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THE OUTER GENERALIZED INVERSE OF AN EVEN-ORDER TENSOR WITH THE EINSTEIN PRODUCT THROUGH THE MATRIX UNFOLDING AND TENSOR FOLDING*

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Abstract. Necessary and sufficient conditions for the existence of the outer inverse of a tensor with the Einstein product are studied. This generalized inverse of a tensor unifies several generalized inverses of tensors introduced recently in the literature, including the weighted Moore-Penrose, the Moore-Penrose, and the Drazin inverses. The outer inverse of a tensor is expressed through the matrix unfolding of a tensor and the tensor folding. This expression is used to find a characterization of the outer inverse through group inverses, establish the behavior of outer inverse under a small perturbation, and show the existence of a full rank factorization of a tensor and obtain the expression of the outer inverse using full rank factorization. The tensor reverse rule of the weighted Moore-Penrose and Moore-Penrose inverses is examined and equivalent conditions are also developed.

Key words. Einstein product, Full rank factorization, Matrix folding, Outer inverse, Perturbation.

AMS subject classifications. 15A69, 15A09.

1. Introduction.

For a positive integer N, let $[N] = \{1, 2, \ldots, N\}$. An order k tensor $\mathcal{A} = (A_{i_1\ldots i_k}) \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$ is a multidimensional array with $I_1 I_2 \cdots I_k$ entries over complex field $\mathbb{C}$, where $i_j \in [I_j], j \in [k]$. Given $\mathcal{A} = (A_{i_1\ldots i_k})$ and $\mathcal{B} = (B_{i_1\ldots i_k}) \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$ and a scalar $\alpha \in \mathbb{C}$, with the standard addition $\mathcal{A} + \mathcal{B} = (A_{i_1\ldots i_k} + B_{i_1\ldots i_k})$ and the scalar product $\alpha A = (\alpha A_{i_1\ldots i_k})$, $\mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$ is a vector space. The vector space $\mathbb{C}^n$ and matrix space $\mathbb{C}^{I_1\times I_2}$ are special examples of tensor spaces.

For tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_k\times J_{k+1}\times\cdots\times J_m}$ with $m \geq k$, the Einstein product $A *_k B$ of tensors $\mathcal{A}$ and $\mathcal{B}$ is a tensor in $\mathbb{C}^{I_1\times\cdots\times I_k\times J_{k+1}\times\cdots\times J_m}$ defined in [9] by

$$(\mathcal{A} *_k \mathcal{B})_{i_1\ldots i_k j_{k+1}\ldots j_m} = \sum_{j_r\in[J_r],r\in[k]} A_{i_1\ldots i_k j_1\ldots j_k} B_{j_1\ldots j_k j_{k+1}\ldots j_m}.$$

This tensor product satisfies the associative law. When $m = k$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_k}$, $\mathcal{A} *_k \mathcal{B}$ is in $\mathbb{C}^{I_1\times\cdots\times I_k}$. Thus, the $2k$-order tensor $\mathcal{A}$ corresponds to a linear operator $L_{\mathcal{A}}$ from tensor space $\mathbb{C}^{J_1\times\cdots\times J_k}$ to tensor space $\mathbb{C}^{I_1\times\cdots\times I_k}$.

An alternative product of two tensors $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$ of order $m \geq 2$ and $\mathcal{B} \in \mathbb{C}^{n_2\times n_3\times\cdots\times n_{k+1}}$ of order $k \geq 1$ is introduced in [26, 4] and various topics such as the inverse, rank, similarity, and congruence under this product can be found in [26, 4, 43]. For a survey of many interesting topics of tensors, including the modal-k product and its applications, refer to [5, 6, 10, 16, 21, 23, 34, 41].

In this paper, we work with the even-order tensor spaces with the Einstein product. For a scalar α in $\mathbb{C}$,

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$\bar{\alpha}$ is the conjugate of α. For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$, the conjugate transpose $\mathcal{A}^*$ of $\mathcal{A}$ is a tensor in $\mathbb{C}^{J_1\times\cdots\times J_k\times I_1\times\cdots\times I_k}$ with its elements defined by $(\mathcal{A}^*)_{j_1\dots j_k i_1\dots i_k} = \bar{A}_{i_1\dots i_k j_1\dots j_k}$. The diagonal tensor $\mathcal{D}$ in $\mathbb{C}^{N_1\times\cdots\times N_k\times N_1\times\cdots\times N_k}$ is the tensor with its entries defined by

$$(\mathcal{D})_{i_1\dots i_k j_1\dots j_k} = \begin{cases} d_{i_1\cdots i_k} & \text{if } i_r = j_r \in [N_r], \text{ for } r \in [k] \\ 0 & \text{otherwise,} \end{cases}$$

where $d_{i_1\cdots i_k}$ is a complex number. If all the diagonal entries $d_{i_1\cdots i_k} = 1$, then the diagonal tensor $\mathcal{D}$ is called the identity tensor, denoted by $\mathcal{I}$. The identity tensor depends on the dimensions $N_1, N_2, \dots, N_k$ of all orders. For simplicity, we will not indicate its dependency on the dimension of each order and use $\mathcal{I}$ to denote both identity tensors in $\mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ and $\mathbb{C}^{J_1\times\cdots\times J_k\times J_1\times\cdots\times J_k}$. It is easy to show that $(\mathcal{A} * \mathcal{B})^* = \mathcal{B}^* * \mathcal{A}^*$ and $\mathcal{I} * \mathcal{A} = \mathcal{A} * \mathcal{I} = \mathcal{A}$ for $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_k\times L_1\times\cdots\times L_k}$ [30].

