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$(0,1)$-MATRICES AND DISCREPANCY*

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Abstract. Let m and n be positive integers, and let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be nonnegative integral vectors. Let $A(R,S)$ be the set of all $m \times n$ $(0,1)$-matrices with row sum vector R and column vector S. Let R and S be nonincreasing, and let $F(R)$ be the $m \times n$ $(0,1)$-matrix where for each i, the i^{th} row of $F(R,S)$ consists of r_i 1's followed by $n - r_i$ 0's. Let $A \in A(R,S)$. The discrepancy of A, $disc(A)$, is the number of positions in which $F(R)$ has a 1 and A has a 0. In this paper, we investigate the possible discrepancy of A^t versus the discrepancy of A. We show that if the discrepancy of A is ℓ, then the discrepancy of the transpose of A is at least $\frac{\ell}{2}$ and at most 2ℓ. These bounds are tight.

Key words. Ferrers matrix, Row-dense matrix, Discrepancy, Linear preserver, Strong linear preserver.

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The theory of $(0,1)$-matrices plays an important role in the analysis of biological networks. Some obvious ones are the prey–predator models, the climate–growth models, the pollinator–plant models, etc. In the study of plant species versus biological pollinators, a bipartite graph is an obvious tool for analysis. To study the bipartite graph, we often use the reduced adjacency matrix (a $(0,1)$-matrix) which is also called the *biadjacency* matrix [1, 2].

A nested bipartite network has a reduced adjacency matrix that is equivalent to a Ferrers matrix. See [3]. A measure of the 'closeness' of a bipartite network to a nested one is the discrepancy, defined as the number of 1's in the reduced adjacency matrix that must be interchanged with a 0 in the same row to yield a Ferrers matrix. See [4]

In this article, we will consider the discrepancy of a matrix versus the discrepancy of its transpose.

1. Preliminaries.

Definition 1.1. *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be sequences of length m and n of nonnegative integers from $\{0, 1, 2, \ldots, n\}$ and $\{0, 1, 2, \ldots, m\}$, respectively. Let $\mathcal{A}(R,S)$ denote the set of all $m \times n$ $(0,1)$-matrices with r_i 1's in row i and s_j 1's in row j.*

Note that $\mathcal{A}(R,S)$ is empty if $r_1 + r_2 + \ldots + r_m \neq s_1 + s_2 + \ldots + s_n$. Thus, throughout this article, we shall assume that $r_1 + r_2 + \ldots + r_m = s_1 + s_2 + \ldots + s_n$.

Definition 1.2. *Let R and S be monotone decreasing sequences of length m and n of nonnegative integers from $\{0, 1, 2, \ldots, n\}$ and $\{0, 1, 2, \ldots, m\}$, respectively. Let $F_{R,n}$ denote the unique matrix in $\mathcal{A}(R,S)$ whose i^{th} row consists of r_i 1's followed by $n - r_i$ 0's. The matrix $F_{R,n}$ is called a Ferrers matrix. [Note that the Ferrers matrix only relies on the sequence R and the number of columns because, given R, S is fixed.*

Necessarily, the j^{th} column of $F_{R,n}$ consists of s_j 1's followed by $m - s_j$ 0's. Thus, it is easily seen that $(F_{R,n})^t$, the transpose of $F_{R,n}$, is the Ferrers matrix $F_{S,m} \in \mathcal{A}(S,R)$.

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Note that every $1 \times n$ and $m \times 1$ matrix of 0's and 1's which has nonincreasing row and column sums is a Ferrers matrix, and the transpose of a Ferrers matrix is a Ferrers matrix. So, henceforth, we assume that $2 \leq \min\{m, n\}$.

Given an $m \times n$ matrix A of 0's and 1's which has nonincreasing row and column sums, the discrepancy $disc(A)$ or $\mathrm{BR}(A)$ is a measure of how near that matrix is to a Ferrers matrix.

DEFINITION 1.3. *Let $A \in \mathcal{A}(R, S)$ with R and S monotone decreasing sequences. The* discrepancy *of A, $disc(A)$, is the minimum number of 1's exchanged with 0's in the same row of A that yields a Ferrers matrix, or equivalently, the discrepancy of A is the number of entries in A that are equal 0 and such that the corresponding entry of $F_{R,n}$ is 1.*

That is, $disc(A)$ is the number of 1's in A that are outside the support of $F_{R,n}$.

