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SYMMETRY OF CYCLIC WEIGHTED SHIFT MATRICES WITH PIVOT-REVERSIBLE WEIGHTS*

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Abstract. It is proved that every cyclic weighted shift matrix with pivot-reversible weights is unitarily similar to a complex symmetric matrix.

Key words. Cyclic weighted shift, Pivot-reversible weights, Symmetric matrices, Numerical range.

AMS subject classifications. 15A60, 15B99, 47B37.

1. Introduction.

Let A be an $n \times n$ complex matrix. The numerical range of A is defined and denoted by

$$W(A) = \{\xi^* A \xi : \xi \in \mathbb{C}^n,\, \xi^*\xi = 1\}.$$

Toeplitz [16] and Hausdorff [9] firstly introduced this set, and proved the fundamental convex theorem of the numerical range (cf. [14]). The numerical range and its related subjects have been extensively studied. From the viewpoint of algebraic curve theory, Kippenhahn [12] characterized that $W(A)$ is the convex hull of the real affine part of the dual curve of the curve $F_A(x, y, z) = 0$, where $F_A(x, y, z)$ is the homogeneous polynomial associated with A defined by

$$F_A(x, y, z) = \det(zI_n + x\Re(A) + y\Im(A)),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/2i$. Fiedler [6] conjectured the inverse problem that there exists a pair of $n \times n$ Hermitian matrices H, K satisfying

$$F(x, y, z) = \det(zI_n + xH + yK),$$

whenever $F(x, y, z)$ is a homogeneous polynomial of degree n for which the equation $F(-\cos\theta, -\sin\theta, z) = 0$ in z has n real roots for any angle $0 \le \theta \le 2\pi$. Helton and Vinnikov[10] proved that this conjecture is true and such pair H, K can be obtained by real symmetric matrices. In other words, $F(x, y, z) = F_{H+iK}(x, y, z)$, where the representation matrix $H + iK$ is a complex symmetric matrix. The result provides an intensive interest on symmetric matrices in this direction. Garcia et al. [7] and references therein investigated conditions for matrices unitarily similar to complex symmetric matrices. Some classes of matrices are known being unitarily similar to complex symmetric matrices, such as Toeplitz matrices [1], unitary boarding matrices [4], and cyclic weighted matrices with reversible positive weights $b_1, b_2, \ldots, b_2, b_1$ [11]. As a non-normal analogue of unitary matrices, cyclic weighted shift matrices and their numerical ranges have been studied by a number of authors in the past few years (cf. [2, 3, 5, 8, 13, 17]). A cyclic weighted shift matrix

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with weights $a_1, a_2, \ldots, a_n$ is an $n \times n$ matrix of the following form

$$\begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \ddots & a_{n-1} \\ a_n & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

which is denoted by $S = S(a_1, a_2, \ldots, a_n)$.

In this paper, we obtain a new class of matrices which are unitarily similar to symmetric matrices, namely, the class of cyclic weighted shift matrices with pivot-reversible positive weights. The definition of pivot-reversible weights is defined in Section 2.

2. Reversible weights with pivots. Let $S = S(a_1, a_2, \ldots, a_n)$ be a cyclic weighted shift matrix. It is easy to see that $S(a_1, a_2, \ldots, a_n)$ is unitarily similar to $e^{i\phi} S(|a_1|, |a_2|, \ldots, |a_n|)$ via a unitary diagonal matrix for some angle $\phi \in [0, 2\pi)$. Moreover, the cyclic weighted shift $S(a_1, a_2, \ldots, a_n)$ is unitarily similar to $S(a_n, a_1, a_2, \ldots, a_{n-1})$ which is clockwise rotating its weights (cf.[8]). If $a_k = 0$ for some $1 \leq k \leq n$, then by clockwise rotating its weights, $S(a_1, a_2, \ldots, a_n)$ is unitarily similar to the upper triangular weighted shift matrix $S(a_{k+1}, \ldots, a_n, a_1, \ldots, a_{k-1}, 0)$. In this case, the numerical range of S is a circular disc (cf.[2, 17]). Hence, in the following of this paper, we may assume that the weights of a cyclic weighted shift matrix are positive. A cyclic weighted shift $S(a_1, a_2, \ldots, a_n)$ is called reversible if its weights are ordered by $a_1, a_2, \ldots, a_m, a_{m+1}, a_m, \ldots, a_2, a_1$ when $n = 2m + 1$ is odd, and $a_1, a_2, \ldots, a_m, a_m, \ldots, a_2, a_1$ if $n = 2m$ is even. Symmetry of cyclic weighted shift matrices with reversible weights is obtained in [11], and the determinantal representation of their ternary forms is studied in [5].

