Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

ELA

THE REFLEXIVE RE-NONNEGATIVE DEFINITE SOLUTION TO A QUATERNION MATRIX EQUATION*

QING-WEN WANG† AND FEI-ZHANG‡

Abstract. In this paper a necessary and sufficient condition is established for the existence of the reflexive re-nonnegative definite solution to the quaternion matrix equation $AXA^* = B$, where $*$ stands for conjugate transpose. The expression of such solution to the matrix equation is also given. Furthermore, a necessary and sufficient condition is derived for the existence of the general re-nonnegative definite solution to the quaternion matrix equation $A_1X_1A_1^* + A_2X_2A_2^* = B$. The representation of such solution to the matrix equation is given.

Key words. Quaternion matrix equation, Reflexive matrix, Re-nonnegative definite matrix, Reflexive re-nonnegative definite matrix.

AMS subject classifications. 65F05, 15A24, 15A33, 15A57.

1. Introduction. Throughout this paper, we denote the real number field by $\mathbb{R}$, the complex number field by $\mathbb{C}$, the real quaternion algebra by

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3 \ \in \mathbb{R}\},$$

the set of all $m\times n$ matrices over $\mathbb{H}$ by $\mathbb{H}^{m\times n}$, the set of all $m\times n$ matrices in $\mathbb{H}^{m\times n}$ with rank r by $\mathbb{H}_r^{m\times n}$, the set of all $n\times n$ invertible matrices over $\mathbb{H}$ by $\mathbb{GL}_n$. $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^*, $A^\dagger$, $\dim \mathcal{R}(A)$, I_i and $\mathrm{Re}[b]$ stand for the column right space, the left row space, the conjugate transpose, the Moore-Penrose inverse of $A \in \mathbb{H}^{m\times n}$, the dimension of $\mathcal{R}(A)$, an $i\times i$ identity matrix, and the real part of a quaternion b, respectively. By [1], for a quaternion matrix A, $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$, which is called the rank of A and denoted by $r(A)$. Moreover, A^{-*} denote the inverse matrix of A^* if A is invertible. Obviously, $A^{-*} = (A^{-1})^* = (A^*)^{-1}$. $\mathcal{H}_n = \{A \in \mathbb{H}^{n\times n} \mid A^* = A\}$. Because $\mathbb{H}$ is not commutative, one cannot directly extend various results on $\mathbb{C}$ to $\mathbb{H}$. General properties of matrices over $\mathbb{H}$ can be found in [2].

*Received by the editors July 31, 2007. Accepted for publication February 24, 2008. Handling Editor: Michael Neumann.

†Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China (wqw858@yahoo.com.cn). Supported by the Natural Science Foundation of China (60672160), Shanghai Pujiang Program (06PJ14039), and Shanghai Leading Academic Discipline Project (J50101).

‡Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

We consider the classical matrix equation

$$AXA^* = B. \tag{1.1}$$

The general Hermitian positive semidefinite solutions of (1.1) have been studied extensively for years. For instance, Khatri and Mitra [3], Baksalary [4], Groβ [5], Zhang and Cheng [6] derived the general Hermitian positive semidefinite solution to matrix equation (1.1), respectively. Moreover, Dai and Lancaster [7] studied the similar problem and emphasized the importance of (1.1) within the real setting. Liao and Bai [8] investigated the bisymmetric positive semidefinite solution to matrix equation (1.1) by studying the symmetric positive semidefinite solution of the matrix equation

$$A_1X_1A_1^* + A_2X_2A_2^* = B, \tag{1.2}$$

which was also investigated by Deng and Hu [9] over the real number field.

The definition of reflexive matrix can be found in [10]: A complex square matrix A is reflexive if $A = PAP$, where P is a Hermitian involution, i.e., $P^* = P$, $P^2 = I$. Obviously, a reflexive matrix is a generalization of a centrosymmetric matrix. The reflexive matrices with respect to a Hermitian involution matrix P have been widely used in engineering and scientific computations (see, for instance, [10]). In 1996, Wu and Cain [11] defined the re-nonnegative definite matrix: $A \in \mathbb{C}^{n\times n}$ is re-nonnegative definite if $\mathrm{Re}[x^*Ax] \geq 0$ for every nonzero vector $x \in \mathbb{C}^{n\times 1}$. However, to our knowledge, the reflexive re-nonnegative definite solution to (1.1) and the re-nonnegative definite solution to (1.2) have not been investigated yet so far. Motivated by the work mentioned above and keeping the interests and wide applications of quaternion matrices in view (e.g. [12]$-$[27]), we in this paper consider the reflexive re-nonnegative definite solution to (1.1) and re-nonnegative definite solution to (1.2) over $\mathbb{H}$.