As seen in the following example, the discrepancy of a $(0, 1)$-matrix is not independent of permutation of columns which maintains the nonincreasing nature of the columns.

EXAMPLE 1.4. Consider the two matrices:

$$A = \begin{bmatrix} 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 1 & \mathbf{0} & \mathbf{1} \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \mathbf{0} & \mathbf{1} & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \mathbf{0} & 0 & \mathbf{1} \end{bmatrix}.$$

Both are in $A((3, 3, 3, 2, 2, 2), (6, 3, 3, 3))$ and both can be reduced to the Ferrers matrix $F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ by exchanging the bold 1's and 0's in each row. The discrepancy of A is 4, while the discrepancy of A' is 3. Note that A' is achieved from A by permuting the last two columns.

Note that the discrepancy of A^t and the discrepancy of A'^t are both three. So the discrepancy of the transpose of A is not necessarily the same as the discrepancy of A. A natural question is: given that the discrepancy of A is k, how large or small can the discrepancy of A^t be? In this article, we shall answer that question.

If the discrepancy of A is 0, A is a Ferrers matrix and so is the transpose so that whenever the discrepancy of a matrix is 0, the discrepancy of its transpose is also 0. If the discrepancy is 1, that is no longer the case, as seen in the following example.

EXAMPLE 1.5. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ so that $A^t = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Both A and A^t are in $\mathcal{A}(2, 2, 1|2, 2, 1)$, and $disc(A) = 1$ while $disc(A^t) = 2$. Thus, transpose does not preserve discrepancy 1 for $\min\{m, n\} \geq 3$.

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Note that if the sequences R and S are monotone decreasing, and if $\min\{m,n\} \leq 2$, then the discrepancy of any matrix and the discrepancy of its transpose are the same.

Definition 1.6. *Let $A \in \mathcal{A}(R,S)$. A* row-exchangeable pair*, denoted $(i|j,k)$, refers to a pair of indices in the same row of A (the i^{th}) such that $j < k$, $a_{i,j} = 0$ and $a_{i,k} = 1$. A* proper set of row-exchangeable pairs *is a set of row-exchangeable pairs such that the replacement of each first indexed entry with a 1 and the second indexed entry with 0 (exchanging the values of the entries) yields a matrix whose discrepancy is one less and this replacement for all row-exchangeable pairs in a proper row-exchangeable set yields a Ferrers matrix.*

A column-exchangeable pair*, denoted $(p,q|j)$, refers to a pair of indices in the same column of A such that $p < q$, $a_{p,j} = 0$ and $a_{q,j} = 1$. A* proper set of column-exchangeable pairs *is a set of column-exchangeable pairs such that the replacement of each first indexed entry with a 1 and the second indexed entry with 0 (exchanging the values of the entries) yields a matrix whose transpose has discrepancy one less and this replacement for all column-exchangeable pairs in a proper column-exchangeable set yields a Ferrers matrix.*

It is easily seen that the cardinality of a proper row-exchangeable set for A is the discrepancy of A. The cardinality of a proper column-exchangeable set for A is the discrepancy of A^t.

Definition 1.7. *Let $A \in \mathcal{A}(R,S)$ and let $(i|j,k)$ be a row-exchangeable pair, and $(u,v|z)$ be a column-exchangeable pair. The notation $A \Leftarrow (i|j,k)$ (resp. $A \Leftarrow (u,v|z)$) will represent the matrix $\overline{A}$ where $\overline{a}_{i,j} = 1, \overline{a}_{i_k} = 0$ and $\overline{a}_{r,s} = a_{r,s}$ otherwise (resp. the matrix $\overline{A}$ where $\overline{a}_{u,z} = 1, \overline{a}_{v,z} = 0$ and $\overline{a}_{r,s} = a_{r,s}$ otherwise).*

If $\mathcal{X} = \{pr_v \mid v = 1,2,\ldots,r\}$ is a labeled set of row- or column-exchangeable pairs for A, let $A \Leftarrow \mathcal{X}$ denote the matrix $A^{(r)}$ where $A^{(1)} = A \Leftarrow pr_1, A^{(2)} = A^{(1)} \Leftarrow pr_2, \ldots, A^{(r)} = A^{(r-1)} \Leftarrow pr_r$.

