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C–NORMAL OPERATORS*

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Abstract. A new class of operators, larger than C–symmetric operators and different than normal one, named C–normal operators is introduced. Basic properties are given. Characterizations of this operators in finite dimensional spaces using a relation with conjugate normal matrices are presented. Characterizations of Toeplitz operators and composition operators as C–normal operators are given. Bunches of examples are presented.

Key words. C–symmetric operators, C–skew–symmetric operators, Toeplitz operators, Conjugate normal matrices, Composition operators, Truncated Toeplitz operators.

AMS subject classifications. 15A57, 47B15.

1. Introduction and main definition. Let $\mathcal{H}$ be a complex Hilbert space and denote by $L(\mathcal{H})$ (by $LA(\mathcal{H})$, respectively) the algebra (the space, respectively) of all bounded linear (antilinear, respectively) operators in the space $\mathcal{H}$. The theory of selfadjoint and normal operators has been developed for many years. However, there are many operators which do not belong to those classes. On the other hand, a complex Hilbert space can be equipped with additional structure given by *conjugation* C, i.e., antilinear isometric involution; ($C \in LA(\mathcal{H})$, $C^2 = I$ and $\langle h, g\rangle = \langle Cg, Ch\rangle$ for all $h, g \in \mathcal{H}$). Such a structure naturally appears in physics, see [8]. On the other hand, conjugations are related to adjoint operators in the antilinear sense. Following Wigner (see [17]), for antilinear operator $X \in LA(\mathcal{H})$, there is the unique antilinear operator $X^\sharp$ called the *antilinear adjoint* of X such that

$$\langle Xx, y\rangle = \overline{\langle x, X^\sharp y\rangle} \quad \text{for all } x, y \in \mathcal{H}. \tag{1.1}$$

The antilinear operator X is called antilinear selfadjoint if $X^\sharp = X$. Conjugations are the examples of such operators since $C^\sharp = C$.

Having a conjugation C on a space $\mathcal{H}$, an operator T can be called *C–symmetric* if $CAC = A^*$, see [9]. It turned out, see [4, Lemma 5.1], that operator $A \in L(\mathcal{H})$ is C–symmetric if and only if AC is antilineary selfadjoint, i.e., $(AC)^\sharp = AC$. The C–symmetric operators have applications in physics especially in the quantum mechanics and the spectral analysis; let us recall monograph [14] and paper [1]. Authors send the reader to [8] for more of *Mathematical and physical aspects of complex symmetric operators*. It is worth to mention that C–symmetric operators have got interesting properties which was intensively studied, see [9, 10]. For more references, see [8]. On the other hand, many natural operators belong to this class: truncated Toeplitz, Voltera operators, normal operators and many others.

It is natural to search for the larger class of operators than C–symmetric ones. Having in mind classical

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selfadjoint and normal operators, it is natural to put forward the following:

Definition 1.1. An operator $N \in L(\mathcal{H})$ is called *C–normal* if

$$NC(NC)^\sharp = (NC)^\sharp(NC). \tag{1.2}$$

The definition refers to definition of normality for antilinear operators, see [17]. Namely, an antilinear operator $X \in LA(\mathcal{H})$ is called *antylinearly normal* if

$$X\,X^\sharp = X^\sharp X. \tag{1.3}$$

After stating the main definition the aim of the paper is to give equivalent conditions and basic properties of C–normal operator, Section 2. The next section is devoted to C–normal operators in finite dimensional Hilbert spaces. Section 3 shows the relation between C–normal operators and conjugate normal matrices; in fact we fully characterized the C–normal operators. The following sections concern finding a class of examples in various natural Hilbert spaces having a natural conjugations. Section 4 concerns multiplications operators in L^2 type spaces. Section 5 concerns Hardy space H^2 with some natural conjugation. Section 6 deals with composition operators. Especially interesting there are classes of C–normal operators being neither normal (in classical sense), nor C–symmetric, nor C–skew–symmetric. Theorems 6.6 and 7.3 give collections of such operators. Authors think that this paper proves that C–normal operators form widely enough class of operators. On the other hand, we hope there will be many theorems and properties of classical normal operators which can be moved to this new class and which should be of the future investigations.

2. Equivalent conditions and basic examples. Let $\mathcal{H}$ be a complex Hilbert space with conjugation C. An operator $A \in L(\mathcal{H})$ is called *C–symmetric* if $CAC = A^*$. It is called *C–skew–symmetric* if $CAC = -A^*$. The immediate consequence of the definition of C–normality (Definition 1.1) is that C–symmetric operators and C–skew–symmetric operators are C–normal.

The paper concentrates on examples of C–normal operators which are neither C–cymmetric nor C–skew–symmetric, but let us recall two classes of C–symmetric operators, so also C–normal, to give a feeling to the reader how large and important is the class of C–normal operators.

Example 2.1. Let C be a conjugation in $\mathbb{C}^n$ given by $C(z_1,\ldots,z_n) = (\bar z_n, \bar z_{n-1},\ldots,\bar z_1)$. The operators are C–symmetric if and only if its matrix is symmetric according to "second diagonal". (Notations are in Sections 3 and 4. This is an immediate consequence of Lemma 4.1.)

Let m be the normalized Lebesgue measure on the unit circle $\mathbb{T}$ and let us consider space $L^2 = L^2(\mathbb{T}, m)$. The Hardy space H^2 is a subspace of those elements of L^2 which have negative Fourier coefficient equal to 0. One of the most interesting examples of C–symmetric, hence also C–normal, operators are truncated Toeplitz operators (TTO). (See [7] for more details about TTO.)

Example 2.2. By Beurling's theorem all subspaces which are invariant for the unilateral shift S in the Hardy space H^2 ($Sf(z) = zf(z)$ for $f \in H^2$) can be written as θH^2, where θ is an inner function. Consider, so-called, the *model space* $K^2_\theta = H^2 \ominus \theta H^2$ and the orthogonal projection $P_\theta \colon L^2 \to K^2_\theta$. A *truncated Toeplitz operator* A^θ_φ with a symbol $\varphi \in L^2$ is defined as

$$A^\theta_\varphi \colon D(A^\theta_\varphi) \subset K^2_\theta \to K^2_\theta; \quad A^\theta_\varphi f = P_\theta(\varphi f)$$

for $f \in D(A^\theta_\varphi) = \{f \in K^2_\theta : \varphi f \in L^2\}$. If A^θ_φ is bounded, it naturally extends to the operator in $L(\mathcal{H})$. The model space K^2_θ is equipped with natural conjugation C_θ, $C_\theta f = \theta \bar z \bar f$ for $f \in K^2_\theta$. Denote by $\mathcal{T}(\theta)$ the

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set of all bounded truncated Toeplitz operators on K^2_θ. As it was shown in [16, 7], operators from $\mathcal{T}(\theta)$ are C_θ–symmetric, hence C_θ–normal.