The paper is organized as follows. In Section 2, we first present a criterion that a 3×3 partitioned quaternion matrix is re-nonnegative definite. Then we establish a criterion that a quaternion matrix is reflexive re-nonnegative definite. In Section 3, we establish a necessary and sufficient condition for the existence of re-nonnegative definite solution to (1.2) over $\mathbb{H}$ as well as an expression of the general solution. In Section 4, based on the results obtained in Section 3, we present a necessary and sufficient condition for the existence and the expression of the reflexive re-nonnegative definite solution to (1.1) over $\mathbb{H}$. In closing this paper, we in Section 5 give a conclusion and a further research topic related to this paper.

2. Reflexive re-nonnegative definite quaternion matrix. In this section, we first present a criterion that a 3×3 partitioned quaternion matrix is re-nonnegative definite, then derive a criterion that a quaternion matrix is reflexive re-nonnegative definite.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

Throughout, the set of all $n \times n$ reflexive quaternion matrices with a Hermitian involution $P \in \mathbb{H}^{n\times n}$ is denoted by $\mathbb{RH}^{n\times n}(P)$, i.e.,

$$\mathbb{RH}^{n\times n}(P) = \left\{ A \in \mathbb{H}^{n\times n} \mid A = PAP \right\}.$$

Definition 2.1. Let $A \in \mathbb{H}^{n\times n}$. Then
(1) $A \in \mathcal{H}_n$ is said to be nonnegative definite, abbreviated nnd, if $x^*Ax \geq 0$ for any nonzero vector $x \in \mathbb{H}^{n\times 1}$. The set of all $n \times n$ nnd matrices is denoted by $\mathbb{SP}_n$.
(2) A is re-nonnegative definite, abbreviated rennd, if $\mathrm{Re}[x^*Ax] \geq 0$ for every nonzero vector $x \in \mathbb{H}^{n\times 1}$. The set of all rennd matrices in $\mathbb{H}^{n\times n}$ is denoted by $\mathbb{SP}_n^*$.
(3) A is called reflexive re-nonnegative definite matrix if $A \in \mathbb{RH}^{n\times n}(P) \cap \mathbb{SP}_n^*$. The set of all $n \times n$ reflexive re-nonnegative definite matrices is denoted by $\mathbb{RSP}_n^*(P)$.

For any quaternion matrix A, it can be uniquely expressed as

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) \overset{\text{def}}{=} \mathcal{H}(A) + \mathcal{S}(A).$$

It is easy to prove the following.

Lemma 2.2. *Let $Q \in \mathbb{GL}_n, A \in \mathbb{H}^{n\times n}$. Then*
(1) $A \in \mathbb{SP}_n^*$ *if and only if* $\mathcal{H}(A) \in \mathbb{SP}_n$.
(2) $A \in \mathbb{SP}_n \Longleftrightarrow Q^*AQ \in \mathbb{SP}_n$.
(3) $A \in \mathbb{SP}_n^* \Longleftrightarrow Q^*AQ \in \mathbb{SP}_n^*$.
(4) $A \in \mathbb{RSP}_n^*(P) \Longleftrightarrow Q^*AQ \in \mathbb{RSP}_n^*(P)$.

The following lemma is due to Albert [28] which can be generalized to $\mathbb{H}$.

Lemma 2.3. *Let $A \in \mathbb{H}^{n\times n}$ be partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{H}_n$$

where $A_{ii} \in \mathcal{H}_{n_i}$ $(n_1 + n_2 = n)$. Then the following statements are equivalent:
(1) $A \in \mathbb{SP}_n$.
(2) $r(A_{11}) = r(A_{11}, A_{12})$, A_{11} *and* $A_{22} - A_{12}^*A_{11}^{\dagger}A_{12}$ *are nnd.*
(3) $r(A_{22}) = r(A_{12}^*, A_{22})$, A_{22} *and* $A_{11} - A_{12}A_{22}^{\dagger}A_{12}^*$ *are nnd.*

The following lemma follows from Lemma 2.2 and Lemma 2.3.

Lemma 2.4. *Let $A \in \mathbb{H}^{n\times n}$ be partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{ii} \in \mathbb{H}^{n_i\times n_i}$ $(n_1 + n_2 = n)$. Then the following statements are equivalent:
(1) $A \in \mathbb{SP}_n^*$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

(2) $r\left(A_{11}^* + A_{11}\right) = r\left(A_{11}^* + A_{11}, A_{12} + A_{21}^*\right)$, A_{11} *and* $A_{22} - \frac{1}{2}(A_{12}^* + A_{21})(A_{11} + A_{11}^*)^\dagger(A_{12} + A_{21}^*)$ *are rennd.*
(3) $r\left(A_{22} + A_{22}^*\right) = r\left(A_{12}^* + A_{21}, A_{22} + A_{22}^*\right)$, A_{22} *and* $A_{11} - \frac{1}{2}(A_{12} + A_{21}^*)(A_{22} + A_{22}^*)^\dagger(A_{12}^* + A_{21})$ *are rennd.*

The case of complex matrices of Lemma 2.4 can also be found in [11].

Now we present a criterion that a 3×3 partitioned Hermitian quaternion matrix is nnd.