Definition 1.8. *Let $A \in \mathcal{A}(R,S)$ and let $\mathcal{X} = \{(i_v|j_v,k_v) \mid v = 1,2,\ldots,r\}$ be a labeled set of row exchangeable pairs for A. Define $\mathcal{Y}(\mathcal{X}) = \{(i_v,p_v|j_v) \mid v = 1,2,\ldots,r\} \cup \{(i_v,q_v|k_v) \mid v = 1,2,\ldots,r\}$ where p_v and q_v are defined as follows:*

> *Beginning with $v = 1$, let p_1 be the index of the last row of A above row i_1 containing a 1 in column j_1, if all the entries of A in column j_1 above the row i_1 are 0, let $p_1 = m+1$. let q_1 be the index of the first row of A below i_1 containing a 0 in column k_1, if all the entries of A in column k_1 below the row i_1 are 1, let $q_1 = 0$. Let $A^{(1)} = A \Leftarrow (i_1|j_1,k_1)$. Let p_2 be the index of the last row of $A^{(1)}$ above row 1_2 containing a 1 in column j_2, if all the entries of $A^{(1)}$ in column j_2 above the row i_2 are 0, let $p_2 = m+1$. Let q_2 be the index of the first row of $A^{(1)}$ below row i_2 containing a 0 in column k_2, if all the entries of $A^{(1)}$ in column k_2 below the row i_2 are 1, let $q_2 = 0$. Let $A^{(2)} = A^{(1)} \Leftarrow (i_2|j_2,k_2)$. Continue in this way to get p_v and q_v, $v = 3,\ldots,r$.*

Example 1.9. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$. Then $\mathcal{X} = \{(1|4,6),(2|3,6)\}$. So $i_1 = 1, j_1 = 4$, and $k_1 = 6$. Therefore $p_1 = 3$ and $q_1 = 0$. From this, we get that the first row-exchangeable-pair gives the single column-exchangeable pair $(1,3|4)$. Now, $A^{(1)} = A \Leftarrow (1|4,6) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$. From the second row-exchangeable pair, we have $i_2 = 2, j_2 = 3$, and $k_2 = 6$. Therefore, $p_2 = 3$ and $q_2 = 1$. We now have two column-exchangeable pairs corresponding to the second row-exchangeable-pair: $(2,3|3)$ and

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$(2,1|6)$. However, since $2 \not< 1$, $(2,1|6)$ is not a column-exchangeable-pair. Now, $A^{(2)} = A^{(1)} \Leftarrow (2,3|6) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$, Which is a Ferrers matrix.

From the above, $\mathcal{Y}(\mathcal{X}) = \{(1,3|4), (2,3|3)\}$. Applying the column-exchangeable pairs to A, we get

$A \Leftarrow (1,3|4) = A_{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$, and

$A_{(1)} \Leftarrow (2,3|3) = A_{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, also a Ferrers matrix, but not the one above.

2. $disc(A)$ vs. $disc(A^t)$.

THEOREM 2.1. *Let $A \in \mathcal{A}(R,S)$ for R and S monotone decreasing sequences of length m and n, respectively. If $\mathcal{X}$ is a proper set of row-exchangeable pairs, then $\mathcal{Y}(\mathcal{X})$ contains a proper set of column-exchangeable pairs for A.*

Proof. We proceed by induction on the cardinality of a proper set of row-exchangeable pairs, or equivalently on the discrepancy of A.

Suppose that $A \in \mathcal{A}(R,S)$ and the discrepancy of A is 1 and let $\{(i|j.k)\}$ be a proper set of row-exchangeable pairs. That is the exchangeable 0 is entry (i,j) and the exchanged 1 is the (i,k) entry, then the only rows of A^t that might not be a sequence of 1's followed by 0's are the j^{th} and k^{th} rows. Thus, only the j^{th} and k^{th} rows can contain exchangeable 1's, and then each of them can contain at most one exchangeable 1. Thus, a proper set of column-exchangeable pairs has cardinality at most 2, and the discrepancy of A^t is at most 2.

Now assume that any matrix in $\mathcal{A}(R,S)$ of discrepancy $\ell - 1$ contains a proper set of column-exchangeable pairs of cardinality at most $2\ell - 2$.