We have the following equivalent conditions:

Theorem 2.3. *Let C be a conjugation on $\mathcal{H}$ and let $N \in L(\mathcal{H})$. The followings conditions are equivalent:*

(1) *N is C–normal,*
(2) *N^* is C–normal,*
(3) *CNC is C–normal,*
(4) *CN^*C is C–normal,*
(5) *$CNN^* = N^*NC$,*
(6) *$CN^*N = NN^*C$,*
(7) *$CN(CN)^\sharp = (CN)^\sharp(CN)$,*
(8) *$||NCh|| = ||N^*h||$,*
(9) *$||N^*Ch|| = ||Nh||$,*
(10) *$N_+ \stackrel{df}{=} \frac{1}{2}(CN + N^*C)$ and $N_- \stackrel{df}{=} \frac{1}{2}(CN - N^*C)$ commute,*
(11) *$N^+ \stackrel{df}{=} \frac{1}{2}(NC + CN^*)$ and $N^- \stackrel{df}{=} \frac{1}{2}(NC - CN^*)$ commute.*

Proof. We prove, for instance, equivalences (1) and (5), (1) and (6). Let's assume (1). From (1.1) and (1.2) we have following:

$$NCCN^* = CN^*NC,$$

and from Definition 1.1,

$$NN^* = CN^*NC.$$

Then, by covering the above equation from the left side by C, we get condition (5). Furthermore, by covering the above equation from the right side by C, we get condition (6). ∎

Lemma 2.4. *Let C be a conjugation in $\mathcal{H}$. If $N \in L(\mathcal{H})$ is C–normal, then $N_L = CNCN$ and $N_R = NCNC$ are normal.*

Example 2.5. The reverse implication is not true, which follows from the following example. Let $\mathcal{H} = \mathbb{C}^3$, $C(z_1, z_2, z_3) = (\bar{z}_3, \bar{z}_2, \bar{z}_1)$ and

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next, we will present some results on relations between C–normal operators and unitary ones.

Proposition 2.6. *Let C be a conjugation in $\mathcal{H}$ and $U \in L(\mathcal{H})$ be a unitary operator. Then:*

(1) *U is C–normal,*
(2) *UCU^* is a conjugation,*
(3) *CUC is unitary,*
(4) *if T is C–symmetric, then UTU^* is UCU^*–symmetric,*
(5) *if N is C–normal, then UNU^* is UCU^*–normal,*
(6) *moreover, if U is C–symmetric, then*
    (a) *if T is C–symmetric, then UTU^* is C–symmetric,*
    (b) *if N is C–normal, then UNU^* is C–normal.*

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PROPOSITION 2.7. *Let C be a conjugation in $\mathcal{H}$ and let $U \in L(\mathcal{H})$ be unitary operator. An operator N is C–normal if and only if U^*NCUC ($U^*C\,NUC$, respectively) is C–normal.*

It is a consequence of the following:

LEMMA 2.8. *Let $X \in LA(\mathcal{H})$ and let $U \in L(\mathcal{H})$ be unitary operator. If X is antilinearly normal, then U^*XU is also antilinearly normal.*

Proof. The direct computation shows that

$$\begin{aligned}(U^*XU)\,(U^*XU)^\sharp &= U^*XUU^*(U^*X)^\sharp \;=\; U^*XX^\sharp U\\ &= U^*X^\sharp XU \;=\; U^*X^\sharp U\,U^*X^\sharp U\\ &= (U^*XU)^\sharp(U^*XU). \qquad \square\end{aligned}$$

Let $h, g \in \mathcal{H}$ then, by $h \otimes g \in L(\mathcal{H})$ we will denote rank one operator given by $(h \otimes g)x = \langle x, g\rangle\, h$ for $x \in \mathcal{H}$.

LEMMA 2.9. *Let C be a conjugation in $\mathcal{H}$. Let $x, y, h, g \in \mathcal{H}$. Then:*

(1) $(h \otimes g)^* = g \otimes h$,
(2) $C(h \otimes g)C = Ch \otimes Cg$,
(3) $(h \otimes g)(x \otimes y) = \langle x, g\rangle\, h \otimes y$.

Let $\mathcal{H}$ be a complex Hilbert space with conjugation C. Direct calculations show that all C–normal rank–one operators have the form $h \otimes Ch$, where $h \in \mathcal{H}$. This operators are C–cymmetric, see [13]. Hence, there can be found interesting examples among rank–two or rank–three operators. Let $\dim \mathcal{H} \geqslant 3$. Then, by [8, Lemma 2.1], there is an orthonormal basis $\{e_k\}$ such that $Ce_k = e_k$. Denote $h = \frac{1}{\sqrt{2}}(e_1 + ie_2)$, $g = e_3$ then h, Ch, g are orthonormal. Let us consider two operators

$$A_1 = h \otimes h + h \otimes Ch + Ch \otimes h - Ch \otimes Ch, \tag{2.4}$$

$$A_2 = h \otimes Ch + g \otimes h + 2\, g \otimes g + 2\, Ch \otimes h - Ch \otimes g. \tag{2.5}$$

A direct calculation, using Lemma 2.9, shows that operators A_1 and A_2 are neither C–symmetric, nor C–skew–symmetric, but they are C–normal. Moreover, the operator A_2 is neither selfadjoint nor normal.

3. Finite dimensional case. Let $\mathbf{M}_n$ denote the algebra of all $n \times n$ complex matrices. Except the algebra structure, which was recalled, there are some operations on matrices which are defined as follows; let $M = [a_{jk}] \in \mathbf{M}_n$, then we denote

$$\overline{M} = [\bar{a}_{jk}], \quad M^t = [a_{kj}], \quad M^* = [\bar{a}_{kj}], \quad M^s = [a_{n-j+1\;\,n-k+1}].$$

We will call the matrix *unitary* if its columns (or rows) form an orthonormal basis.

Let us recall relations between antilinear operators and matrices. Let $X \in LA(\mathbb{C}^n)$. Let $e_1, \ldots, e_n$ be an orthonormal basic in $\mathbb{C}^n$. There is a matrix $M_X = [a_{jk}]$ such that for any $x = \sum_{k=1}^n \langle x, e_k\rangle e_k \in \mathbb{C}^n$ we have

$$Xx = \sum_{j=1}^n \Big(\sum_{k=1}^n a_{jk}\; \overline{\langle x, e_k\rangle}\Big) e_j.$$

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Moreover, $a_{jk} = \langle Xe_k, e_j\rangle$. The matrix M_X will be called a *matrix representation* of antilinear operator X as to basis $e_1, \ldots, e_n$. (The standard matrix for linear operator $T \in L(\mathbb{C}^n)$ is also denoted by M_T.) The following properties hold.