For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$, if there exists a tensor $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ such that $\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A} = \mathcal{I}$, then $\mathcal{A}$ is said to be invertible and the $\mathcal{B}$ is called the inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{-1}$ [3]. For a general tensor $\mathcal{A}$ in $\mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$, its inverse may not exist. But it is shown in [30] that there exists a unique $\mathcal{X}$ in $\mathbb{C}^{J_1\times\cdots\times J_k\times I_1\times\cdots\times I_k}$ satisfying

$$\mathcal{A} * \mathcal{X} * \mathcal{A} = \mathcal{A}, \tag{1.1}$$

$$\mathcal{X} * \mathcal{A} * \mathcal{X} = \mathcal{X}, \tag{1.2}$$

$$(\mathcal{A} * \mathcal{X})^* = \mathcal{A} * \mathcal{X}, \tag{1.3}$$

$$(\mathcal{X} * \mathcal{A})^* = \mathcal{X} * \mathcal{A}. \tag{1.4}$$

The unique $\mathcal{X}$, denoted by $\mathcal{A}^\dagger$, is called the Moore-Penrose inverse of $\mathcal{A}$. Obviously, we have $\mathcal{A}^\dagger = \mathcal{A}^{-1}$ if $\mathcal{A}$ is invertible.

A tensor $\mathcal{X} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k\times I_1\times I_2\times\cdots\times I_k}$ is called a $\{2\}$-inverse of $\mathcal{A}$ if $\mathcal{X}$ satisfies the equation (1.2). We will consider the $\{2\}$-inverse of $\mathcal{A}$ with prescribed range and nullspace, i.e., the outer inverse of $\mathcal{A}$. This generalized inverse will unify the three very important classes of generalized inverses of $\mathcal{A}$ introduced recently in the literature [2, 33]: the weighted Moore-Penrose inverse [13], the Moore-Penrose inverse [30, 18], the Drazin inverse [14, 24], as well as the Core and the Core-EP inverse [25, 8, 17, 19], the CMP and the DMP inverse [20, 32]. The expressions of these generalized inverses are to be obtained through the matrix unfolding of a tensor. We study the full rank factorization of a tensor and express the outer inverse using the factorization. A characterization of the outer inverse of a tensor through group inverse, the behavior of outer inverse under a small perturbation, and the reverse order rule of the weighted Moore-Penrose inverse of tensors are also presented.

2. The generalized inverses through its matrix unfolding. In this section, we will introduce the generalized inverses through its matrix unfolding.

Definition 2.1. For $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ in $[I_1] \times [I_2] \times \cdots \times [I_k]$, we write $(i_1, \dots, i_k) < (j_1, \dots, j_k)$, and say $(i_1, \dots, i_k)$ is less than $(j_1, \dots, j_k)$ if there exists an integer l in $\{1, 2, \dots, k\}$ such that $i_l < j_l$ and $i_t = j_t$ for $l + 1 \le t \le k$.

Under this ordering, we have $(1, 1) < (2, 1) < (1, 2) < (2, 2)$. We also define a map $h : [I_1] \times [I_2] \times \cdots \times [I_k] \to [I_1 I_2 \cdots I_k]$ by

$$h(\mathbf{i}, \mathbf{I}) = i_1 + (i_2 - 1)I_1 + (i_3 - 1)I_1 I_2 + \cdots + (i_k - 1)I_1 I_2 \cdots I_{k-1}$$

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where $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and $\mathbf{I} = (I_1, I_2, \ldots, I_k)$.

For $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$, let $\mathbf{i} = (i_1, i_2, \ldots, i_k)$, $\mathbf{j} = (j_1, j_2, \ldots, j_k)$, $\mathbf{I} = (I_1, I_2, \ldots, I_k)$, and $\mathbf{J} = (J_1, J_2, \ldots, J_k)$, and define the transformation $f : \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k} \longrightarrow \mathbb{C}^{I_1\cdots I_k\times J_1\cdots J_k}$ with $f(\mathcal{A}) = \mathbf{A}$ defined component-wise as

$$\mathcal{A}_{i,j} \xrightarrow{f} \mathbf{A}_{h(i,I),h(j,J)}. \tag{2.5}$$

Here, the matrix $\mathbf{A}$ is the matrix unfolding of a $2k$-order tensor $\mathcal{A}$ with the first k orders for the row and the last k orders for the column ordered according to Definition 2.1 with the rows from top to bottom and columns from left to right and $\mathcal{A}$ is the folding of a matrix $\mathbf{A}$ into a tensor in $\mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$. Though the matrix unfolding f and tensor folding f^{-1} depend on $\mathbf{I}$ and $\mathbf{J}$, we may omit their dependency for simplicity in the remainder of the paper.