Let $\mathcal{X}$ be a proper set of row-exchangeable pairs for A and enumerate $\mathcal{X}$ so that $\mathcal{X} = \{(i_v|j_v,k_v) \mid v = 1,\dots,\ell\}$. Let $\overline{A} = A \Leftarrow (i_\ell|j_\ell,k_\ell)$, so that $\overline{A}$ has discrepancy $\ell - 1$ and a proper set of row-exchangeable pairs, $\overline{\mathcal{X}} = \{(i_v|j_v,k_v) \mid v = 1,\dots,\ell-1\}$. By induction, $\overline{\mathcal{Y}} = \mathcal{Y}(\overline{\mathcal{X}})$ has a proper set of column-exchangeable pairs, call it $\mathcal{Z}$, so that $\overline{A} \Leftarrow \mathcal{Z}$ is a Ferrers matrix.

Consider $A \Leftarrow \mathcal{Z}$. If $B = A \Leftarrow \mathcal{Z}$ is not a Ferrers matrix, the only columns of B that are not a column of 1's followed by a column of 0's might be the columns j_ℓ and k_ℓ, and then the only exception would in each case be in the i_ℓ row. In this case, there would be at most one column-exchangeable pair in the j_ℓ column and one in the k_ℓ column. Thus, these column-exchangeable pairs together with $\mathcal{Z}$ would contain a proper set of column-exchangeable pairs for A of cardinality at most 2ℓ. ∎

COROLLARY 2.2. *Let $A \in \mathcal{A}(R,S)$ for R and S monotone decreasing sequences of length m and n respectively. If $disc(A) = \ell$, then $\lceil \frac{\ell}{2} \rceil \leq disc(A^t) \leq 2\ell$.*

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Proof. Suppose that the discrepancy of A is ℓ. The above theorem shows that the discrepancy of A^t is at most 2ℓ. Now, if the discrepancy of A^t is k, then the discrepancy of $A^{t^t} = A$ is at most $2k$. Since the discrepancy is an integer, we have $\lceil \frac{\ell}{2} \rceil \leq disc(A^t) \leq 2\ell$. $\square$

The following example shows that the above bounds are always achievable for $(0,1)$-matrices of order $(\ell+1) \times 2\ell$ or larger.

EXAMPLE 2.3. Let $\ell \geq 2$, $A = \begin{bmatrix} K_\ell & I_\ell \\ \mathbf{j}_\ell^t & \mathbf{0}_\ell^t \end{bmatrix}$, and let $B = A + E_{\ell+1,\ell+1}$ where $K_\ell = J_\ell \setminus I_\ell$ is the matrix af all 1's except for the diagonal entries, all of which are 0's, $\mathbf{j}_\ell$ is the ℓ-vector of all 1's, I_ℓ is the $\ell \times \ell$ identity matrix, $\mathbf{0}_\ell$ is the ℓ-vector of all 0's, and $E_{r,s}$ is the matrix with a 1 in the (r,s) entry and 0 elsewhere. Then, both A and B have discrepancy ℓ, whereas A^t has discrepancy $2\ell - 1$ and B^t has discrepancy 2ℓ. If $C = A^t$ has discrepancy k, then C^t has discrepancy $\ell = \frac{k+1}{2} = \lceil \frac{k}{2} \rceil$, and if $D = B^t$ has discrepancy d then $D^t = B$ has discrepancy $\ell = \frac{d}{2} = \lceil \frac{d}{2} \rceil$.

3. A Generalization.

3. A Generalization. In the previous sections, we required the $(0,1)$-matrices to be in $\mathcal{A}(R,S)$ with R and S nonincreasing. We now generalize to any $(0,1)$-matrix.

Let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a row vector with n entries, each a 0 or a 1. ($\vec{a} \in \mathbb{B}^n = \mathcal{M}_{1,n}(\mathbb{B})$). The vector $\vec{a}$ is said to be *left justified* with k 1's if $a_1 = a_2 = \ldots = a_k = 1$ and $a_{k+1} = a_{k+2} = \ldots = a_n = 0$ and label it $\vec{f}^k$. The *discrepancy* of a vector $\vec{a} \in \mathcal{M}_{1,n}(\mathbb{B})$ with k 1's is the number of entries of $\vec{a}$ that are 1 and the corresponding entry of $\vec{f}^k$ is 0. That is equivalent to the minimum number of pairs of entries that must be exchanged to yield a left justified vector. For example, if $\vec{a} = (1\,0\,0\,1\,0\,1\,1\,0)$, then $\vec{a}$ has discrepancy 2 since exchanging the entries 2 and 5 and the entries 3 and 6, we get the left justified vector $(1\,1\,1\,1\,0\,0\,0\,0)$.