LEMMA 2.5. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & X_{23} \\ A_{13}^* & X_{23}^* & A_{33} \end{pmatrix} \in \mathcal{H}_n$$

where $A_{11} \in \mathcal{H}_{r_1}, A_{22} \in \mathcal{H}_{r_2}, A_{33} \in \mathcal{H}_{n-r_1-r_2}$. *Then there exists* $X_{23} \in \mathbb{H}^{r_2 \times (n-r_1-r_2)}$ *such that* $A \in \mathbb{SP}_n$ *if and only if the following*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \overset{\text{def}}{=} A_1, \begin{pmatrix} A_{11} & A_{13} \\ A_{13}^* & A_{33} \end{pmatrix} \overset{\text{def}}{=} A_2$$

are all nnd.

Proof. If there exists a quaternion matrix X_{23} such that $A \in \mathbb{SP}_n$, then by Lemma 2.3, A_1 is nnd. Clearly, $A_2 \in \mathcal{H}_{n-r_2}$ by $A \in \mathcal{H}_n$. For an arbitrary nonzero column vector $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, it follows from $A \in \mathbb{SP}_n$ that

$$\alpha^* A_2 \alpha = \begin{pmatrix} \alpha_1^* & 0^* & \alpha_2^* \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & X_{23} \\ A_{13}^* & X_{23}^* & A_{33} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_2 \end{pmatrix} \geq 0.$$

Therefore, A_2 is also nnd.

Conversely, if A_1 and A_2 are all nnd, then A_{11}, $A_{22} - A_{12}^* A_{11}^\dagger A_{12}$, $A_{33} - A_{13}^* A_{11}^\dagger A_{13}$ are all nnd by Lemma 2.3. Taking $X_{23} = A_{12}^* A_{11}^\dagger A_{13}$ and

$$P = \begin{pmatrix} I_{r_1} & -A_{11}^\dagger A_{12} & -A_{11}^\dagger A_{13} \\ 0 & I_{r_2} & 0 \\ 0 & 0 & I_{n-r_1-r_2} \end{pmatrix}$$

yields that

$$P^* A P = \operatorname{diag}(A_{11}, A_{22} - A_{12}^* A_{11}^\dagger A_{12}, A_{33} - A_{13}^* A_{11}^\dagger A_{13}).$$

Hence by Lemma 2.2 and Lemma 2.3, A is nnd. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

Based on the above lemma, we now present a criterion that a 3×3 partitioned quaternion matrix is rennd. The result plays an important role in constructing a rennd matrix.

THEOREM 2.6. *Assume that*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & X_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \in \mathbb{H}^{n \times n}$$

where $A_{11} \in \mathbb{H}^{r_1 \times r_1}, A_{22} \in \mathbb{H}^{r_2 \times r_2}, A_{33} \in \mathbb{H}^{(n-r_1-r_2)\times(n-r_1-r_2)}$. *Then there exists* $X_{23} \in \mathbb{H}^{r_2 \times (n-r_1-r_2)}$ *such that* $A \in \mathbb{SP}_n^*$ *if and only if the following*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \stackrel{\text{def}}{=} A_1, \begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix} \stackrel{\text{def}}{=} A_2$$

are all rennd.

Proof. Put $B_{11} = A_{11} + A_{11}^*, B_{12} = A_{21}^* + A_{12}, B_{13} = A_{13} + A_{31}^*, B_{22} = A_{22} + A_{22}^*$, $B_{23} = X_{23} + A_{32}^*$, $B_{33} = A_{33} + A_{33}^*$. Then

$$2\mathcal{H}(A_1) = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}, 2\mathcal{H}(A_2) = \begin{pmatrix} B_{11} & B_{13} \\ B_{13}^* & B_{33} \end{pmatrix}$$

$$2\mathcal{H}(A) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{12}^* & B_{22} & B_{23} \\ B_{13}^* & B_{23}^* & B_{33} \end{pmatrix}.$$

Hence the theorem follows immediately from Lemma 2.2 and Lemma 2.5. ∎

Now we turn our attention to establish a criterion that a quaternion matrix is reflexive rennd.

For $A \in \mathbb{H}^{n \times n}$, a quaternion λ is said to be a right (or left) eigenvalue of A if $Ax = x\lambda$ (or $Ax = \lambda x$) for some nonzero vector $x \in \mathbb{H}^{n \times 1}$. In this paper, we only use the right eigenvalue (simply say eigenvalue). It can be verified easily that the eigenvalues of an involution $P \in \mathbb{H}^{n \times n}$ are 1 and -1.

In [29], a practical method to represent an involutory quaternion matrix was given as follows.