We introduce a new type of cyclic weighted shifts, namely, the class of cyclic weighted shift matrices with pivot-reversible positive weights. A matrix of even size $n = 2m$ is a cyclic weighted shift matrix with two-pivot-reversible weights if its weights are ordered as $a_1, a_2, \ldots, a_m, a_{m+1}, a_m, a_{m-1}, \ldots, a_2$, the two pivots are a_1 and a_{m+1}. A matrix of odd size $n = 2m - 1$ is a cyclic weighted shift matrix with one-pivot-reversible weights if its weights are ordered as $a_1, a_2, \ldots, a_m, a_m, a_{m-1}, \ldots, a_2$, the one pivot is a_1. Figure 1 displays the graph of a reversible cyclic weighted shift with two pivots a_1 and a_6 for even size $n = 10$. Figure 2 shows the odd size $n = 9$ with one pivot a_1.

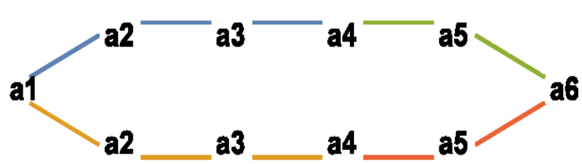

Figure 1. Reversible weights with two pivots a_1 and a_6.

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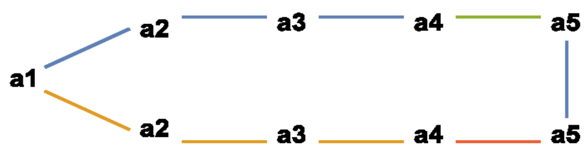

Figure 2. Reversible weights with one pivot a_1.

Assume that A is a 2×2 complex matrix. Then there is a unitary matrix U and and an angle θ such that

$$U\Re(A)U^* = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \quad \text{and} \quad U\Im(A)U^* = \begin{pmatrix}\mu_1 & \nu e^{i\theta}\\ \nu e^{-i\theta} & \mu_2\end{pmatrix}$$

for some real numbers $\lambda_1,\lambda_2,\mu_1,\mu_2,\nu$, and thus,

$$\begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}(UAU^*)\begin{pmatrix}1 & 0\\ 0 & e^{-i\theta}\end{pmatrix} = \begin{pmatrix}\lambda_1+i\mu_1 & i\nu\\ i\nu & \lambda_2+i\mu_2\end{pmatrix}$$

is a complex symmetric matrix. Hence, in the following of this paper we assume the size of a matrix $n\geq 3$.

Theorem 2.1. *Every cyclic weighted shift matrix with pivot-reversible positive weights is unitarily similar to a complex symmetric matrix.*

Proof. Let S be an $n\times n$ cyclic weighted shift matrices with pivot-reversible positive weights. Suppose $n=2m-1$ is odd. Then $S=S(a_1,a_2,\dots,a_{m-1},a_m,a_m,a_{m-1},\dots,a_2)$. By the cycling property [8], $S(a_1,a_2,\dots,a_m,a_m,a_{m-1},\dots,a_2)$ is unitarily similar to $S(a_m,\dots,a_2,a_1,a_2,\dots,a_m)$. According to the result in [11], the type of reversible matrix $S(a_m,\dots,a_2,a_1,a_2,\dots,a_m)$ is unitarily similar to a complex symmetric matrix.

