B) = \mathcal{H}$ implies that

$$\begin{aligned} \mathcal{H} = \overline{\mathcal{R}(A)} + \mathcal{R}(B) &= \mathcal{R}(AP_{\mathcal{N}(B)}) + \mathcal{R}((\alpha A+\beta B)|_{\mathcal{T}}) + \mathcal{R}(B) \subseteq \\ &\subseteq \mathcal{R}(AP_{\mathcal{N}(B)}) + \mathcal{R}(A|_{\mathcal{T}}) + \mathcal{R}(B|_{\mathcal{T}}) + \mathcal{R}(B) \subseteq \mathcal{R}(A) + \mathcal{R}(B) \subseteq \mathcal{H}. \end{aligned}$$

We have thus proven that if $\alpha A+\beta B$ is right invertible conditions (i) and (ii) hold.

Conversely, it is easy to see that if conditions (i) and (ii) hold, the operator matrix given by (3.5) satisfies the conditions of Theorem 1.6, so $\alpha A+\beta B$ is right invertible as well. ∎

Remark 3.4. It is not hard to prove that this theorem holds even if we take any arbitrary decomposition

$$\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{M} = \mathcal{N} \oplus \mathcal{N}(B),$$

where $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{H}$ complementary to $\overline{\mathcal{R}(A)}$ and $\mathcal{N}(B)$.

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In conjunction with Theorems 3.1 and 3.3, we can examine some special cases where left (right) invertiblity of the linear combination is independent of the choice of scalars $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

THEOREM 3.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators such that $\mathcal{R}(A) = \mathcal{R}(AP_{\mathcal{N}(B)})$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha A + \beta B$ is left invertible if and only if the following conditions hold:*

(i) $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$;
(ii) $\mathcal{R}(A)$ *is closed;*
(iii) $\mathcal{R}(P_{\mathcal{R}(A)^\perp} B)$ *is closed;*
(iv) $\mathcal{R}(B) \cap \mathcal{R}(A) = \{0\}$.

Proof. Let us first note that since $\mathcal{R}(A) = \mathcal{R}(AP_{\mathcal{N}(B)})$, the subspace $\mathcal{S} = \mathcal{R}(AP_{\mathcal{N}(B)})^\perp \cap \overline{R(A)} = \{0\}$. First, assume that $\alpha A + \beta B$ is left invertible, this means that the following conditions from Theorem 3.1 hold, that is:

1. $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$,
2. $\mathcal{R}(AP_{\mathcal{N}(B)})$ is closed.
3. $\mathcal{N}(B)^\perp = \mathcal{R}(B^* P_{\mathcal{R}(A)^\perp}) + \mathcal{R}((\alpha A + \beta B)^*|_{\mathcal{S}})$.

It is clear that Condition 2. is equivalent to $\mathcal{R}(A)$ being closed. Since $\mathcal{S} = \{0\}$, Condition 3. becomes

$$\mathcal{N}(B)^\perp = \mathcal{R}(B^* P_{\mathcal{R}(A)^\perp}),$$

which is equivalent to B_2^* being right invertible which in turn implies that B_2 is left invertible, which means that $\mathcal{R}(B_2) = \mathcal{R}(P_{\mathcal{R}(A)^\perp} B)$ is closed and $\mathcal{T} = \{0\}$. The equation $\mathcal{T} = \{0\}$ implies that $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ (here we already used the fact that $\mathcal{R}(A)$ is closed).

Conversely, if conditions $(i) - (iv)$ are satisfied we easily see that the conditions of Theorem 3.1 are satisfied as well. □

THEOREM 3.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators such that $\mathcal{R}(A) = \mathcal{R}(AP_{\mathcal{N}(B)})$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha A + \beta B$ is right invertible if and only if the following conditions hold:*

(i) $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{H}$;
(ii) $\overline{\mathcal{R}(A)} = \mathcal{R}(A) + \mathcal{R}(B|_{\mathcal{T}})$,

where $\mathcal{T} = \mathcal{N}(B_2) = B^{-1}(\overline{\mathcal{R}(A)}) \cap \mathcal{N}(B)^\perp$.