Lemma 3.1. *Let $X, Y \in LA(\mathbb{C}^n)$ and $T \in L(\mathbb{C}^n)$. Let M_X, M_Y, M_T be a matrix representation of operators X, Y, T as to certain orthonormal basis $e_1, \ldots, e_n$, respectively. Then:*

(1) $M_{XT} = M_X\ \overline{M}_T$,
(2) $M_{TX} = M_T\ M_X$,
(3) $M_{XY} = M_X\ \overline{M}_Y$,
(4) $M_{X^\sharp} = M_X^t$.

There is quite large literature concerning conjugate normal matrices.

Definition 3.2 ([6]). Matrix $M \in \mathbf{M}_n(\mathbb{C})$ is *conjugate normal* if

$$MM^* = \overline{M^*M}.$$

The theorem bellow shows the relationships between antilinearly normal operators and conjugate normal matrices.

Theorem 3.3. *Let $X \in LA(\mathbb{C}^n)$. Then X is antilineary normal if and only if the matrix M_X is conjugate normal.*

Proof. The antilinear operator X is antilinearly normal, if (1.3) is fulfilled, which is equivalent to

$$M_{X\,X^\sharp} = M_{X^\sharp X}.$$

By Lemma 3.1, we have

$$M_X\overline{M}_{X^\sharp} = M_{X^\sharp}\overline{M}_X$$

and

$$M_X M_X^* = \overline{M^*}_X\overline{M}_X.$$

Remark 3.4. Let $M \in \mathbf{M}_n$ be a conjugate normal matrix and M_u be an unitary matrix. As it was observed in [6, Condition 4.13], the matrix $M_u M M_u^t$ was also conjugate normal. On the other hand, having fixed orthonormal basis, if matrix M is the matrix of some antilinear operator $X \in LA(\mathbb{C}^n)$, i.e., $M = M_X$ and matrix M_u is a matrix of unitary operator $U \in L(\mathbb{C}^n)$, i.e., $M_u = M_U$ then, by Lemma 3.1, $M_u M M_u^t = M_U M_X M_U^t = M_{UXU^*}$ and UXU^* is antilinear normal (see Theorem 3.3, or else Lemma 2.8).

Recall after [5, 6] the following theorem characterizes conjugate normal matrices.

Theorem 3.5. *Let matrix $M \in \mathbf{M}_n$ be conjugate normal. Then there is unitary matrix $M_u \in \mathbf{M}_n$ such that matrix $M_d = M_u M M_u^t$, where M_d is block diagonal matrix with block diagonal matrices $(M_d)'_i$ of size 1×1 and $(M_d)''_j$ of size 2×2 of a form*

$$(M_d)'_i = [r_i], \quad r_i \geqslant 0 \quad \textit{and} \quad (M_d)''_j = \begin{bmatrix} s_j & t_j \\ -t_j & s_j \end{bmatrix}, \quad s_j \geqslant 0,\ t_j \in \mathbb{R}.$$

The consequence of the above is the following characterization of C–normal operators:

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THEOREM 3.6. *Let C be a conjugation in $\mathbb{C}^n$. Let $N \in L(\mathbb{C}^n)$ be a C–normal operator. Then, there is unitary operator $U \in L(\mathbb{C}^n)$ such that*

(1) $N = U^* (DC) (CUC)$, *noticing that* $U^*, DC, CUC \in L(\mathbb{C}^n)$, *or*
(2) $N = (UC)^\sharp (DC) CU$, *noticing that* $(UC)^\sharp, CU \in LA(\mathbb{C}^n)$ *and* $DC \in L(\mathbb{C}^n)$,

where D is block diagonal operator given by block diagonal matrices $(M_d)'_i$ of size 1×1 and $(M_d)''_j$ of size 2×2 of a form

$$(M_d)'_i = [r_i], \quad r_i \geqslant 0 \quad \textit{and} \quad (M_d)''_j = \begin{bmatrix} s_j & t_j \\ -t_j & s_j \end{bmatrix}, \quad s_j \geqslant 0,\ t_j \in \mathbb{R}.$$

Proof. Operator N is C–normal, and thus, NC is antilinearly normal. Let us fix some orthonormal basis in $\mathbb{C}^n$, for example canonical one. Hence, by Theorem 3.3, the matrix M_{NC} of NC is conjugate normal. Now by Theorem 3.5 there is a unitary matrix M_u and specific block diagonal matrix M_d described in Theorem 3.5 such that $M_d = M_u M_{NC} M_u^t$. Let $D \in LA(\mathbb{C}^n)$ be an antilinear operator represented by matrix M_d and $U \in L(\mathbb{C}^n)$ be the unitary operator represented by the matrix M_u. Then, $M_D = M_U M_{NC} M_U^t = M_{UNCU^*}$ by Lemma 3.1. Hence, $D = UNCU^*$ and we get (1). Condition (2) can be proved similarly starting with CN. ∎

4. Case of canonical conjugation in $\mathbb{C}^n$. Let C_{z^n} be a *canonical* conjugation in $\mathbb{C}^n$ given by $C_{z^n}(z_1, \ldots, z_n) = (\bar{z}_n, \bar{z}_{n-1}, \ldots, \bar{z}_1)$. Recall the model spaces defined in Example 2.2. If we consider the inner function $\theta(z) = z^n$ then $\mathbb{C}^n$ can be seen as a model space $\mathbb{C}^n = H^2 \ominus z^n H^2$. Moreover, the conjugation C_{z^n} is exactly the conjugation C_θ with $\theta = z^n$ considered in Example 2.2.

LEMMA 4.1. *Let $T \in L(\mathbb{C}^n)$ and $M_T = [a_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$. Then $M_{C_{z^n} T C_{z^n}} = [\bar{a}_{n-i+1\ \ n-j+1}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$. That means*

$$C_{z^n} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} C_{z^n} = \begin{bmatrix} a_{nn} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{11} \end{bmatrix}.$$

By the second diagonal of the matrix $M = [a_{ij}] \in M_{nn}$ we will mean the set of elements a_{ij} such that $i + j = n + 1$.

THEOREM 4.2. *Let $N \in L(\mathbb{C}^n)$ be C_{z^n}–normal operator. Then, there is a unitary operator $U \in L(\mathbb{C}^n)$ and the operator $\tilde{D} \in L(\mathbb{C}^n)$ having a matrix representation concentrated on the second diagonal given by block diagonal matrices $(M'_d)_i$ of the size 1×1 and $(M''_d)_j$ of the size 2×2 of the form $(M'_d)_i = [r_i]$, $r_i \geqslant 0$ and $(M''_d)_j = \begin{bmatrix} t_j & s_j \\ s_j & -t_j \end{bmatrix}$, $s_j \geqslant 0$, $t_j \in \mathbb{R}$ such that*

(1) $N = U\, \tilde{D}\, (C_{z^n} U^* C_{z^n})$, *which can be written using matrix representation as,*
(2) $M_N = M_U\, M_{\tilde{D}}\, (M_U^s)^t$.