LEMMA 2.7. (Theorem 2.1 in [29]) *Suppose that* $P \in \mathbb{H}^{n \times n}$ *is a nontrivial involution and* $K = \begin{pmatrix} I_n + P \\ I_n \end{pmatrix}$, *then we have the following:*

(i) K *can be reduced into* $\begin{pmatrix} N & 0 \\ \Phi & M \end{pmatrix}$, *where* N *is a full column rank matrix of size*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

$n \times r$ and $r = r(I_n + P)$, by applying a sequence of elementary column operations on K.
(ii) Put $T = \begin{pmatrix} N, & M \end{pmatrix}$, then

$$P = T \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} T^{-1}. \tag{2.1}$$

By Lemma 2.7, $\mathcal{R}(N)$ is the eigenspace corresponding to the eigenvalue 1 of P and $\mathcal{R}(M)$ the eigenspace corresponding to the eigenvalue -1 of P. Since P is Hermitian, it can be orthogonally diagonalized. To see this, we perform the Gram-Schmidt process to the columns of N and M, respectively. Suppose that the corresponding orthonormal column vectors are as follows:

$$V = \{\alpha_1, \cdots, \alpha_r\}, \, W = \{\beta_1, \cdots, \beta_{n-r}\}$$

It is clear that $PV = V$, $PW = -W$ and $U = \begin{pmatrix} V, & W \end{pmatrix}$ is unitary. Hence it follows from $P^* = P$ that

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*. \tag{2.2}$$

Therefore, similar to Theorem 2.2 in [29], we get the following lemma.

LEMMA 2.8. *Let $A \in \mathbb{H}^{n\times n}$. Then $A \in \mathbb{RH}^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} U^* \tag{2.3}$$

where $A_1 \in \mathbb{H}^{r\times r}$, $A_2 \in \mathbb{H}^{(n-r)\times(n-r)}$, U is defined as (2.2).

At the end of the section, we present a criterion that a quaternion matrix is reflexive rennd.

THEOREM 2.9. *Let $A \in \mathbb{H}^{n\times n}$. Then $A \in \mathbb{RSP}_n^*(P)$ if and only if*

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} U^* \tag{2.4}$$

where $A_1 \in \mathbb{SP}_r^*, A_2 \in \mathbb{SP}_{n-r}^*$ and U is defined as (2.2).

Proof. If $A \in \mathbb{RSP}_n^*(P)$, then $A \in \mathbb{RH}^{n\times n}(P)$. Hence by Lemma 2.8, A can be expressed as (2.4) where $A_1 \in \mathbb{H}^{r\times r}, A_2 \in \mathbb{H}^{(n-r)\times(n-r)}$. It follows from $A \in \mathbb{RSP}_n^*(P)$, Lemma 2.2 and Lemma 2.4 that $A_1 \in \mathbb{SP}_r^*, A_2 \in \mathbb{SP}_{n-r}^*$.

Conversely, by Lemmas 2.2, 2.4 and 2.8, it is easy to verify that the matrix A with the form of (2.4) is reflexive rennd, where $A_1 \in \mathbb{SP}_r^*, A_2 \in \mathbb{SP}_{n-r}^*$. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

3. The re-nonnegative definite solution to (1.2). In this section, given $A_1 \in \mathbb{H}^{m\times r}$, $A_2 \in \mathbb{H}^{m\times (n-r)}$ and $B \in \mathbb{H}^{m\times m}$, we investigate the rennd solution to the matrix equation (1.2).

In order to investigate the rennd solution to (1.2), we recall the following lemma.

LEMMA 3.1. ([30]) *Let* $A_1 \in \mathbb{H}^{m\times r}_{r_{a_1}}, A_2 \in \mathbb{H}^{m\times(n-r)}_{r_{a_2}}$. *Then there exist* $P \in \mathbb{GL}_m, Q \in \mathbb{GL}_r, T \in \mathbb{GL}_{n-r}$ *such that*

$$PA_1Q = \begin{array}{c} \begin{pmatrix} I_{r_{a_1}} & 0 \\ 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} r_{a_1} \\ m - r_{a_1} \end{array} , \tag{3.1}$$

$$PA_2T = \begin{array}{cc} \begin{pmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_t \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} s \\ r_{a_1} - s \\ t \\ m - r_{a_1} - t \end{array} \\ \begin{array}{ccc} s & n - r - r_{a_2} & t \end{array} & \end{array} \tag{3.2}$$

where

$$r_{ab} = r\begin{pmatrix} A_1, & A_2 \end{pmatrix}, s = r_{a_1} + r_{a_2} - r_{ab}, \quad t = r_{ab} - r_{a_1}.$$

In some way, Lemma 3.1 can be regarded as an extension of the generalized singular value decomposition (GSVD) on a complex matrix pair (see [31]-[33]). It is worth pointing out that there is a good collection of literature, [8] and [9] for example, on the GSVD approach for solving matrix equations.

Now we propose an algorithm for finding out P, Q and T in Lemma 3.1.