For $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k\times L_1\times L_2\times\cdots\times L_k}$, it is observed in [3] that both h and f are bijections for given I and J and that

$$f(\mathcal{A} * \mathcal{B}) = f(\mathcal{A}) \cdot f(\mathcal{B}) \quad \text{and} \quad f^{-1}(\mathbf{A}\cdot\mathbf{B}) = f^{-1}(\mathbf{A}) * f^{-1}(\mathbf{B}) \tag{2.6}$$

for $\mathbf{A} = f(\mathcal{A}), \mathbf{B} = f(\mathcal{B})$ where $\cdot$ refers to the usual matrix multiplication.

Any $\mathcal{B} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k}$ can be viewed as an even-order tensor in $\mathbb{C}^{J_1\times J_2\times\cdots\times J_k\times L_1\times\cdots\times L_k}$ with $L_l = 1$ for $l = 1, \ldots, k$. In this case, $f(\mathcal{B})$ is actually a vector in $\mathbb{C}^{J_1J_2\cdots J_k\times 1} = \mathbb{C}^{J_1J_2\cdots J_k}$. That is, the operator $f(\cdot)$ turns tensors of order k into column vectors ordered according to Definition 2.1. With this extension, (2.6) now even holds for a $\mathcal{B} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k}$. Throughout the paper, any $\mathcal{B} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k}$ will always be treated as an oven-order tensor in $\mathbb{C}^{J_1\times J_2\times\cdots\times J_k\times 1\times\cdots\times 1}$.

Note that the transformation f, which preserves the structures of the tensor space of even-order with the Einstein product and the matrix space with the usual matrix multiplication as indicated in (2.6), allows us to explore many interesting properties of tensors through those of matrices. Indeed, the singular value decomposition (SVD) of a tensor $\mathcal{A} \in \mathbb{C}^{I\times J\times I\times J}$, the inversion of higher-order tensor, and multilinear systems were studied in [3, 7, 35, 36, 42] through the transformation f and the SVD of a general even-order tensor $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$ was given in [30].

In view of (2.6), we have

$$\mathcal{A} * \mathcal{B} = \mathcal{I} \quad \text{if and only if} \quad f(\mathcal{A}) \cdot f(\mathcal{B}) = \mathbf{I}.$$

for "square" tensors $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times I_1\times I_2\times\cdots\times I_k}$ and $\mathbf{I}$ above is $f(\mathcal{I})$, the identity matrix of order $I_1I_2\cdots I_k$. Thus, the condition that $\mathcal{A}^{-1}$ exists as a tensor is equivalent to the condition that $[f(\mathcal{A})]^{-1}$ exists as a matrix. Moreover, $[f(\mathcal{A})]^{-1} = f(\mathcal{B})$, that is,

$$\mathcal{A}^{-1} = \mathcal{B} = f^{-1}\left([f(\mathcal{A})]^{-1}\right). \tag{2.7}$$

The expression (2.7) connects the inverse of a tensor $\mathcal{A}$ with that of its matrix unfolding $f(\mathcal{A})$. We may use it to compute the inverse of an invertible tensor. We illustrate the procedure through an example.

Example 2.2. Let $\mathcal{A} \in \mathbb{R}^{2\times2\times2\times2}$ in [1] where

$$a_{ij11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_{ij21} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad a_{ij12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad a_{ij22} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

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First form $\mathbf{A} = f(\mathcal{A})$ as follows.

$$\mathbf{A} = f(\mathcal{A}) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

A simple calculation indicates that the matrix $\mathcal{A} = f(\mathcal{A})$ is nonsingular and its inverse is

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Finally, applying the inverse transformation f^{-1} onto $\mathbf{A}^{-1}$, we have $\mathcal{A}^{-1} = f^{-1}\left(\mathbf{A}^{-1}\right)$, i.e.,

$$\mathcal{A}^{-1}_{ij11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{A}^{-1}_{ij21} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{A}^{-1}_{ij12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{A}^{-1}_{ij22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next we study the generalized inverses of tensors through their matrix unfoldings under the transformation f with a focus on the $\{2\}$-inverse of a tensor with prescribed range T and nullspace S. We give a necessary and sufficient condition for the existence and uniqueness of such a tensor by extending an analogous result from matrix spaces to tensor spaces. More importantly, we establish an expression similar to the one in (2.7) for this inverse. As special cases, expressions like (2.7) for the weighted Moore-Penrose inverse, the Moore-Penrose inverse, and the Drazin inverse of a tensor are obtained.