Proof. By Theorem 3.6 (1) there is a unitary operator $U \in L(\mathbb{C}^n)$ and decomposition such that $N = U\,(DC_{z^n})\,(C_{z^n} U^* C_{z^n})$ where $DC_{z^n} \in L(\mathbb{C}^n)$ $(C_{z^n} U^* C_{z^n}) \in L(\mathbb{C}^n)$. Define $\tilde{D} = DC_{z^n} \in L(\mathbb{C}^n)$ and applying Lemma 3.1 the operator $\tilde{D}$ has got a suitable representation. Hence, we get (1). Applying Lemma 4.1, we obtain (2). ∎

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EXAMPLE 4.3. For $n = 3$, having a canonical conjugation $C_{z^3}(z_1, z_2, z_3) = (\bar{z}_3, \bar{z}_2, \bar{z}_1)$, all C_{z^3}–normal operators have the matrix representation $M_U M_{\tilde{D}}(M_U^s)^t$, where M_U is any unitary matrix and $M_{\tilde{D}} = \begin{bmatrix} 0 & 0 & r \\ t & s & 0 \\ s & -t & 0 \end{bmatrix}$, $r \geqslant 0$, $s \geqslant 0$, $t \in \mathbb{R}$ or $M_{\tilde{D}} = \begin{bmatrix} 0 & 0 & r_1 \\ 0 & r_2 & 0 \\ r_3 & 0 & 0 \end{bmatrix}$, $r_1, r_2, r_3 \geqslant 0$.

5. C–normal operators on L^2 spaces. Now, we would like to find examples of C–normal operators in L^2 spaces. Direct calculation shows the following:

PROPOSITION 5.1. *Let (X, μ) be a measure space. Let $L^2(X, \mu)$ be a space of complex valued functions with conjugation C given by $Cf(x) = \overline{f(x)}$. Let $\varphi \in L^\infty$ and M_φ be a multiplication operator on $L^2(X, \mu)$, $M_\varphi f = \varphi f$. Then M_φ is C–symmetric, thus also C–normal.*

Recall that any normal operator $N \in L(\mathcal{H})$ is unitary equivalent to the multiplication operator M_φ, i.e., $M_\varphi = UNU^*$, where $U \in L(\mathcal{H}, L^2(X, \mu))$ is unitary. Let C be a conjugation in H such that $(UCU^*)f(x) = \overline{f(x)}$. Then N is C–normal. On the other hand, we have the following

EXAMPLE 5.2. Consider $L^2[0, 1]$. A conjugation C on $L^2[0, 1]$ is given by $(Cf)(t) = \overline{f(1-t)}$, $t \in [0, 1]$. Let $\varphi \in L^\infty$ and consider $M_\varphi \in L(L^2[0, 1])$, $M_\varphi f = \varphi f$. It turns out, that operator M_φ is C–normal if and only if $|\varphi|^2(t) = |\varphi|^2(1-t)$.

PROPOSITION 5.3. *Let $M_\varphi \in L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})\, dx)$ and $\varphi \in L^\infty$. Let conjugation C be given by $Cf(x) = \overline{f(-x)}$. It turns out, that the operator M_φ is C–normal if and only if $|\varphi|^2$ is an even mapping.*

6. C–normal Toeplitz operators on Hardy spaces. In the following section, we would like to characterize C–symmetric, C–skew–symmetric, C–normal operators in the Hardy space H^2. Recall that $L^2 = L^2(\mathbb{T}, m)$ and the Hardy space H^2 is its subspace of those elements of L^2 which have negative Fourier coefficient equal to 0. Now, we will consider Toeplitz operators. Let $\varphi \in L^\infty = L^\infty(\mathbb{T}, m)$ and define the *Toeplitz operator with symbol* φ as

$$T_\varphi f = P_{H^2}(\varphi f).$$

Note also after [2, Theorem 9] that conditions for a Toeplitz operator to be selfadjoint (i.e., a symbol have to be real) or to be normal (i.e., a symbol have to be linear function of a real function) are very restrictive. In the following section, we will show that, the classes of C–symmetric, C–skew–symmetric, C–normal operators Toeplitz operators are much more wider. In fact, we fully characterize these classes of operators with respect to some natural conjugations.

First natural conjugation (see [15, p. 103]) which can be studied is given by

$$(C_0 f)(z) = \overline{f(\bar{z})} \quad \text{for } f \in H^2. \tag{6.6}$$

In [11], for a given real ξ, θ, there was also considered more general conjugation given by

$$(C_{\xi,\theta} f)(z) = e^{i\xi} \cdot \overline{f(e^{i\theta}\, \bar{z})}. \tag{6.7}$$

The Hardy space has the natural basis $e_k(z) = z^k$, $k = 0, 1, \ldots$ Note that $C_{\xi,\theta} e_k = e^{i\xi} \cdot e^{-ik\theta} e_k$, $k \in \mathbb{Z}_+$.

LEMMA 6.1. *Let $C_{\xi,\theta}$, $\xi, \theta \in \mathbb{R}$, be a conjugation on H^2 given by $(C_{\xi,\theta} f)(z) = e^{i\xi} \cdot \overline{f(e^{i\theta}\, \bar{z})}$. Let an operator $T \in L(H^2)$ be given by a matrix $[a_{lk}]_{k,l \geqslant 0}$ as to the basis $\{e_k\}_{k \in \mathbb{Z}_+}$, i.e $a_{lk} = \langle Te_k, e_l \rangle$. Then*

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(1) *the operator* $C_{\xi,\theta}TC_{\xi,\theta}$ *has a matrix* $[b_{lk}]_{k,l\geqslant 0}$, $b_{lk}=e^{i(k-l)\theta}\,\bar{a}_{lk}$,
(2) *the operator* T *is* $C_{\xi,\theta}$*–symmetric if and only if* $a_{lk}=e^{i(k-l)\theta}a_{kl}$, $k,l\geqslant 0$*; in particular* a_{ll} *are arbitrary,*
(3) *the operator* T *is* $C_{\xi,\theta}$*–skew–symmetric if and only if* $a_{lk}=-e^{i(k-l)\theta}a_{kl}$, $k,l\in\mathbb{Z}_+$*; in particular* $a_{ll}=0$.