ALGORITHM 3.2. *Let*

$$K = \begin{pmatrix} I_m & A_1 & A_2 \\ 0 & 0 & I_{n-r} \\ 0 & I_r & 0 \end{pmatrix}.$$

Apply a sequence of elementary row operations on the submatrices $\begin{pmatrix} I_m & A_1 & A_2 \end{pmatrix}$ *of* K, *and a sequence of elementary column operations on the submatrices* $\begin{pmatrix} A_1 \\ 0 \\ I_r \end{pmatrix}$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

and $\begin{pmatrix} A_2 \\ I_{n-r} \end{pmatrix}$ *of* K*, respectively, till we obtain the following*

$$\begin{pmatrix} P & \begin{pmatrix} I_{r_{a_1}} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_t \\ 0 & 0 & 0 \end{pmatrix} \\ 0 & 0 & T \\ 0 & Q & 0 \end{pmatrix}.$$

Then P, Q *and* T *are obtained.*

Now we consider (1.2). Let

$$Q^{-1}X_1Q^{-*} = \begin{array}{cc} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} & \begin{array}{c} r_{a_1} \\ r-r_{a_1} \end{array} \\ \begin{array}{cc} r_{a_1} & r-r_{a_1} \end{array} & \end{array}, \tag{3.3}$$

$$T^{-1}X_2T^{-*} = \begin{array}{cc} \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} & \begin{array}{c} s \\ n-r-r_{a_2} \\ t \end{array} \\ \begin{array}{ccc} s & n-r-r_{a_2} & t \end{array} & \end{array}, \tag{3.4}$$

$$PBP^* = \begin{array}{cc} \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} & \begin{array}{c} s \\ r_{a_1}-s \\ t \\ m-r_{a_1}-t \end{array} \\ \begin{array}{cccc} s & r_{a_1}-s & t & m-r_{a_1}-t \end{array} & \end{array}. \tag{3.5}$$

Lemma 3.3. *Let* $A_1 \in \mathbb{H}^{m\times r}$*,* $A_2 \in \mathbb{H}^{m\times(n-r)}$ *and* $B \in \mathbb{H}^{m\times m}$*. Then matrix equation (1.2) is consistent if and only if*

$$B_{\alpha 4} = 0, B_{4\beta} = 0, \alpha, \beta = 1, 2, 3, 4; B_{32} = 0, B_{23} = 0. \tag{3.6}$$

In that case, the general solution of (1.2) can be expressed as

$$X_1 = Q\begin{pmatrix} \begin{pmatrix} B_{11}-Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}Q^*, \tag{3.7}$$

$$X_2 = T\begin{pmatrix} Y_{11} & Y_{12} & B_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ B_{31} & Y_{32} & B_{33} \end{pmatrix}T^*, \tag{3.8}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

where X_{12}, X_{21}, X_{22}; $Y_{11}, Y_{12}, Y_{21}, Y_{22}, Y_{23}, Y_{32}$ *are arbitrary quaternion matrices whose sizes are determined by (3.3) and (3.4).*

Proof. Obviously, matrix equation (1.2) is equivalent to the following matrix equation

$$PA_1QQ^{-1}X_1Q^{-*}Q^*A_1^*P^* + PA_2TT^{-1}X_2T^{-*}T^*A_2^*P^* = PBP^*. \tag{3.9}$$

If (1.2) is consistent, then by (3.1)−(3.5) and (3.9), we have the following

$$\begin{pmatrix} X_{11} + \begin{pmatrix} Y_{11} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} Y_{13} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} Y_{31} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} Y_{33} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix}$$

yielding

$$X_{11} = \begin{pmatrix} B_{11} - Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, Y_{13} = B_{13}, Y_{31} = B_{31}, Y_{33} = B_{33},$$

and (3.6) holds. Therefore X_1 and X_2 can be expressed as (3.7) and (3.8), respectively.

Conversely, if (3.6) holds, then it can be verified that the matrices X_1 and X_2 with the form (3.7) and (3.8), respectively, consist of a solution of (1.2). □

THEOREM 3.4. *Let* $A_1 \in \mathbb{H}^{m\times r}$, $A_2 \in \mathbb{H}^{m\times(n-r)}$ *and* $B \in \mathbb{H}^{m\times m}$ *be given. Put*

$$C = \frac{1}{2}(B_{12}+B_{21}^*)(B_{22}+B_{22}^*)^\dagger(B_{12}^*+B_{21}), D = \frac{1}{2}(B_{13}+B_{31}^*)(B_{33}+B_{33}^*)^\dagger(B_{13}^*+B_{31}),$$

and $L = B_{11} - C - D$. *Then matrix equation (1.2) has a solution* $X_1 \in \mathbb{SP}_r^*, X_2 \in \mathbb{SP}_{n-r}^*$ *if and only if (3.6) holds,*

$$r(B_{22}+B_{22}^*, B_{21}+B_{12}^*) = r(B_{22}+B_{22}^*), r(B_{33}+B_{33}^*, B_{31}+B_{13}^*) = r(B_{33}+B_{33}^*), \tag{3.10}$$

and $L \in \mathbb{SP}_s^*, B_{22} \in \mathbb{SP}_{r_{a_1}-s}^*, B_{33} \in \mathbb{SP}_t^*$. *In that case, the general solution* $X_1 \in \mathbb{SP}_r^*$, $X_2 \in \mathbb{SP}_{n-r}^*$ *of (1.2) can be expressed as the following, respectively,*