Suppose $n=2m$ is even. Then $S=S(a_1,a_2,\dots,a_m,a_{m+1},a_m,\dots,a_2)$. Denote two Hermitian matrices

$$H = 2\Re(S(a_1,a_2,\dots,a_m,a_{m+1},a_m,\dots,a_2))$$

and

$$K = 2\Im(S(a_1,a_2,\dots,a_m,a_{m+1},a_m,\dots,a_2)).$$

Then

$$S(a_1,a_2,\dots,a_m,a_{m+1},a_m,\dots,a_2) = \frac{H}{2}+i\frac{K}{2}.$$

Let

$$f_1 = \frac{1}{\sqrt{2}}(1,1,\underbrace{0,\dots,0}_{n-2})^T,$$

$$f_k = \frac{1}{\sqrt{2}}(\underbrace{0,\dots,0}_{k},1,\underbrace{0,\dots,0}_{n-2k},1,\underbrace{0,\dots,0}_{k-2})^T, \quad 2\leq k\leq m,$$

and

$$g_1 = \frac{1}{\sqrt{2}}(1,-1,\underbrace{0,\dots,0}_{n-2})^T,$$

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$$g_k = \frac{1}{\sqrt{2}}(\underbrace{0,\ldots,0}_{k}, (-1)^k, \underbrace{0,\ldots,0}_{n-2k}, (-1)^{k+1}, \underbrace{0,\ldots,0}_{k-2})^T, \quad 2 \le k \le m.$$

Then, $\{f, \ldots, f_m, g_1, \ldots, g_m\}$ is an orthonormal basis for $\mathbb{C}^n$. After a direct computation, we have that

$$\begin{aligned}
Hf_1 &= a_1 f_1 + a_2 f_2,\\
Hf_k &= a_k f_{k-1} + a_{k+1} f_{k+1},\ 2 \le k \le m-1,\\
Hf_m &= a_m f_{m-1} + a_{m+1} f_m,\\
Hg_1 &= -a_1 g_1 - a_2 g_2,\\
Hg_k &= -a_k g_{k-1} - a_{k+1} g_{k+1},\ 2 \le k \le m-1,\\
Hg_m &= -a_m g_{m-1} - a_{m+1} g_m.
\end{aligned}$$

With respect to the orthonormal basis $\{f, \ldots, f_m, g_1, \ldots, g_m\}$, H is expressed as a block matrix

$$\begin{pmatrix} H_0 & 0_m \\ 0_m & -H_0 \end{pmatrix},$$

where

$$H_0 = \begin{pmatrix}
a_1 & a_2 & 0 & 0 & \cdots & 0 & 0\\
a_2 & 0 & a_3 & 0 & \cdots & 0 & 0\\
0 & a_3 & 0 & a_4 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & a_m\\
0 & 0 & 0 & 0 & \cdots & a_m & a_{m+1}
\end{pmatrix}.$$

Similarly, we also have that

$$\begin{aligned}
Kf_1 &= i(-a_1 g_1 + a_2 g_2),\\
Kf_k &= i((-1)^k a_k g_{k-1} + (-1)^{k+1} a_{k+1} g_{k+1}),\ 2 \le k \le m-1,\\
Kf_m &= i((-1)^m a_m g_{m-1} + (-1)^{m+1} a_{m+1} g_m),\\
Kg_1 &= -i(-a_1 f_1 + a_2 f_2),\\
Kg_k &= -i((-1)^k a_k f_{k-1} + (-1)^{k+1} a_{k+1} f_{k+1}),\ 2 \le k \le m-1,\\
Kg_m &= -i((-1)^m a_m f_{m-1} + (-1)^{m+1} a_{m+1} f_m).
\end{aligned}$$

With respect to the orthonormal basis $\{f, \ldots, f_m, g_1, \ldots, g_m\}$, K is expressed as a block matrix

$$\begin{pmatrix} 0_m & -K_0 \\ K_0 & 0_m \end{pmatrix},$$

where

$$K_0 = i \begin{pmatrix}
-a_1 & a_2 & 0 & 0 & \cdots & 0 & 0\\
a_2 & 0 & -a_3 & 0 & \cdots & 0 & 0\\
0 & -a_3 & 0 & a_4 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & (-1)^m a_m\\
0 & 0 & 0 & 0 & \cdots & (-1)^m a_m & (-1)^{m+1} a_{m+1}
\end{pmatrix}.$$

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Hence, the Hermitian matrices H and K are simultaneously unitarily similar to the respective block matrices

$$\begin{pmatrix} H_0 & 0_m \\ 0_m & -H_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0_m & -K_0 \\ K_0 & 0_m \end{pmatrix}.$$