Proof. To see (1), let us compute

$$\begin{aligned}b_{lk}&=\langle C_{\xi,\theta}TC_{\xi,\theta}e_k,e_l\rangle=\langle C_{\xi,\theta}e_l,TC_{\xi,\theta}e_k\rangle=\overline{\langle TC_{\xi,\theta}e_k,C_{\xi,\theta}e_l\rangle}\\&=\overline{\langle Te^{i\xi}e^{-ik\theta}e_k,e^{i\xi}e^{-il\theta}e_l\rangle}=e^{i(k-l)\theta}\,\overline{\langle Te_k,e_l\rangle}=e^{i(k-l)\theta}\,\bar{a}_{lk}.\end{aligned}$$

Conditions (2) and (3) follows from (1) and appropriate definitions. ◻

Corollary 6.2. *Let* C_0 *be a conjugation on* H^2 *given by* $(C_0f)z=\overline{f(\bar{z})}$, $f\in H^2$*. Let* $T\in L(H^2)$ *be given by the matrix* $[a_{kl}]_{k,l\geqslant 0}$ *according to the basis* $\{e_k\}_{k\in\mathbb{Z}_+}$*. Then,* T *is* C_0*–symmetric if and only if* $a_{kl}=a_{lk}$, $k,l=0,1,2,\ldots$, *and* T *is* C_0*–skew–symmetric if and only if* $a_{ll}=0$, $a_{kl}=-\,a_{lk}$, $k,l=0,1,2,\ldots,$

Proposition 6.3. *Let* $\varphi\in L^\infty$ *have a Fourier expansion* $\varphi(z)=\sum_{-\infty}^{+\infty}\widehat{\varphi}(n)z^n$*. The Toeplitz operator* T_φ *has the matrix* $[a_{lk}]_{k,l=0,1,2,\ldots}$ *and* $a_{lk}=\widehat{\varphi}(l-k)$*. Then:*

(1) *the operator* $C_{\xi,\theta}T_\varphi C_{\xi,\theta}$ *has matrix* $[b_{lk}]$ *with* $b_{lk}=e^{i(k-l)\theta}\overline{\widehat{\varphi}(l-k)}$,
(2) *the Toeplitz operator* T_φ *is* $C_{\xi,\theta}$*–symmetric if and only if* $\widehat{\varphi}(-k)=e^{ik\theta}\widehat{\varphi}(k)$, $k\in\mathbb{Z}$*; in particular* $\widehat{\varphi}(0)$ *is arbitrary,*
(3) *the operator* T_φ *is* $C_{\xi,\theta}$*–skew–symmetric if and only if* $\widehat{\varphi}(-k)=-e^{ik\theta}\widehat{\varphi}(k)$, $k\in\mathbb{Z}$*; in particular* $\widehat{\varphi}(0)=0$ *if* $Arg\theta\neq\pi$ *and* $\widehat{\varphi}(0)$ *is arbitrary if* $Arg\,\theta=\pi$.

Proposition 6.4. *Let* $C_{\xi,\theta}$, $\xi,\theta\in\mathbb{R}$*, be a conjugation on* H^2 *given by* $(C_{\xi,\theta}f)(z)=e^{i\xi}\overline{f(e^{i\theta}\bar{z})}$*. Let* $\varphi\in L^\infty$, $\varphi(z)=\sum_{n=-\infty}^{+\infty}\widehat{\varphi}(n)z^n$ *and denote* $\varphi_+(z)=\sum_{n=1}^{+\infty}\widehat{\varphi}(n)z^n$, $\varphi_-(z)=\sum_{n=-\infty}^{-1}\widehat{\varphi}(n)z^n$*. If* T_φ *is* $C_{\xi,\theta}$*–normal then there is* η, $|\eta|=1$ *such that*

$$\widehat{\varphi}(-k)=\eta e^{ik\theta}\widehat{\varphi}(k)\quad\textit{for}\ \ k=1,2,\ldots,\tag{6.8}$$

or equivalently, there is η, $|\eta|=1$ *such that*

$$\varphi_-=\eta\;e^{i\xi}\;\overline{C_{\xi,\theta}\varphi_+}.\tag{6.9}$$

Remark 6.5. Let us consider $\varphi,\psi\in L^\infty$ with the Fourier expansion $\varphi(z)=\sum_{n=-\infty}^{+\infty}\widehat{\varphi}(n)z^n$ and $\psi(z)=\sum_{n=-\infty}^{+\infty}\widehat{\psi}(n)z^n$, respectively. Let T_φ,T_ψ be Toeplitz operators on H^2. The operator $T_\varphi T_\psi$ is not always a Toeplitz operator. In fact, as it was shown in [2] that

$$\langle T_\varphi T_\psi e_{k+1},e_{l+1}\rangle-\langle T_\varphi T_\psi e_k,e_l\rangle=\widehat{\varphi}(l+1)\;\widehat{\psi}(-k-1).\tag{6.10}$$

Proof of Proposition 6.4. Applying Remark 6.5, we have

$$\begin{aligned}\langle(S^*T_{\bar{\varphi}}T_\varphi S-T_{\bar{\varphi}}T_\varphi)e_k,e_l\rangle&=\langle T_{\bar{\varphi}}T_\varphi Se_k,Se_l\rangle-\langle T_{\bar{\varphi}}T_\varphi e_k,e_l\rangle\\&=\langle T_{\bar{\varphi}}T_\varphi e_{k+1},e_{l+1}\rangle-\langle T_{\bar{\varphi}}T_\varphi e_k,e_l\rangle=\overline{\widehat{\varphi}(-l-1)}\;\widehat{\varphi}(-k-1).\end{aligned}\tag{6.11}$$

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On the other hand, also using Lemma 6.1 and Remark 6.5, we get

$$\begin{aligned}\langle (S^* C_{\xi,\theta} T_\varphi T_{\bar\varphi} C_{\xi,\theta} S - C_{\xi,\theta} T_\varphi T_{\bar\varphi} C_{\xi,\theta}) e_k, e_l\rangle &= \langle C_{\xi,\theta} T_\varphi T_{\bar\varphi} C_{\xi,\theta} e_{k+1}, e_{l+1}\rangle - \langle C_{\xi,\theta} T_\varphi T_{\bar\varphi} C_{\xi,\theta} e_k, e_l\rangle \\ &= e^{i(k-l)\theta}\, \overline{\langle T_\varphi T_{\bar\varphi} e_{k+1}, e_{l+1}\rangle} - e^{i(k-l)\theta}\, \overline{\langle T_\varphi T_{\bar\varphi} e_k, e_l\rangle} \\ &= e^{i(k-l)\theta}\, \overline{\widehat{\varphi}(l+1)}\, \widehat{\varphi}(k+1).\end{aligned}$$