To this end, we need to collect a few useful results related to the mapping f.

First, it is easily seen from the definition of f (2.5) that

$$f(\mathcal{A}^*) = (f(\mathcal{A}))^* \quad \text{and} \quad f(\alpha\mathcal{A} + \beta\mathcal{B}) = \alpha f(\mathcal{A}) + \beta f(\mathcal{B}) \tag{2.8}$$

for any scalars α, β in $\mathbb{C}$ and tensors $\mathcal{A}, \mathcal{B}$ in $\mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$. In view of (2.8), it is easy to verify that

$$f(\mathcal{L}) \equiv \{f(\mathcal{X}) : \ \mathcal{X} \in \mathcal{L}\}$$

is a subspace of $\mathbb{C}^{N_1N_2\cdots N_k}$ if and only if $\mathcal{L}$ is a subspace of $\mathbb{C}^{N_1\times\cdots\times N_k}$. Moreover, due to the fact that f is a bijection, we have

$$\dim(\mathcal{L}) = \dim(f(\mathcal{L})). \tag{2.9}$$

It can also be easily verified that

$$\mathcal{L}_1 = \mathcal{L}_2 \quad \text{if and only if} \quad \mathrm{f}(\mathcal{L}_1) = \mathrm{f}(\mathcal{L}_2) \tag{2.10}$$

for subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathbb{C}^{N_1\times N_2\times\cdots\times N_k}$.

Consider the range and nullspace of $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$ defined by

$$\mathrm{R}(\mathcal{A}) = \{\mathcal{A} * \mathcal{X} : \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_k}\} \quad \text{and} \quad \mathrm{N}(\mathcal{A}) = \{\mathcal{X} \in \mathbb{C}^{\mathrm{J}_1\times\cdots\times\mathrm{J}_k} : \mathcal{A} * \mathcal{X} = 0\}.$$

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It is easily seen from (2.6) that

$$f(\mathrm{R}(\mathcal{A})) = \{f(\mathcal{A}) \cdot f(\mathcal{X}) : f(\mathcal{X}) \in \mathbb{C}^{J_1 J_2 \cdots J_k}\} = \mathrm{R}(f(\mathcal{A})). \tag{2.11}$$

In view of (2.6), $\mathcal{A} * \mathcal{X} = 0$ is equivalent to $f(\mathcal{A}) \cdot f(\mathcal{X}) = 0$ which, together with (2.6), further implies

$$f(\mathrm{N}(\mathcal{A})) = \{f(\mathcal{X}) \in \mathbb{C}^{J_1 J_2 \cdots J_k} : f(\mathcal{A}) \cdot f(\mathcal{X}) = 0\} = \mathrm{N}(f(\mathcal{A})). \tag{2.12}$$

Combining (2.9) with (2.11), we have

$$\mathrm{rank}(f(\mathcal{A})) = \dim(\mathrm{R}(f(\mathcal{A}))) = \dim(f(\mathrm{R}(\mathcal{A}))) = \dim(\mathrm{R}(\mathcal{A})). \tag{2.13}$$

Replacing $\mathcal{A}$ by $\mathcal{A}^*$ in (2.13), with the help of (2.8), we have

$$\dim(\mathrm{R}(\mathcal{A}^*)) = \mathrm{rank}(f(\mathcal{A}^*)) = \mathrm{rank}([f(\mathcal{A})]^*) = \mathrm{rank}(f(\mathcal{A})).$$

Therefore, we have $\dim(\mathrm{R}(\mathcal{A})) = \dim(\mathrm{R}(\mathcal{A}^*))$, which is equal to the rank of $f(\mathcal{A})$. We comment that the fact that $\dim(\mathrm{R}(\mathcal{A})) = \dim(\mathrm{R}(\mathcal{A}^*))$ was first observed in [14, Lemma 2.2].

The rank of the tensor $\mathcal{A}$ is defined as the dimension of $\mathrm{R}(\mathcal{A})$. With this definition, we have proved the following result.

THEOREM 2.3. *The rank of a tensor $\mathcal{A}$, i.e., $\dim(\mathrm{R}(\mathcal{A}))$, is equal to the rank of its matrix unfolding $f(\mathcal{A})$. Both $\mathcal{A}$ and $\mathcal{A}^*$ have the same rank.*

The set of all tensors in $\mathbb{C}^{I_1 \times \cdots \times I_k \times J_1 \times \cdots \times J_k}$ with rank r is denoted by $\mathbb{C}_r^{I_1 \times \cdots \times I_k \times J_1 \times \cdots \times J_k}$.