$$X_1 = Q\begin{pmatrix} N & -X_{21}^* + (N+N^*)U_1 \\ X_{21} & F + \frac{1}{2}U_1^*(N^*+N)U_1 \end{pmatrix}Q^*, \tag{3.11}$$

$$X_2 = TMT^*, \tag{3.12}$$

with

$$N = \begin{pmatrix} E+C & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{3.13}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

$$M = \begin{pmatrix} G+D & -Y_{21}^* + (G+D+G^*+D^*)U_2 & B_{13} \\ Y_{21} & K + \frac{1}{2}U_2^*(G+D+G^*+D^*)U_2 & Y_{23} \\ B_{31} & Y_{32} & B_{33} \end{pmatrix}, \tag{3.14}$$

where $E, G \in \mathbb{SP}_r^*$ *are arbitrary but satisfy*

$$E + G = L; \tag{3.15}$$

$F \in \mathbb{SP}^*_{n-r_{a_1}}, K \in \mathbb{SP}^*_{n-r-r_{a_2}}; Y_{23} \in \left\{ Y_{23} \in \mathbb{H}^{(n-r-r_{a_2})\times t} \middle| M \in \mathbb{SP}^*_{n-r} \right\}$,
$Y_{32} \in \mathbb{H}^{t\times(n-r-r_{a_2})}, U_1 \in \mathbb{H}^{r_{a_1}\times(n-r_{a_1})}, U_2 \in \mathbb{H}^{s\times(n-r-r_{a_2})}$ *are all arbitrary.*

Proof. If matrix equation (1.2) has a solution $X_1 \in \mathbb{SP}_r^*, X_2 \in \mathbb{SP}^*_{n-r}$, then by Lemma 3.3, (3.6) holds and X_1, X_2 has the form of (3.7) and (3.8), respectively. Hence by Lemma 2.2,

$$\begin{pmatrix} Y_{11} & Y_{12} & B_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ B_{31} & Y_{32} & B_{33} \end{pmatrix} \in \mathbb{SP}^*_{n-r} \tag{3.16}$$

and

$$\begin{pmatrix} \begin{pmatrix} B_{11} - Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbb{SP}_r^*. \tag{3.17}$$

For (3.16), it follows from Theorem 2.6 that

$$\begin{pmatrix} Y_{11} & B_{13} \\ B_{31} & B_{33} \end{pmatrix} \in \mathbb{SP}^*_{s+t}, \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \in \mathbb{SP}^*_{n-r-r_{a_2}+s}.$$

By Lemma 2.4, $B_{33} \in \mathbb{SP}_t^*$ and the last equation of (3.10) holds. Moreover,

$$Y_{11} - D \overset{\text{def}}{=} G$$

is also rennd, i.e.,

$$Y_{11} = G + D \tag{3.18}$$

where G is rennd;

$$r(Y_{11} + Y_{11}^*, Y_{12} + Y_{21}^*) = r(Y_{11} + Y_{11}^*), \tag{3.19}$$

and

$$Y_{22} - \frac{1}{2}(Y_{21} + Y_{12}^*)(Y_{11} + Y_{11}^*)^\dagger (Y_{21}^* + Y_{12}) \overset{\text{def}}{=} K \tag{3.20}$$

is rennd.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

It follows from (3.19) that the matrix equation $(Y_{11}+Y_{11}^*)X = Y_{12}+Y_{21}^*$ is consistent for X. Let U_2 is an any solution of the matrix equation. Then by (3.18),

$$Y_{12} = -Y_{21}^* + (G+D+G^*+D^*)U_2. \tag{3.21}$$

Hence by (3.20),

$$\begin{aligned} Y_{22} &= K + \frac{1}{2}U_2^*(Y_{11}+Y_{11}^*)(Y_{11}+Y_{11}^*)^\dagger(Y_{11}+Y_{11}^*)U_2 \\ &= K + \frac{1}{2}U_2^*(Y_{11}+Y_{11}^*)U_2 \\ &= K + \frac{1}{2}U_2^*(G+D+G^*+D^*)U_2. \end{aligned} \tag{3.22}$$

Accordingly, it follows from (3.18), (3.20), (3.21) and (3.22) that (3.16) can be expressed as (3.14) implying X_2 can be expressed as (3.12).