The last equality follows from (6.11) for $T_\varphi T_{\bar\varphi}$. If T_φ is $C_{\xi,\theta}$–normal, by Theorem 2.3 (5), subtracting both sides we get

$$e^{i(k-l)\theta}\, \overline{\widehat{\varphi}(l+1)}\, \widehat{\varphi}(k+1) = \overline{\widehat{\varphi}(-l-1)}\, \widehat{\varphi}(-k-1) \tag{6.12}$$

for $k, l = 0, 1, 2, \ldots$

Assume for the while that $\widehat{\varphi}(k) \neq 0$, $k = \pm 1, \pm 2, \ldots$ Thus,

$$\overline{\left(\frac{\widehat{\varphi}(-l)}{e^{il\theta}\ \widehat{\varphi}(l)}\right)} = \left(\frac{\widehat{\varphi}(-k)}{e^{ik\varphi}\ \widehat{\varphi}(k)}\right)^{-1} \tag{6.13}$$

for $k, l = 1, 2, \ldots$ Hence, there is η such that $\frac{\widehat{\varphi}(-k)}{e^{ik\varphi}\widehat{\varphi}(k)} = \eta$ for $k = 1, 2, \ldots$ Moreover, by (6.13), we get $|\eta| = 1$. Thus,

$$\widehat{\varphi}(-k) = \eta\, e^{ik\theta}\, \widehat{\varphi}(k) \quad \text{for} \ \ k = 1, 2, \ldots \tag{6.14}$$

If $\widehat{\varphi}(k) = 0$ and (6.14) is fulfilled, then $\widehat{\varphi}(-k) = 0$ and (6.12) holds. ∎

THEOREM 6.6. *Let $C_{\xi,\theta}$, $\xi, \theta \in \mathbb{R}$, be a conjugation on H^2 given by $(C_{\xi,\theta} f)(z) = e^{i\xi}\overline{f(e^{i\theta}\bar z)}$. Let $\varphi \in L^\infty$, $\varphi(z) = \sum_{n=-\infty}^{+\infty} \widehat{\varphi}(n) z^n$ and denote $\varphi_+(z) = \sum_{n=1}^{+\infty} \widehat{\varphi}(n) z^n$, $\varphi_-(z) = \sum_{n=-\infty}^{-1} \widehat{\varphi}(n) z^n$. Then T_φ is $C_{\xi,\theta}$–normal if and only if there is η, $|\eta| = 1$ such that*

$$\varphi_- = \eta\, e^{i\xi}\, \overline{C_{\xi,\theta}\varphi_+} \qquad \textit{and} \tag{6.15}$$

$$(\eta - \bar\eta)\varphi_+\, C_{\xi,\theta}\varphi_+ + \overline{\widehat{\varphi}(0)}(\eta - 1) e^{i\xi} \varphi_+ - \widehat{\varphi}(0)(\bar\eta - 1) C_{\xi,\theta}\varphi_+ = 0. \tag{6.16}$$

Denote by $\varphi^\theta_\sim(z) = e^{-i\xi}\, C_{\xi,\theta}\varphi_+(z) = \overline{\varphi_+(e^{i\theta}\bar z)}$. Easy to see that $\overline{\varphi^\theta_\sim} = \bar\varphi^\theta_\sim$.

LEMMA 6.7. *With the notation above, the following hold:*

(1) $C_{\xi,\theta} T_{\varphi_+} C_{\xi,\theta} = T_{\varphi^\theta_\sim}$,
(2) $C_{\xi,\theta} T_{\bar\varphi_+} C_{\xi,\theta} = T_{\bar\varphi^\theta_\sim}$,
(3) $C_{\xi,\theta} T_{\varphi^\theta_\sim} C_{\xi,\theta} = T_{\varphi_+}$,
(4) $C_{\xi,\theta} T_{\bar\varphi^\theta_\sim} C_{\xi,\theta} = T_{\bar\varphi_+}$.

Proof. To see (1), let us calculate for $f, g \in H^2$:

$$\begin{aligned}\langle C_{\xi,\theta} T_{\varphi_+} C_{\xi,\theta} f, g\rangle &= \langle C_{\xi,\theta} g, T_{\varphi_+} C_{\xi,\theta} f\rangle = \langle C_{\xi,\theta} g, P_{H^2} M_{\varphi_+} C_{\xi,\theta} f\rangle \\ &= \langle C_{\xi,\theta} g, M_{\varphi_+} C_{\xi,\theta} f\rangle = \int e^{i\xi}\, \overline{g(e^{i\theta}\bar z)}\ \overline{\varphi_+(z)\, e^{i\xi}\, \overline{f(e^{i\theta}\bar z)}}\, dm(z) \\ &= \int \bar\varphi_+(z) f(e^{i\theta}\bar z)\, \overline{g(e^{i\theta}\bar z)}\, dm(z).\end{aligned}$$

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Let us substitute $\omega = e^{i\theta}\bar{z}$. Then $z = e^{i\theta}\bar{\omega}$. Thus,

$$\langle C_{\xi,\theta}T_{\varphi_+}C_{\xi,\theta}f,g\rangle = \int \overline{\varphi_+(e^{i\theta}\bar{\omega})}\ f(\omega)\,\overline{g(\omega)}\,dm(\omega) = \langle T_{\varphi^\theta_\sim}f,g\rangle.$$

Property (3) follows from (1) since $(\varphi^\theta_\sim)^\theta_\sim = \varphi$ and (2), (4) follows from (1) and (3) taking $\bar{\varphi}$ instead of φ. ∎

Proof of Theorem 6.6. Let us apply Proposition 6.4 and by (6.9) operator T_φ being $C_{\xi,\theta}$–normal has to be represented as

$$T_\varphi = T_{\varphi_+} + \widehat{\varphi}(0)I + \eta e^{i\xi}\ T_{\overline{C_{\xi,\theta}\varphi_+}} = T_{\varphi_+} + \widehat{\varphi}(0)I + \eta T_{\bar{\varphi}^\theta_\sim}.$$

Therefore,

$$T^*_\varphi = T_{\bar{\varphi}_+} + \overline{\widehat{\varphi}(0)}I + \bar{\eta}T_{\varphi^\theta_\sim}.$$

Let us calculate:

$$\begin{aligned} T_\varphi T^*_\varphi = {} & T_{\varphi_+}T_{\bar{\varphi}_+} + \overline{\widehat{\varphi}(0)}\,T_{\varphi_+} + \bar{\eta}\,T_{\varphi_+}T_{\varphi^\theta_\sim} + \widehat{\varphi}(0)\,T_{\bar{\varphi}_+} + |\widehat{\varphi}(0)|^2\ I \\ & + \widehat{\varphi}(0)\,\bar{\eta}\,T_{\varphi^\theta_\sim} + \eta T_{\bar{\varphi}^\theta_\sim}T_{\bar{\varphi}_+} + \overline{\widehat{\varphi}(0)}\,\eta\,T_{\bar{\varphi}^\theta_\sim} + |\eta|^2\,T_{\bar{\varphi}^\theta_\sim}T_{\varphi^\theta_\sim}. \end{aligned}$$