THEOREM 2.4. *Let $\mathcal{A} \in \mathbb{C}_r^{I_1 \times \cdots \times I_k \times J_1 \times \cdots \times J_k}$, $\mathcal{T}$ be a subspace of $\mathbb{C}^{J_1 \times J_2 \times \cdots \times J_k}$ of a dimension $t \le r$ and $\mathcal{S}$ a subspace of $\mathbb{C}^{I_1 \times I_2 \times \cdots \times I_k}$ of dimension $I_1 I_2 \cdots I_k - t$. Then $\mathcal{A}$ has a $\{2\}$-inverse $\mathcal{X}$ such that $\mathrm{R}(\mathcal{X}) = \mathcal{T}$ and $\mathrm{N}(\mathcal{X}) = \mathcal{S}$ if and only if*

$$\mathcal{A} * \mathcal{T} \oplus \mathcal{S} = \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_k}, \tag{2.14}$$

in which case, X is unique and denoted by $\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}$. Moreover, $\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}$ exists if and only if $[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}$ exists and

$$\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}} = f^{-1}\left([f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\right). \tag{2.15}$$

Proof. Under the transformation f, (1.2) is equivalent to

$$f(\mathcal{X}) \cdot f(\mathcal{A}) \cdot f(\mathcal{X}) = f(\mathcal{X}).$$

Thus, $\mathcal{X}$ is a $\{2\}$-inverse of the tensor $\mathcal{A}$ if and only if $f(\mathcal{X})$ is a $\{2\}$-inverse of the matrix $f(\mathcal{A}) \in \mathbb{C}^{I_1 I_2 \cdots I_k \times J_1 J_2 \cdots J_k}$. Notice that $\mathrm{rank}(f(\mathcal{A})) = \mathrm{rank}(\mathcal{A}) = r$. It is seen from (2.11) and (2.12) that

$$\mathrm{R}(\mathcal{X}) = \mathcal{T} \quad \text{if and only if} \quad \mathrm{R}(\mathrm{f}(\mathcal{X})) = \mathrm{f}(\mathcal{T})$$

and

$$\mathrm{N}(\mathcal{X}) = \mathcal{S} \quad \text{if and only if} \quad \mathrm{N}(\mathrm{f}(\mathcal{X})) = \mathrm{f}(\mathcal{S}).$$

Moreover, we also have $\dim(f(\mathcal{T})) = \dim(\mathcal{T}) = t \le r$ and $\dim(f(\mathcal{S})) = \dim(\mathcal{S}) = I_1 I_2 \cdots I_k - t$. Thus, $\mathcal{A}$ has a $\{2\}$-inverse $\mathcal{X}$ such that $\mathrm{R}(\mathcal{X}) = \mathcal{T}$ and $\mathrm{N}(\mathcal{X}) = \mathcal{S}$ if and only if $f(\mathcal{A})$ has a $\{2\}$-inverse $f(\mathcal{X})$ such

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that $\mathrm{R}(f(\mathcal{X})) = f(\mathcal{T})$ and $\mathrm{N}(f(\mathcal{X})) = f(\mathcal{S})$ which, in view of [33, Theorem 1.3.8, p. 24] or [2, Theorem 14, p. 72], is equivalent to

$$(f(\mathcal{A}) \cdot f(\mathcal{T})) \oplus f(\mathcal{S}) = \mathbb{C}^{I_1 I_2 \cdots I_k}. \tag{2.16}$$

Finally, the condition (2.16) is equivalent to (2.14), due to (2.6) and (2.8). Thus, we have $f(\mathcal{X}) = [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}$ which further deduces the result in (2.15). ∎

Theorem 2.4 is the extension of [33, Theorem 1.3.8, p. 24] from matrix spaces to even-order tensor spaces with the Einstein product. The expression (2.15) is a direct extension of the formula (2.7) for the regular inverse to the $\{2\}$-inverse with prescribed range and null space. Based on this formula, $\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}$ can be obtained by first obtaining the $\{2\}$-inverse of its matrix unfolding $f(\mathcal{A})$ with the prescribed range $f(\mathcal{T})$ and nullspace $f(\mathcal{S})$ and then folding back to its original tensor space through f^{-1}. One may employ the methods in [28, 12, 37] or any other ones in the literature for the $\{2\}$-inverse of the matrix $f(\mathcal{A})$ with the prescribed range $f(\mathcal{T})$ and nullspace $f(\mathcal{S})$ for such a task.

Now, let us turn to the three special cases: the weighted Moore-Penrose inverse, the Moore-Penrose inverse, and the Drazin inverse of an even-order tensor.

For $\mathcal{P} \in \mathbb{C}^{N_1\times\cdots\times N_k\times N_1\times\cdots\times N_k}$, if there exists a unitary tensor $\mathcal{U}$ such that

$$\mathcal{P} = \mathcal{U} * \mathcal{D} * \mathcal{U}^*, \tag{2.17}$$

where $\mathcal{D}$ is a diagonal tensor with positive diagonal entries, then $\mathcal{P}$ is said to be Hermitian positive definite [13]. In view of (2.6) and (2.8), the factorization (2.17) is equivalent to

$$f(\mathcal{P}) = f(\mathcal{U}) \cdot f(\mathcal{D}) \cdot [f(\mathcal{U})]^*.$$

Thus, $\mathcal{P}$ is a Hermitian positive definite tensor if and only if $f(\mathcal{P})$ is a Hermitian positive definite matrix in $\mathbb{C}^{N_1\cdots N_k\times N_1\cdots N_k}$.