Choose the unitary matrix

$$U = \begin{pmatrix} I_m & 0_m \\ 0_m & iI_m \end{pmatrix},$$

then we have that

$$U \begin{pmatrix} H_0 & 0_m \\ 0_m & -H_0 \end{pmatrix} U^* = \begin{pmatrix} H_0 & 0_m \\ 0_m & -H_0 \end{pmatrix},$$

$$U \begin{pmatrix} 0_m & -K_0 \\ K_0 & 0_m \end{pmatrix} U^* = \begin{pmatrix} 0_m & iK_0 \\ iK_0 & 0_m \end{pmatrix}.$$

The two matrices on the right-hand sides are real symmetric. This proves the matrix $S(a_1, a_2, \ldots, a_m, a_{m+1}, a_m, \ldots, a_2)$ is unitarily similar to a complex symmetric matrix. □

It is shown in [5] that when n is odd, for any $n \times n$ cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$ there exists a reversible cyclic weighted shift matrix $S(b_1, b_2, \ldots, b_2, b_1)$ such that

$$F_{S(a_1,a_2,\ldots,a_n)}(x, y, z) = F_{S(b_1,b_2,\ldots,b_2,b_1)}(x, y, z).$$

It follows that the numerical ranges $W(S(a_1, a_2, \ldots, a_n)) = W(S(b_1, b_2, \ldots, b_2, b_1))$. When $n = 2m$ is even, we deal with the same problem that for an $n \times n$ cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$, does there exist a 2-pivot reversible cyclic weighted shift matrix $S(b_1, b_2, \ldots, b_m, b_{m+1}, b_m, \ldots, b_2)$ satisfying

$$F_{S(a_1,a_2,\ldots,a_n)}(x, y, z) = F_{S(b_1,b_2,\ldots,b_m,b_{m+1},b_m,\ldots,b_2)}(x, y, z)?$$

This problem may relate to the Helton-Vinnikov theorem (cf. [15, Theorems 6,7]). So far, we are unable to solve this problem. In the following, we confirm only for the case $n = 4$.

Theorem 2.2. *Let $S(a_1, a_2, a_3, a_4)$ be a 4×4 cyclic weighted shift matrix with positive weights. Then there exist two-pivot reversible positive weights b_1, b_2, b_3, b_2 such that*

$$F_{S(a_1,a_2,a_3,a_4)}(x, y, z) = F_{S(b_1,b_2,b_3,b_2)}(x, y, z).$$

Proof. According to a formula in [8], the ternary form $F_{S(a_1,a_2,a_3,a_4)}(x, y, z)$ is given by

$$\begin{aligned} &16F_{S(a_1,a_2,a_3,a_4)}(z, -\cos\theta, -\sin\theta) \\ &= 16z^4 - 4(a_1^2 + a_2^2 + a_3^3 + a_4^2)z^2 + a_1^2a_3^2 + a_2^2a_4^2 - 2a_1a_2a_3a_4\cos(4\theta). \end{aligned}$$

Hence, to prove the existence of 2-pivot reversible positive weights b_1, b_2, b_3, b_2, it suffices to find positive numbers b_1, b_2, b_3 satisfying the following three conditions

$$b_1b_2^2b_3 = a_1a_2a_3a_4,$$

$$b_1^2b_3^2 + b_2^4 = a_1^2a_3^2 + a_2^2a_4^2,$$

$$b_1^2 + 2b_2^2 + b_3^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2.$$

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We change the variables: $A_j = a_j^2$ and $B_j = b_j^2$, $j = 1, 2, 3, 4$. Then the above three conditions are rewritten as

$$B_1 B_3 B_2^2 = A_1 A_2 A_3 A_4,$$
$$B_1 B_3 + B_2^2 = A_1 A_3 + A_2 A_4,$$
$$B_1 + B_3 + 2B_2 = A_1 + A_2 + A_3 + A_4.$$

Firstly, we choose $B_2 = \sqrt{A_2 A_4} = a_2 a_4$. Then, the three conditions imply that

$$B_1 B_3 = A_1 A_3 \quad \text{and} \quad B_1 + B_3 = A_1 + A_3 + (\sqrt{A_2} - \sqrt{A_4})^2.$$