Hence, by Lemma 6.7, we get

$$\begin{aligned} C_{\xi,\theta}T_\varphi T^*_\varphi C_{\xi,\theta} = {} & T_{\varphi^\theta_\sim}T_{\bar{\varphi}^\theta_\sim} + \widehat{\varphi}(0)\,T_{\varphi^\theta_\sim} + \eta\,T_{\varphi^\theta_\sim}T_{\varphi_+} + \overline{\widehat{\varphi}(0)}\,T_{\bar{\varphi}^\theta_\sim} + |\widehat{\varphi}(0)|^2\ I \\ & + \overline{\widehat{\varphi}(0)}\eta\,T_{\varphi_+} + \bar{\eta}\,T_{\bar{\varphi}_+}T_{\bar{\varphi}^\theta_\sim} + \widehat{\varphi}(0)\bar{\eta}\,T_{\bar{\varphi}_+} + T_{\bar{\varphi}_+}T_{\varphi_+}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} T^*_\varphi T_\varphi = {} & T_{\bar{\varphi}_+}T_{\varphi_+} + \widehat{\varphi}(0)\,T_{\bar{\varphi}_+} + \eta\,T_{\bar{\varphi}_+}T_{\bar{\varphi}^\theta_\sim} + \overline{\widehat{\varphi}(0)}\,T_{\varphi_+} + |\widehat{\varphi}(0)|^2\,I \\ & + \eta\overline{\widehat{\varphi}(0)}\,T_{\bar{\varphi}^\theta_\sim} + \bar{\eta}\,T_{\varphi^\theta_\sim}T_{\varphi_+} + \bar{\eta}\widehat{\varphi}(0)\,T_{\varphi^\theta_\sim} + T_{\varphi^\theta_\sim}T_{\bar{\varphi}^\theta_\sim}. \end{aligned}$$

Since φ_+ is analytic and $\bar{\varphi}_+$ is coanalytic, by [2], we have the following:

$$\begin{aligned} C_{\xi\theta}T_\varphi T^*_\varphi C_{\xi\theta} - T^*_\varphi T_\varphi = {} & (\eta-\bar{\eta})T_{\varphi^\theta_\sim\varphi_+} + (\bar{\eta}-\eta)T_{\bar{\varphi}_+\bar{\varphi}^\theta_\sim} \\ & + (\widehat{\varphi}(0) - \bar{\eta}\widehat{\varphi}(0))T_{\varphi^\theta_\sim} + (\overline{\widehat{\varphi}(0)} - \eta\overline{\widehat{\varphi}(0)})T_{\bar{\varphi}^\theta_\sim} \\ & + (\overline{\widehat{\varphi}(0)}\eta - \overline{\widehat{\varphi}(0)})T_{\varphi_+} + (\widehat{\varphi}(0)\bar{\eta} - \widehat{\varphi}(0))T_{\bar{\varphi}_+}. \end{aligned} \tag{6.17}$$

The condition for the operator T_φ to be $C_{\xi,\theta}$–normal is that the operator above has to be zero. In fact the operator above is a Toeplitz one with the symbol (let say) $\psi \in L^\infty \subset L^2$. Thus, the symbol ψ has to be a zero. Hence, the analytic and co-analytic part, which are complex adjoint one to the other, of ψ have to be 0. Extracting the analytical part of the function ψ we get:

$$\begin{aligned} 0 & = (\eta-\bar{\eta})\varphi_+\varphi^\theta_\sim + \overline{\widehat{\varphi}(0)}\,(\eta-1)\varphi_+ + \widehat{\varphi}(0)(1-\bar{\eta})\varphi^\theta_\sim \\ & = (\eta-\bar{\eta})e^{-i\xi}\varphi_+\,C_{\xi,\theta}\varphi_+ + \overline{\widehat{\varphi}(0)}(\eta-1)\varphi_+ - \widehat{\varphi}(0)(\bar{\eta}-1)e^{-i\xi}C_{\xi,\theta}\varphi_+. \end{aligned}$$

Hence, we get (6.16).

Arguing the other direction, if (6.15) and (6.16) are fulfilled the operator considered in (6.17) have to be zero. ∎

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EXAMPLE 6.8. If, in Theorem 6.6, the existing η is real, then we have the following cases:

(1) Let $\eta = 1$ then (6.16) is fulfilled and (6.15) means that operator T_φ is $C_{\xi,\theta}$–symmetric, see Lemma 6.3, (2).
(2) Let $\eta = -1$ and $\widehat{\varphi}(0) = 0$ then (6.16) is fulfilled and (6.15) with $\widehat{\varphi}(0) = 0$ means that operator T_φ is $C_{\xi,\theta}$-skew–symmetric, see Lemma 6.3, (3).
(3) For $\eta = -1$, $\widehat{\varphi}(0) \neq 0$, $Arg\theta \neq \pi$, condition (6.16) is equivalent to

$$\overline{\widehat{\varphi}(0)}\,\varphi_+ = \widehat{\varphi}(0)\,e^{-i\xi}\,C_{\xi,\theta}\varphi_+ = \widehat{\varphi}(0)\varphi_\sim^\theta. \tag{6.18}$$

Hence, in this case, the operator T_φ is $C_{\xi,\theta}$–normal (but neither $C_{\xi,\theta}$–symmetric nor $C_{\xi,\theta}$–skew–symmetric) for $\varphi \in L^\infty$ if

$$\begin{aligned} &\hat{\varphi}(-k) = -e^{ik\theta}\hat{\varphi}(k) \quad \text{for} \quad k = 1, 2, \ldots, \quad \text{and} \\ &Arg\hat{\varphi}(k) \overset{\mathrm{mod}\, 2\pi}{=} Arg\hat{\varphi}(0) - \tfrac{k}{2}\theta \quad \text{for} \quad k = 1, 2, \ldots \end{aligned}$$

It is worth to notice the special case of Theorem 6.6.

COROLLARY 6.9. *Let C_0, be a conjugation on H^2 given by $(C_0 f)(z) = \overline{f(\bar{z})}$ for $f \in H^2$. Let $\varphi \in L^\infty$ and $\varphi = \varphi_- + \widehat{\varphi}(0) + \varphi_+$. Then, the Toeplitz operator T_φ is C_0–normal if and only if there is η, $|\eta| = 1$ such that*

(1) $\varphi_- = \eta\overline{C_0\varphi_+}$, *and*
(2) $(\eta - \bar{\eta})\varphi_+\,C_0\varphi_+ + \overline{\widehat{\varphi}(0)}(\eta - 1)\varphi_+ - \widehat{\varphi}(0)(\bar{\eta} - 1)C_0\varphi_+ = 0.$

EXAMPLE 6.10. Let $s \in (-1;1)$ and let $\varphi(z) = \frac{-s\bar{z}}{1-is\bar{z}} + (\frac{1}{2} + \frac{1}{2}i) + \frac{isz}{1-isz}$. Conditions (1) and (2) of Corollary are fulfilled for $\eta = i$. Thus, T_φ is C_0–normal but neither C_0–symmetric nor C_0–skew–symmetric by Lemma 6.3.