If $\mathcal{M}$ and $\mathcal{N}$ are Hermitian positive definite in $\mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ and $\mathbb{C}^{J_1\times\cdots\times J_k\times J_1\times\cdots\times J_k}$ respectively, then there exists a unique $\mathcal{X}$ in $\mathbb{C}^{J_1\times\cdots\times J_k\times I_1\times\cdots\times I_k}$ satisfying

$$\mathcal{A} * \mathcal{X} * \mathcal{A} = \mathcal{A}, \tag{2.18}$$

$$\mathcal{X} * \mathcal{A} * \mathcal{X} = \mathcal{X}, \tag{2.19}$$

$$(\mathcal{M} * \mathcal{A} * \mathcal{X})^* = \mathcal{M} * \mathcal{A} * \mathcal{X}, \tag{2.20}$$

$$(\mathcal{N} * \mathcal{X} * \mathcal{A})^* = \mathcal{N} * \mathcal{X} * \mathcal{A} \tag{2.21}$$

and the unique solution is the weighted Moore-Penrose inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{\dagger}_{\mathcal{M}\mathcal{N}}$. It is shown in [13, Theorem 3.4] that

$$\mathrm{R}\left(\mathcal{A}^{\dagger}_{\mathcal{M}\mathcal{N}}\right) = \mathrm{R}\left(\mathcal{A}^{\#}\right) \quad \text{and} \quad \mathrm{N}\left(\mathcal{A}^{\dagger}_{\mathcal{M}\mathcal{N}}\right) = \mathrm{N}\left(\mathcal{A}^{\#}\right), \tag{2.22}$$

where

$$\mathcal{A}^{\#} = \mathcal{N}^{-1} * \mathcal{A}^* * \mathcal{M}. \tag{2.23}$$

It is seen from (2.22) and Lemmas 3.2 and 3.3 of [13] that

$$\mathcal{A} * \mathrm{R}\left(\mathcal{A}^{\#}\right) \oplus \mathrm{N}\left(\mathcal{A}^{\#}\right) = \mathbb{C}^{I_1\times\cdots\times I_k}. \tag{2.24}$$

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Moreover, in view of (2.6), (2.7), (2.8), and (2.23) we have

$$f(\mathcal{A}^{\#}) = [f(\mathcal{N})]^{-1} \cdot [f(\mathcal{A})]^{*} \cdot f(\mathcal{M}), \tag{2.25}$$

which, together with Theorem 2.3, implies

$$\dim\left(\mathrm{R}\left(\mathcal{A}^{\#}\right)\right) = \mathrm{rank}\left(f(\mathcal{A}^{\#}\right) = \mathrm{rank}(f(\mathcal{A})). \tag{2.26}$$

It follows from (2.9), (2.12), and (2.26) that

$$\dim\left(\mathrm{N}\left(\mathcal{A}^{\#}\right)\right) = \dim\left(f(\mathrm{N}\left(\mathcal{A}^{\#}\right))\right) = \dim\left(\mathrm{N}\left(f(\mathcal{A}^{\#})\right)\right) = I_1 I_2 \cdots I_k - \mathrm{rank}\left(f(\mathcal{A})\right). \tag{2.27}$$

With the help of (2.22), (2.24), (2.26), and (2.27), it is seen from Theorem 2.4 that

$$\mathcal{A}^{\dagger}_{\mathcal{M}\mathcal{N}} = \mathcal{A}^{(2)}_{\mathrm{R}(\mathcal{A}^{\#}),\mathrm{N}(\mathcal{A}^{\#})} = f^{-1}\left([f(\mathcal{A})]^{(2)}_{f(\mathrm{R}(\mathcal{A}^{\#})),f(\mathrm{N}(\mathcal{A}^{\#}))}\right). \tag{2.28}$$

Define $[f(\mathcal{A})]^{\#} = [f(\mathcal{N})]^{-1} \cdot [f(\mathcal{A})]^{*} \cdot f(\mathcal{M})$. In view of (2.25), we have $f(\mathcal{A}^{\#}) = [f(\mathcal{A})]^{\#}$. Equations (2.11), (2.12), and (2.25) imply that

$$f(\mathrm{R}(\mathcal{A}^{\#})) = \mathrm{R}(f(\mathcal{A}^{\#})) = \mathrm{R}([f(\mathcal{A})]^{\#}) \quad \text{and} \quad \mathrm{f}(\mathrm{N}(\mathcal{A}^{\#})) = \mathrm{N}(\mathrm{f}(\mathcal{A}^{\#})) = \mathrm{N}([\mathrm{f}(\mathcal{A})]^{\#}),$$

which together with (2.28) imply

$$\mathcal{A}^{\dagger}_{\mathcal{M}\mathcal{N}} = f^{-1}\left([f(\mathcal{A})]^{(2)}_{\mathrm{R}([f(\mathcal{A})]^{\#}),\mathrm{N}([f(\mathcal{A})]^{\#})}\right) = f^{-1}\left([f(\mathcal{A})]^{\dagger}_{f(\mathcal{M}),f(\mathcal{N})}\right), \tag{2.29}$$

where the last equality directly comes from [38, Lemma 1.2].