As above, let $\vec{\mathbf{0}}_k$ denote the vector of k 0's, and let $\vec{\mathbf{j}}_k$ denote the vector of k 1's.

EXAMPLE 3.1. Let k be any positive integer and $A = \begin{bmatrix} \vec{\mathbf{j}}_k & \vec{\mathbf{0}}_k & \vec{\mathbf{j}}_k \\ \vec{\mathbf{0}}_k & \vec{\mathbf{j}}_k & \vec{\mathbf{0}}_k \end{bmatrix}$. Then, $disc(A) = 2k$. Let $P_{2,3}$ be the permutation that interchanges rows 2 and 3 upon multiplication on the left of a conformal matrix, or that interchanges columns 2 and 3 upon multiplication on the right of a conformal matrix. Then, $AP_{2,3} = \begin{bmatrix} \vec{j}_k & \vec{j}_k & \vec{0}_k \\ \vec{0}_k & \vec{0}_k & \vec{j}_k \end{bmatrix}$ which has discrepancy k.

DEFINITION 3.2. *Let m and n be positive integers, $S = (s_1, s_2, \ldots, s_n)$ be a sequence of nonnegative integers from $\{0, 1, 2, \ldots, m\}$, and let $R = (r_1, r_2, \ldots, r_m)$ be a sequence of nonnegative integers from $\{0, 1, 2, \ldots, n\}$. Let $\mathcal{P}_n(S)$ be the set of all permutations Q such that $A \in \mathcal{A}(R,S)$ if and only if $AQ \in \mathcal{A}(R,S)$.*

That is, $\mathcal{P}_n(S)$ is the set of permutations that fix the column sums of any matrix in $\mathcal{A}(R,S)$ for any sequence R.

The following definition and label is from Berger and Schreck [2].

DEFINITION 3.3. *Let m and n be positive integers, $S = (s_1, s_2, \ldots, s_n)$ be a sequence of nonnegative integers from $\{0, 1, 2, \ldots, m\}$, and let $R = (r_1, r_2, \ldots, r_m)$ be a sequence of nonnegative integers from $\{0, 1, 2, \ldots, n\}$. If R and S are monotone decreasing and $A \in \mathcal{A}(R,S)$, the* isomorphic discrepancy *of A, $d_I(A)$, is* $\min\{disc(AQ) | Q \in \mathcal{P}_n(S)\}$.

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EXAMPLE 3.4. Recall the matrices in Example 1.4:

$$A = \begin{bmatrix} 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 1 & \mathbf{0} & \mathbf{1} \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \mathbf{0} & \mathbf{1} & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \mathbf{0} & 0 & \mathbf{1} \end{bmatrix}.$$

Note that the minimum discrepancy of AQ is achieved from A by permuting the last two columns.

LEMMA 3.5. *l Let A be an $m \times n$ matrix of 0's and 1's. Then, $disc(A) = disc(PA)$ for all $P \in \mathcal{P}_m$.*

Proof. The discrepancy of any matrix only involves exchanges in rows, the order of the rows does not change the discrepancy. □

THEOREM 3.6. *Let A be an $m \times n$ matrix in $\mathcal{A}(R,S)$ with R and S nonincreasing. If $disc(A) = k$, then $d_I(A) \geq \frac{1}{4}k$.*

Proof. Suppose that $d_I(A) = \ell$ and $disc(A) = k$. Then, there exists a $Q \in \mathcal{P}_n(S)$ such that $disc(AQ) = \ell$. Now, by Corollary 2.2, $\ell = disc(AQ) \geq \frac{1}{2}disc((AQ)^t) = \frac{1}{2}disc(Q^tA^t) = \frac{1}{2}discA^t$, the last equality follows from Lemma 3.5. Thus, $\ell \geq \frac{1}{2}disc(A^t) \geq \frac{1}{2}(\frac{1}{2}disc((A^t)^t)) = \frac{1}{4}disc(A) = \frac{1}{4}k$. That is, $d_I(A) \geq \frac{1}{4}disc(A)$.

As only examples are known that show that there exist matrices whose isomorphic discrepancy of A is at least one half the discrepancy of A, the following problem is proposed:

Problem: *Let R and S be nonincreasing. Find a matrix in $\mathcal{A}(R,S)$ whose discrepancy is greater than twice the isomorphic discrepancy,*
OR
Show that for any $A \in \mathcal{A}(R,S)$ $d_I(A) \geq \frac{1}{2}disc(A)$.

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