Consider the quadratic equation

$$t^2 - (A_1 + A_3 + (\sqrt{A_2} - \sqrt{A_4})^2)t + A_1 A_3 = 0.$$

If the two roots of this equation are positive which can be chosen for B_1 and B_3, then the proof is completed. In fact, the discriminant of the quadratic equation is given by

$$\begin{aligned} D &= (A_1 + A_2 + A_3 + A_4 - 2\sqrt{A_2 A_4})^2 - 4A_1 A_3 \\ &= \left((a_1 + a_3)^2 + (a_2 - a_4)^2\right)\left((a_1 - a_3)^2 + (a_2 - a_4)^2\right). \end{aligned}$$

Hence, the discriminant D is positive except $a_1 = a_3$ and $a_2 = a_4$. In this exceptional case, by letting $b_1 = b_3 = a_1 = a_3$, $b_2 = a_2 = a_4$, all the required conditions on b_1, b_2, b_3 are satisfied. ∎

3. Fourier transform method. In this section, we provide an alternative proof of Theorem 2.1 through Fourier transform method which is used in [11] to prove the symmetry of reversible cyclic weighted shift matrices. Define an $n \times n$ unitary transformation matrix $U = (u_{k\ell})$:

$$u_{k\ell} = \frac{1}{\sqrt{n}} \omega^{(k-1)(\ell-1)},$$

where $\omega = e^{2\pi i/n}$. Let $A = (a_{pq})$ be an $n \times n$ matrix. The Fourier transform $B = (b_{k\ell}) = U^* A U$ of A is given by

$$b_{k\ell} = \frac{1}{n} \sum_{p,q=1}^{n} \omega^{-(k-1)(p-1)} \omega^{(\ell-1)(q-1)} a_{pq}.$$

Denote $\tilde{B} = nB$.

Theorem 3.1. *Let $n = 2m$ and $A = (a_{pq})$ be an $n \times n$ two-pivot-reversible cyclic weighted shift matrix with weights $a_1, a_2, \ldots, a_m, a_{m+1}, a_m, \ldots, a_2$. Then $V^* \tilde{B} V$ is a complex symmetric matrix, where*

$$V = \operatorname{diag}(\omega^{-1/2}, \omega^{-1}, \omega^{-3/2}, \ldots, \omega^{-(n-1)/2}).$$

Proof. If $A = (a_{pq})$ is a weighted shift matrix with weights $a_1, a_2, \ldots, a_n$, the matrix $\tilde{B} = nB = (\tilde{b}_{pq})$ is given by

$$\tilde{b}_{pp} = \omega^{p-1} \sum_{j=1}^{p} a_j = \omega^{p-1}(a_1 + a_2 + \cdots + a_n), \quad p = 1, 2, \ldots, n,$$

$$\begin{aligned} \tilde{b}_{p,p+k} &= \sum_{j=1}^{n} a_j \omega^{p+k-1+(j-1)k} = \sum_{j=1}^{n} a_j \omega^{p+jk-1} \\ &= \omega^{p+k-1}(a_1 + a_2 \omega^k + a_3 \omega^{2k} + \cdots + a_n \omega^{(n-1)k}), \end{aligned}$$

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for $1 \le p \le n-1$, $1 \le k \le n-p$, and

$$\begin{aligned}\tilde{b}_{p+k,p} &= \sum_{j=1}^{n} a_j \omega^{p-1-(j-1)k} \\ &= \omega^{p-1}(a_1 + a_2\omega^{-k} + a_3\omega^{-2k} + \cdots + a_n\omega^{-(n-1)k})\end{aligned}$$

for $1 \le p \le n-1$, $1 \le k \le n-p$.

In the case that $n = 2m$ and the weights are 2-pivot reversible $a_1, a_2, \ldots, a_m, a_{m+1}, a_m, \ldots, a_2$, we have that

$$\tilde{b}_{pp} = \omega^{p-1}(a_1 + a_{m+1} + 2a_2 + \cdots + 2a_m), \quad p = 1, 2, \ldots, n,$$

$$\tilde{b}_{p,p+k} = \omega^{p+k-1}\left(a_1 + (-1)^k a_{m+1} + 2a_2\cos(k\pi/m) + 2a_3\cos(2k\pi/m) + \cdots + 2a_m\cos((m-1)k\pi/m)\right),$$

$$\tilde{b}_{p+k,p} = \omega^{p-1}\left(a_1 + (-1)^k a_{m+1} + 2a_2\cos(k\pi/m) + 2a_3\cos(2k\pi/m) + \cdots + 2a_m\cos((m-1)k\pi/m)\right)$$