Also, the expression for the (unweighted) Moore-Penrose inverse of a tensor can be obtained from (2.29) by taking $\mathcal{M}$ and $\mathcal{N}$ to be the identity matrices of proper orders and the fact that $f(\mathcal{I}) = I$.

In summary, we have derived the following result.

THEOREM 2.5. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$. If $\mathcal{M}$ and $\mathcal{N}$ are Hermitian positive definite in $\mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ and $\mathbb{C}^{J_1\times\cdots\times J_k\times J_1\times\cdots\times J_k}$ respectively, then*

$$\mathcal{A}^{\dagger}_{\mathcal{M}\mathcal{N}} = \mathcal{A}^{(2)}_{\mathrm{R}(\mathcal{A}^{\#}),\mathrm{N}(\mathcal{A}^{\#})} = f^{-1}\left([f(\mathcal{A})]^{\dagger}_{f(\mathcal{M}),f(\mathcal{N})}\right). \tag{2.30}$$

In particular, we have

$$\mathcal{A}^{\dagger} = \mathcal{A}^{(2)}_{\mathrm{R}(\mathcal{A}^{*}),\mathrm{N}(\mathcal{A}^{*})} = f^{-1}\left([f(\mathcal{A})]^{\dagger}\right). \tag{2.31}$$

The expression (2.30) for $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ was derived as a special case of the $\{2\}$-inverse with prescribed range and nullspace. Alternatively, it can be shown through the algebraic equations (2.18)-(2.21). $\mathcal{X} = \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ is the unique solution of (2.18)-(2.21). Taking the transformation f on both sides of four equations (2.18)-(2.21), with the help of (2.6) and (2.8), we have

$$f(\mathcal{A}) \cdot f(\mathcal{X}) \cdot f(\mathcal{A}) = f(\mathcal{A}), \tag{2.32}$$

$$f(\mathcal{X}) \cdot f(\mathcal{A}) \cdot f(\mathcal{X}) = f(\mathcal{X}), \tag{2.33}$$

$$(f(\mathcal{M}) \cdot f(\mathcal{A}) \cdot f(\mathcal{X}))^{*} = f(\mathcal{M}) \cdot f(\mathcal{A}) \cdot f(\mathcal{X}), \tag{2.34}$$

$$(f(\mathcal{N}) \cdot f(\mathcal{X}) \cdot f(\mathcal{A}))^{*} = f(\mathcal{N}) \cdot f(\mathcal{X}) \cdot f(\mathcal{A}), \tag{2.35}$$

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which implies that $f(\mathcal{X}) = [f(\mathcal{A})]^{\dagger}_{f(\mathcal{M}),f(\mathcal{N})}$. Therefore, we have $\mathcal{X} = f^{-1}\left([f(\mathcal{A})]^{\dagger}_{f(\mathcal{M}),f(\mathcal{N})}\right)$.

The concept of the index and the Drazin inverse of an even-order tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ was introduced recently in [14]. The smallest non-negative integer p such that $\mathrm{R}\left(\mathcal{A}^{p+1}\right) = \mathrm{R}\left(\mathcal{A}^{p}\right)$, denoted by $\mathrm{Ind}(\mathcal{A})$, is called the index of $\mathcal{A}$. Let $\mathrm{Ind}(\mathcal{A}) = i$. If l is a non-negative integer and $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_k\times I_1\times\cdots\times I_k}$ is such that

$$\mathcal{A}^{l+1} * \mathcal{X} = \mathcal{A}^{l}, \quad \mathcal{X} * \mathcal{A} * \mathcal{X} = \mathcal{X}, \quad \text{and} \quad \mathcal{A} * \mathcal{X} = \mathcal{X} * \mathcal{A}, \tag{2.36}$$

then $l \geq i$ and there exists a unique solution, denoted by $\mathcal{A}_d$, to (2.36). It is shown [14, Theorem 3.2] that

$$\mathrm{R}(\mathcal{A}^{i}) \oplus \mathrm{N}(\mathcal{A}^{i}) = \mathbb{C}^{I_1\times\cdots\times I_k}, \tag{2.37}$$

which implies that $\dim(\mathrm{N}(\mathcal{A}^{i})) = I_1 I_2 \cdots I_k - \dim(\mathrm{R}(\mathcal{A}^{i}))$. Due to the fact that $\mathrm{R}(\mathcal{A}^{i}) \subseteq \mathrm{R}(\mathcal{A})$, we have $\dim(\mathrm{R}(\mathcal{A}^{i})) \leq \dim(\mathrm{R}(\mathcal{A})) = \mathrm{rank}(\mathcal{A})$. Since $\mathcal{A} * \mathrm{R}(\mathcal{A}^{i}) = \mathrm{R}(\mathcal{A}^{i+1}) = \mathrm{R}(\mathcal{A}^{i})$, (2.37) can be re-written as

$$\mathcal{A} * \mathrm{R}(\mathcal{A}^{i}) \oplus \mathrm{N}(\mathcal{A}^{i}) = \mathbb{C}^{I_1\times\cdots\times I_k}.$$