7. Composition operators. Let (X, Σ, μ) be a measure space with a non–negative σ-finite measure μ and consider a space $L^2(X, \Sigma, \mu)$. Then a measurable function $T\colon X \to X$ induces a composition operator $C_T f = f \circ T$. It is known [18] that if C_T is bounded then $\mu \circ T^{-1}$ is absoluty continuous with respect to μ and the Radon-Nikodym derivative $h = \frac{d\mu \circ T^{-1}}{d\mu}$ is essentially bounded. Conversely, if T satisfies this conditions, function T induce bounded linear operator C_T on $L^2(X, \Sigma, \mu)$. It is clear that h is always nonnegative. Note also the basic formula

$$\int C_T f\, d\mu = \int f \circ T\, d\mu = \int f h\, d\mu. \tag{7.19}$$

PROPOSITION 7.1. *Take the conjugation C in $L^2(X, \Sigma, \mu)$ given by $C(f)(x) = \overline{f(x)}$. Assume that C_T is a bounded composition operator given by a measurable function $T\colon X \to X$. Then following are equivalent:*

(1) *C_T is C–normal,*
(2) *C_T is normal.*

Proof. To show equivalence of (1) to (2), we will show that $CC_T^*C_TC = C_T^*C_T$. Let $f, g \in L^2(X, \Sigma, \mu)$ then

$$\begin{aligned} \langle CC_T^*C_TCf, g\rangle = \langle Cg, C_T^*C_TCf\rangle \;&=\; \langle C_TCg, C_TCf\rangle \;=\; \int (Cg \circ T) \cdot \overline{Cf \circ T}\, d\mu \\ &= \int (\bar{g} \circ T)\; (f \circ T)\, d\mu \;=\; \int \bar{g} f\, h\, d\mu = \langle C_T f, C_T g\rangle \;=\; \langle C_T^*C_T f, g\rangle. \end{aligned}$$

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Let us note that $(Cf)x = \overline{f(-x)}$ gives us a conjugation in $L^2(\mathbb{R}, m)$, (m Lebesgue measure). On the other hand, $(Cf)x = \overline{f(1-x)}$ defines a conjugation on the space $L^2([0,1], m)$. Consider the general space $L^2(X, \mu)$, where (X, μ) is a measure space with non-negative measure μ. The above two situations lead to the following:

Proposition 7.2. *Let (X, Σ, μ) be a measure space with a non–negative measure μ and the antilinear operator $C\colon L^2(X, \Sigma, \mu) \to L^2(X, \Sigma, \mu)$ given by $(Cf)(x) = \overline{f(\alpha(x))}$, where $\alpha\colon X \to X$ is measurable. Then, C is conjugation if and only if*

(1) $\alpha^2 = I_X$,
(2) $\mu = \mu \circ \alpha$.

Proof. For $f \in L^2(X, \Sigma, \mu)$ and $x \in X$, we have

$$(C^2 f)(x) = C(Cf)(x) = \overline{Cf(\alpha(x))} = f(\alpha^2(x)).$$

Hence, $C^2 = I$ is equivalent to $\alpha^2 = I_X$. For the second condition, for any $f, g \in L^2(X, \Sigma, \mu)$, let us calculate

$$\langle Cf, Cg\rangle = \int (Cf)(x)\overline{(Cg)(x)}d\mu(x) = \int \overline{f(\alpha(x))} \cdot g(\alpha(x))d\mu(x)$$

and

$$\langle g, f\rangle = \int g(x)\overline{f(x)}d\mu(x).$$

Hence, the equality of two above for all f, g gives $\mu = \mu \circ \alpha^{-1} = \mu \circ \alpha$. ◻

Theorem 7.3. *Let $L^2(X, \Sigma, \mu)$ with conjugation C given by $(Cf)(x) = \overline{f(\alpha(x))}$, i.e., $\alpha\colon X \to X$ be measurable function with $\alpha^2 = I_X$ and $\mu = \mu \circ \alpha$. Assume that C_T is a bounded composition operator given by a measurable function $T\colon X \to X$. Then, the operator C_T is C–normal if and only if*

(1) *$T^{-1}(\Sigma)$ is essentially all Σ, i.e., for a given $\omega \in \Sigma$, there is $\tilde\omega \in \Sigma$ such that $m\big((T^{-1}(\tilde\omega) \setminus \omega) \cup (\omega \setminus T^{-1}(\tilde\omega))\big) = 0$, and*
(2) *$h \circ T = h \circ \alpha$ μ a.e., where $h = \frac{d\mu\circ T^{-1}}{d\mu}$.*

Proof. For $f, g \in L^2(X, \mu)$, we have

$$\begin{aligned}\langle CC_T^* C_T Cf, g\rangle = \langle C_T Cg, C_T Cf\rangle &= \int (Cg \circ T)\, \overline{(Cf \circ T)}\, d\mu \\ &= \int (\bar g \circ \alpha \circ T)\ (f \circ \alpha \circ T)\, d\mu \\ &= \int (\bar g \circ \alpha)\ (f \circ \alpha)\ h\, d\mu = \int f\, \bar g\ (h \circ \alpha^{-1})\, d\mu \circ \alpha^{-1}.\end{aligned}$$

Then, since $\alpha = \alpha^{-1}$,

$$CC_T^* C_T Cf = (h \circ \alpha^{-1}) \cdot f.$$

If f belongs to range of C_T, then $f = C_T f_0$ and

$$\begin{aligned}C_T C_T^* f = C_T C_T^* C_T f_0 &= C_T CCC_T^* C_T CC f_0 \\ &= C_T C\ (CC_T^* C_T C)\ (Cf_0) = C_T C((h \circ \alpha) \cdot (Cf_0)) \\ &= C_T((\bar h \circ \alpha \circ \alpha) \cdot C(Cf_0)) = C_T(h \cdot f_0) \\ &= (h \circ T) \cdot (C_T f_0) = (h \circ T) \cdot f.\end{aligned}$$

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If C_T is C–normal, then

$$(h \circ \alpha) f = (h \circ T) f$$

for all f in range of C_T. The rest of the proof is analogous as the proof of [18, Lemma 2]. $\square$

EXAMPLE 7.4. Let us consider $L^2(\mathbb{R}, m)$ with the conjugation $(Cf)x = \overline{f(-x)}$, $\alpha(x) = -x$. Let $T(x) = -x$ for $x \geqslant 0$ and $T(x) = -2x$ for $x < 0$. Then the Radon–Nikodym derivative $h = \frac{dm \circ T^{-1}}{dm}$ is given by $h(x) = \frac{1}{2}$ for $x \geqslant 0$ and $h(x) = 1$ for $x < 0$. It is clear that $h \circ \alpha = h \circ T$, and thus, C_T is C–normal. Furthermore, $h \neq h \circ T$, and thus, C_T is not normal (see [18, Lemma 2]) and direct calculation shows that it is also always neither C–symmetric nor C–skew–symmetric.

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