for $p = 1, 2, \ldots, n-1$, $k = 1, 2, \ldots, n-p$. Define a diagonal unitary matrix

$$V = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n),$$

where $\beta_j = e^{-j\pi i/n}$, $j = 1, 2, \ldots, n$. Then the $(p, p+k)$ and $(p+k, p)$ entries of the matrix $V^*\tilde{B}V$ are respectively equal to $\beta_p^{-1}\beta_{p+k}\omega^{p+k-1}\gamma$ and $\beta_{p+k}^{-1}\beta_p\omega^{p-1}\gamma$, where

$$\gamma = a_1 + (-1)^k a_{m+1} + 2a_2\cos(k\pi/m) + 2a_3\cos(2k\pi/m) + \cdots + 2a_m\cos((m-1)k\pi/m).$$

Clearly, these two entries are the same since $\beta_{p+k}^2\omega^{p+k} = \beta_p^2\omega^p = 1$, and thus, $V^*\tilde{B}V$ is symmetric. ∎

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REFERENCES

[1] M.T. Chien, J.Z. Liu, H. Nakazato, and T.Y. Tam. Toeplitz matrices are unitarily similar to symmetric matrices. *Linear Multilinear Algebra*, 65:2131–2144, 2017.
[2] M.T. Chien and H. Nakazato. Hyperbolic forms associated with cyclic weighted shift matrices. *Linear Algebra Appl.*, 439:3541–3554, 2013.
[3] M.T. Chien and H. Nakazato. Determinantal representations of hyperbolic forms via weighted shift matrices. *Appl. Math. Comput.*, 258:172–181, 2015.
[4] M.T. Chien and H. Nakazato. Matrices unitarily similar to symmetric matrices. *Proceedings of the 20th International Conference of Applied Mathematics*, Budapest, Hungary, 86–91, 2015.
[5] M.T. Chien and H. Nakazato. Hyperbolic forms admit reversible weighted shift determinantal reprentation. *Applied Math. Comput.*, 362:124595, 2019.
[6] M. Fiedler. Geometry of the numerical range of matrices. *Linear Algebra Appl.*, 37:81–96, 1981.
[7] S.R. Garcia, D.E. Poore, and J.E. Tener. Unitary equivalence to a complex symmetric matrix: Low dimensions. *Linear Algebra Appl.*, 437:271–284, 2012.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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http://repository.uwyo.edu/ela

[8] H.-L. Gau, M.C. Tsai, and H.C. Wang. Weighted shift matrices: Unitary equivalence, reducibility and numerical ranges. *Linear Algebra Appl.*, 438:498–513, 2013.

[9] F. Hausdorff. Das wertevorrat einer bilinearform. *Math. Z.*, 3:314–316, 1919.

[10] J.W. Helton and V. Vinnikov. Linear matrix inequality representations of sets. *Comm. Pure Appl. Math.*, 60:654–674, 2007.

[11] P.R. Huang and H. Nakazato. Cyclic weighted shift matrix with reversible weights. *Ann. Funct. Anal.*, 10:72–80, 2019.

[12] R. Kippenhahn. Über den wertevorrat einer Matrix. *Math. Nachr.*, 6:193–228, 1951.

[13] K. Lentzos, and L. Pasley. Determinantal representations of invariant hyperbolic plane curves. *Linear Algebra Appl.*, 556:108–130, 2018.

[14] C.K. Li and T.Y. Tam (editors). Special Issue on The Numerical Range and Numerical Radius. *Linear Multilinear Algebra*, 57(5), 2009.

[15] D. Plaumann, B. Sturmfels, and C. Vinzant. Computing linear matrix representations of Helton-Vinnikov curves. In: H. Dym et al. (editors), *Mathematical Methods in Systems, Optimizations, and Control*, Festschrift in honor of J. William Helton, *Oper. Theory Adv. Appl.*, Birkhäuser, Basel, 222:259–277, 2012.

[16] O. Toeplitz. Das algebraische Analogon zu einem Satze von Fejér. *Math. Z.*, 2:187–197, 1918.

[17] M.C. Tsai and P.Y. Wu. Numerical range of weighted shift matrices. *Linear Algebra Appl.*, 435:243–254, 2011.