For (3.17), by Lemma 2.4,

$$\begin{pmatrix} B_{11}-Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{SP}^*_{r_{a_1}}. \tag{3.23}$$

Hence it follows from Lemma 2.4 that $B_{22} \in \mathbb{SP}^*_{r_{a_1}-s}$ and the first equation of (3.10) holds, and

$$B_{11} - Y_{11} - C \overset{\text{def}}{=} E \tag{3.24}$$

is rennd. It is implies that (3.23) can be expressed as (3.13). Therefore, (3.15) follows from (3.18) and $L \in \mathbb{SP}^*_s$ from E and G is rennd. By (3.17) and Lemma 2.4,

$$r(N+N^*, X_{12}+X_{21}^*) = r(N+N^*) \tag{3.25}$$

and

$$X_{22} - \frac{1}{2}(X_{21}+X_{12}^*)(N+N^*)^\dagger(X_{21}^*+X_{12}) \overset{\text{def}}{=} F \tag{3.26}$$

is rennd. It follows from (3.25) that the matrix equation $(N+N^*)X = X_{12}+X_{21}^*$ is solvable for X. Assume that U_1 is an any solution of the matrix equation. By (3.26), we have that

$$X_{12} = -X_{21}^* + (N+N^*)U_1 \tag{3.27}$$

and

$$X_{22} = F + \frac{1}{2}U_1^*(N^*+N)U_1.$$

Consequently, X_1 can be expressed as (3.11).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

Conversely, suppose that (3.6), (3.10) and (3.15) hold, where $L \in \mathbb{SP}_s^*, B_{22} \in \mathbb{SP}_{r_{a_1}-s}^*$, $B_{33} \in \mathbb{SP}_t^*$. It can be verified that M, N are all rennd by Theorem 2.6, Lemma 2.4. And the matrices with the form of (3.11) and (3.12) are all rennd by Lemma 2.2 and Lemma 2.4. It is easy to verify that the matrices X_1 and X_2 with the form of (3.11) and (3.12), respectively, are the solution of matrix equation (1.2). □

REMARK 3.5. Setting A_2 vanish in Theorem 3.4, we can get the corresponding results on the rennd solution to matrix equation (1.1) over $\mathbb{H}$.

4. The reflexive re-nonnegative definite solution to (1.1). In this section, we consider the reflexive rennd solution to the matrix equation (1.1), where $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times m}$ are given and $X \in \mathbb{RSP}_n^*(P)$ is unknown.

By Theorem 2.9, we can assume that

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} U^* \tag{4.1}$$

where U is defined as (2.2) and $X_1 \in \mathbb{SP}_r^*$, $X_2 \in \mathbb{SP}_{n-r}^*$.

Suppose that

$$AU = \begin{pmatrix} A_1, & A_2 \end{pmatrix} \tag{4.2}$$

where $A_1 \in \mathbb{H}^{m\times r}$, $A_2 \in \mathbb{H}^{m\times(n-r)}$. Then (1.1) has a solution $X \in \mathbb{RSP}_n^*(P)$ if and only if matrix equation (1.2) has a rennd solution X_1, X_2. By Theorem 3.4, we immediately get the following.

THEOREM 4.1. *Suppose that $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times n}$ are given and C, D, L are the same as in Theorem 3.4, then matrix equation (1.1) has a solution $X \in \mathbb{RSP}_n^*(P)$ if and only if (3.6), (3.10) and (3.15) hold, $L \in \mathbb{SP}_s^*, B_{22} \in \mathbb{SP}_{r_{a_1}-s}^*, B_{33} \in \mathbb{SP}_t^*$. In that case, the general solution $X \in \mathbb{RSP}_n^*(P)$ can be expressed as (4.1) where X_1 and X_2 are as the same as (3.11) and (3.12), respectively.*

5. Conclusions. In this paper we have presented the criteria that a 3×3 partitioned quaternion matrix is re-nonnegative definite and a quaternion matrix is reflexive re-nonnegative definite. A necessary and sufficient condition for the existence and the expression of re-nonnegative definite solution to the matrix equation (1.2) over $\mathbb{H}$ have been established. Using Theorem 3.4, we have established a necessary and sufficient condition for the existence and the expression of the reflexive re-nonnegative definite solution to matrix equation (1.1) over $\mathbb{H}$. In closing this paper, we propose the following two open problems which are related to this paper:
(1) How do we investigate the least-square re-nonnegative definite solution to (1.2)

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

and the least-square reflexive re-nonnegative definite solution to (1.1) over $\mathbb{H}$?
(2) How do we investigate the maximal and minimal ranks of the general reflexive re-nonnegative definite solution to the matrix equation (1.1) over $\mathbb{H}$?

Acknowledgement. The authors would like to thank Professor Michael Neumann and a referee very much for their valuable suggestions and comments, which resulted in a great improvement of the original manuscript.