Proof. If $\alpha A + \beta B$ is right invertible, the conditions of Theorem 3.3 are satisfied. The second condition of Theorem 3.3 in this setting is

$$\overline{R(A)} = \mathcal{R}(A) + \mathcal{R}((\alpha A + \beta B)|_{\mathcal{T}}).$$

It follows now that

$$\overline{\mathcal{R}(A)} \subseteq \mathcal{R}(A) + \mathcal{R}(A|_{\mathcal{T}}) + \mathcal{R}(B|_{\mathcal{T}}) \subseteq \mathcal{R}(A) + \mathcal{R}(B|_{\mathcal{T}}) \subseteq \overline{\mathcal{R}(A)},$$

where we used that $\mathcal{R}(B|_{\mathcal{T}}) \subseteq \overline{\mathcal{R}(A)}$. So, if $\alpha A + \beta B$ is right invertible, conditions (i) and (ii) hold. It is easily verified that if conditions (i) and (ii) hold, the conditions of Theorem 3.3 are satisfied as well. □

THEOREM 3.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators such that $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \{0\}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha A + \beta B$ is right invertible if and only if the following conditions hold:*

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(i) $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathcal{H}$;
(ii) $\mathcal{R}(AP_{\mathcal{N}(B)}) = \overline{\mathcal{R}(A)}$.

Proof. The theorem follows from the fact that in this setting we have that $\mathcal{T} = B^{-1}(\overline{\mathcal{R}(A)}) \cap \mathcal{N}(B)^{\perp} = \{0\}$. ∎

Now conditions for left (right) invertibility are given for the case of orthogonal projections:

THEOREM 3.8. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be given orthogonal projections and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha P + \beta Q$ is left invertible if and only if*

$$\begin{cases} \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}, & \alpha + \beta \neq 0; \\ \mathcal{R}(P) \oplus \mathcal{R}(Q) = \mathcal{H}, & \alpha + \beta = 0. \end{cases}$$

Proof. In this case, the space $\mathcal{H}$ has the following natural orthogonal decompositions $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P) = \mathcal{N}(Q) \oplus \mathcal{R}(Q)$. Let us note that in this setting $\mathcal{T} = \mathcal{R}(P) \cap \mathcal{R}(Q)$ and

$$\mathcal{S} = \mathcal{R}(P(I-Q))^{\perp} \cap \mathcal{R}(P) = \mathcal{N}((I-Q)P) \cap \mathcal{R}(P) = \mathcal{R}(P) \cap \mathcal{R}(Q) = \mathcal{T}.$$

So $\mathcal{R}((\alpha P + \beta Q)|_{\mathcal{T}}) = \mathcal{R}((\alpha P + \beta Q)^*|_{\mathcal{S}}) = \begin{cases} \mathcal{R}(P) \cap \mathcal{R}(Q) & , \alpha \neq -\beta \\ \{0\} & , \alpha = -\beta. \end{cases}$ Using Theorem 3.1 the linear combination $\alpha P + \beta Q$, where $\alpha, \beta \in \mathbb{C}$, is left invertible if and only if

(i) $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$,
(ii) $\mathcal{R}(P(I-Q))$ is closed,
(iii) $\mathcal{R}(Q) = \mathcal{R}(Q(I-P)) + (R(P) \cap \mathcal{R}(Q))$ when $\alpha \neq -\beta$ and $\mathcal{R}(Q) = \mathcal{R}(Q(I-P)$ when $\alpha = -\beta$.

To be more precise condition (iii) in the case when $\alpha \neq -\beta$ can be rewritten as $\mathcal{R}(Q) = \mathcal{R}(Q(I-P)) \oplus (R(P) \cap \mathcal{R}(Q))$ since $\mathcal{R}(Q(I-P)) \cap \mathcal{R}(P) = \{0\}$ (which we saw in the proof of Theorem 2.6).

Let us first assume that $\alpha P + \beta Q$ is left inverible, that is the aforementioned conditions hold.

Condition (i) implies that $\overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$. Condition (ii) is equivalent to $\mathcal{R}((I-Q)P)$ being closed, and using Corollary 2.5 from [16] we have that this is equivalent to $\mathcal{R}(Q) + \mathcal{R}(P)$ being closed. Thus, if $\alpha P + \beta Q$ is left invertible and $\alpha \neq -\beta$, we have that $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$.

Assume on the contrary, $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$ and $\alpha \neq -\beta$. Let us prove that conditions $(i)-(iii)$ are satisfied, which means that $\alpha P + \beta Q$ is left invertible. $\mathcal{R}(P) + \mathcal{R}(Q)$ is closed and that (as already noted) implies that $\mathcal{R}(P(I-Q))$ is closed, so condition (ii) holds. Furthermore, $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$ implies that $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$ so the condition is satisfied as well. It remains to prove that condition (iii) holds as well. We have that the operator Q_2 (from the decomposition given by (3.5)) is right invertible, so Q_2^* is left invertible. Since $\mathcal{R}(Q_2^*) = \mathcal{R}(Q(I-P))$, after applying Corollary 2.5 from [16], we have that $\mathcal{R}(Q_2^*)$ is closed and that

$$\mathcal{R}(Q) = \mathcal{R}(Q_2^*) \oplus (\mathcal{R}(Q_2^*)^{\perp} \cap \mathcal{R}(Q)).$$

Since $\mathcal{R}(Q_2^*)^{\perp} \cap \mathcal{R}(Q) = \mathcal{N}(Q_2) = \mathcal{T}$ we have that

$$\mathcal{R}(Q) = \mathcal{R}(Q(I-P)) \oplus \mathcal{T} = \mathcal{R}(Q(I-P)) \oplus (\mathcal{R}(P) \cap R(Q)),$$

so condition (iii) holds as well.

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Now, assume that $\mathcal{R}(P) \oplus \mathcal{R}(Q) = \mathcal{H}$ and $\alpha = -\beta$. We now know that Conditions (i) and (ii) are satisfied. We commented that Q_2 is right invertible, and since $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, we have that Q_2 is invertible, and so is Q_2^*, so $\mathcal{R}(Q) = \mathcal{R}(Q_2^*) = \mathcal{R}(Q(I-P))$; hence condition (iii) holds as well which completes the proof. ∎

Since from this theorem we see that the linear combinations $\alpha P + \beta Q$ and $\overline{\alpha} P + \overline{\beta} Q$ are simultaneously left invertible, and it is easily concluded that $\alpha P + \beta Q$ will be right invertible if and only if it is left invertible. This gives the following corollaries:

Corollary 3.9. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be given orthogonal projections and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha P + \beta Q$ is right invertible if and only if*

$$\begin{cases} \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}, & \alpha + \beta \neq 0; \\ \mathcal{R}(P) \oplus \mathcal{R}(Q) = \mathcal{H}, & \alpha + \beta = 0. \end{cases}$$

Corollary 3.10. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be given orthogonal projections and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then the following statements are equivalent:*

(i) $\alpha P + \beta Q$ is left (right) invertible,
(ii) $\alpha P + \beta Q$ is invertible.

Remark 3.11. in Corollary 4.3 from [17] it was proven that the sum of orthogonal projections P and Q will be invertible if and only if

$$\mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{R}(P) \cap \mathcal{R}(Q(I-P)) = \{0\} \text{ and } P + Q \text{ has closed range.}$$

This result is equivalent to Corollary 3.9 (for the case of sums of orthogonal projections). Indeed, if $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$, we have that $\overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$. Furthermore, if $R(P+Q)$ is closed, from Lemma 2.4 in [18], we have that $\mathcal{R}(P) + \mathcal{R}(Q)$ is closed as well so finally $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$. Conversely, if $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$, we have that $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$ and using Lemma 2.4 from [18] again, we have that $P + Q$ has closed range.

In Theorem 6.2 of [19], condition (ii) is indeed the corresponding condition of Corollary 3.9.

Some interesting results can be attained for right invertibility of linear combinations of oblique projections as well:

Theorem 3.12. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be given projections and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then $\alpha P + \beta Q$ is right invertible if and only if $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$ and* $\begin{cases} \mathcal{R}(P) = \mathcal{R}(P(I-Q)) + (\mathcal{R}(P) \cap \mathcal{R}(Q)), & \alpha \neq -\beta, \\ \mathcal{R}(P) = \mathcal{R}(P(I-Q)), & \alpha = \beta. \end{cases}$

Proof. Using Theorem 3.3, we see that the linear combination $\alpha P + \beta Q$ will be right invertible if and only if $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$ and $\mathcal{R}(P) = \mathcal{R}(P(I-Q)) + \mathcal{R}((\alpha P + \beta Q)|_{\mathcal{T}})$. It is easy to see that when we utilize the natural decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P) = \mathcal{R}(Q) \oplus N(Q)$, we have that $\mathcal{T} = \mathcal{R}(P) \cap R(Q)$. To complete this proof, it remains to be pointed out that when $\alpha \neq -\beta$ we have that $\mathcal{R}((\alpha P + \beta Q)|_{\mathcal{T}}) = \mathcal{R}(P) \cap \mathcal{R}(Q)$, and when $\alpha = -\beta$ we have that $\mathcal{R}((\alpha P + \beta Q)|_{\mathcal{T}}) = \{0\}$. ∎

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