REFERENCES

[1] T. W. Hungerford. *Algebra*. Spring-Verlag Inc., New York, 1980.
[2] F. Zhang. Quaternions and matrices of quaternions, *Linear Algebra Appl.*, 251:21–57, 1997.
[3] C. G. Khatri and S. K. Mitra. Hermitian and nonnegative definite solutions of linear matrix equations. *SIAM J. Appl. Math.*, 31:579–585, 1976.
[4] J. K. Baksalary. Nonnegative definite and positive definite solutions to the matrix equation $AXA^* = B$. *Linear Multilinear Algebra*, 16:133–139, 1984.
[5] J. Groβ. Nonnegative-define and positive solutions to the matrix equation $AXA^* = B -$ revisited. *Linear Algebra Appl.*, 321:123–129, 2000.
[6] X. Zhang and M. Y. Cheng. The rank-constrained Hermitian nonnegative-definite and positive-definite solutions to the matrix equation $AXA^* = B$. *Linear Algebra Appl.*, 370:163–174, 2003.
[7] H. Dai and P. Lancaster. Linear matrix equations from an inverse problem of vibration theory. *Linear Algebra Appl.*, 246:31–47, 1996.
[8] A. P. Liao and Z. Z. Bai. The constrained solutions of two matrix equations. *Acta Math. Sinica, English Series*, 18(4):671–678, 2002.
[9] Y. B. Deng and X. Y. Hu. On solutions of matrix equation $AXA^T + BXB^T = C$, *J. Comput. Math.*, 23(1):17–26, 2005.
[10] H. C. Chen, *The SAS domain decomposition method for structural analysis*, CSRD Tech. report 754, Center for Supercomputing Research and Development, University of Illinois, Urbana, IL, 1988.
[11] L. Wu and B. Cain. The Re-nonnegative definite solution to the matrix inverse problem $AX = B$. *Linear Algebra Appl.*, 236:137–146, 1996.
[12] S. L. Adler. *Quaternionic Quantum Mechanics and Quantum Fields.* Oxford University Press, Oxford, 1995.
[13] D. I. Merino and V. Sergeichuk. Littlewood's algorithm and quaternion matrices, *Linear Algebra Appl.*, 298:193–208, 1999.
[14] S. J. Sangwine, T. A. Ell, and C. E. Moxey. Vector Phase correlation. *Electronics Letters*, 37(25):1513–1515, 2001.
[15] G. Kamberov, P. Norman, F. Pedit, and U. Pinkall. *Quaternions, spinors, and surfaces*, Contemporary Mathematics, vol. 299, Amer. Math. Soc., Province, 2002.
[16] D. R. Farenick and B. A. F. Pidkowich. The spectral theorem in quaternions. *Linear Algebra Appl.*, 371:75–102, 2003.
[17] C. E. Moxey, S. J. Sangwine, and T.A. Ell. Hypercomplex correlation techniques for vector images. *IEEE Transactions on Signal Processing*, 51(7):1941–1953, 2003.
[18] N. L. Bihan and S. J. Sangwine. *Color image decomposition using quaternion singular value decomposition.* IEEE International conference on visual information engineering (VIE), Guildford, UK, July 2003.
[19] N. L. Bihan and J. Mars. Singular value decomposition of matrices of Quaternions: A New Tool for Vector-Sensor Signal Processing, *Signal Processing* 84(7):1177–1199, 2004.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 88-101, February 2008

[20] Q. W. Wang. Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations. *Comput. Math. Appl.*, 49(5/6):641–650, 2005.

[21] Q. W. Wang. The general solution to a system of real quaternion matrix equations. *Comput. Math. Appl.*, 49(5/6):665–675, 2005.

[22] S. J. Sangwine and N. L. Bihan. Quaternion singular value decomposition based on bidiagonalization to a real or complex matrix using quaternion Householder transformations. *Appl. Math. Comput.*, 182(1):727–738, 2006.

[23] Q. W. Wang. A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity. *Linear Algebra Appl.*, 384:43–54, 2004.

[24] Q. W. Wang, Z. C. Wu, and C. Y. Lin. Extremal ranks of a quaternion matrix expression subject to consistent systems of quaternion matrix equations with applications. *Appl. Math. Comput.*, 182:1755–1764, 2006.

[25] Q. W. Wang, G. J. Song, and C. Y. Lin. Extreme ranks of the solution to a consistent system of linear quaternion matrix equations with an application. *Appl. Math. Comput.*, 189:1517–1532, 2007.

[26] Q. W. Wang, S.W. Yu, and C.Y. Lin. Extreme ranks of a linear quaternion matrix expression subject to triple quaternion matrix equations with applications. *Appl. Math. Comput.*, 195:733–744, 2008.

[27] F. Zhang. Geršgorin type theorems for quaternionic matrices. *Linear Algebra Appl.*, 424:139–153, 2007.

[28] A. Albert. Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM J. Appl. Math.*, 17:434–440, 1969.

[29] Q. W. Wang, H. X. Chang, and C. Y. Lin. P-(skew)symmetric common solutions to a pair of quaternion matrix equations. *Appl. Math. Comput.*, 195:721–732, 2008.

[30] Q. W. Wang. Pairwise matrix decompositions and matrix equations over an arbitrary skew field. *Acta Math. Sinica (Chin. Ser.)*, 39(3):396–403, 1996.

[31] C. F. Van Loan. Generalizing the singular value decomposition. *SIAM J. Numer. Anal.*, 13:76–83, 1976.

[32] C. C. Paige and M. A. Saunders. Towards a generalized singular value decomposition. *SIAM J. Numer. Anal.*, 18:398–405, 1981.

[33] G. W. Stewart. Computing the CS-decomposition of a partitioned orthogonal matrix. *Numer. Math.*, 40:297–306, 1982.