Therefore, it is seen from Theorem 2.4 that

$$\mathcal{A}_d = \mathcal{A}^{(2)}_{\mathrm{R}(\mathcal{A}^{i}),\mathrm{N}(\mathcal{A}^{i})} = f^{-1}\left([f(\mathcal{A})]^{(2)}_{f(\mathrm{R}(\mathcal{A}^{i})),f(\mathrm{N}(\mathcal{A}^{i}))}\right). \tag{2.38}$$

THEOREM 2.6. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times I_1\times I_2\times\cdots\times I_k}$ *and* $\mathrm{Ind}(\mathcal{A}) = i$. *Then we have* $\mathrm{Ind}(f(\mathcal{A})) = i$ *and*

$$\mathcal{A}_d = \mathcal{A}^{(2)}_{\mathrm{R}(\mathcal{A}^{i}),\mathrm{N}(\mathcal{A}^{i})} = f^{-1}\left([f(\mathcal{A})]_d\right). \tag{2.39}$$

Proof. In view of (2.6), (2.11), and (2.12), for any nonnegative integer p, we have

$$f(\mathrm{R}(\mathcal{A}^{p})) = \mathrm{R}(f(\mathcal{A}^{p})) = \mathrm{R}([f(\mathcal{A})]^{p}) \quad \text{and} \quad f(\mathrm{N}(\mathcal{A}^{p})) = \mathrm{N}(f(\mathcal{A}^{p})) = \mathrm{N}([f(\mathcal{A})]^{p}), \tag{2.40}$$

which, together with (2.10), lead to

$$\mathrm{R}(\mathcal{A}^{p+1}) = \mathrm{R}(\mathcal{A}^{p}) \quad \text{is equivalent to} \quad \mathrm{R}([\mathrm{f}(\mathcal{A})]^{\mathrm{p}+1}) = \mathrm{R}([\mathrm{f}(\mathcal{A})]^{\mathrm{p}}).$$

Therefore, we have $\mathrm{Ind}(f(\mathcal{A})) = \mathrm{Ind}(\mathcal{A}) = i$. The expression (2.39) immediately follows from (2.38), (2.40), and the expression for Drazin inverse of the matrix unfolding $f(\mathcal{A})$ [38, Lemma 1.2]

$$[f(\mathcal{A})]_d = [f(\mathcal{A})]^{(2)}_{\mathrm{R}([f(\mathcal{A})]^{i}),\mathrm{N}([f(\mathcal{A})]^{i})}. \qquad \square$$

Again, the expression (2.39) was derived as a special case of the {2}-inverse with prescribed range and nullspace. It can also be proved through the algebraic equations (2.36) which define the Drazin inverse. In view of (2.6) and (2.36), we have

$$[f(\mathcal{A})]^{l+1} \cdot f(\mathcal{X}) = [f(\mathcal{A})]^{l}, \quad f(\mathcal{X}) \cdot f(\mathcal{A}) \cdot f(\mathcal{X}) = f(\mathcal{X}), \quad \text{and} \quad f(\mathcal{A}) \cdot f(\mathcal{X}) = f(\mathcal{X}) \cdot f(\mathcal{A}),$$

implying that $f(\mathcal{X}) = [f(\mathcal{A})]_d$. Thus, expression (2.39) follows immediately.

The expressions (2.30), (2.31), and (2.39) can be viewed as extensions of (2.7) from the regular inverse to the weighted Moore-Penrose, the Moore-Penrose, and the Drazin inverses of an even-order tensor. One application of these expressions is that they can be used to compute these generalized inverses of a tensor through any existing algorithm for the corresponding generalized inverses of matrices. Let us illustrate the procedure through an example which computes the Moore-Penrose inverse of a tensor.

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EXAMPLE 2.7. Let $\mathcal{B} \in \mathbb{R}^{2\times2\times2\times2}$ in [1] where

$$b_{ij11} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad b_{ij21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b_{ij12} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad b_{ij22} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

First form $\mathbf{B} = f(\mathcal{B})$ as follows.

$$\mathbf{B} = f(\mathcal{B}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\mathbf{B} = f(\mathcal{B})$ is singular and its Moore-Penrose inverse is

$$\mathbf{B}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Finally, folding $\mathbf{B}^\dagger$ back to the tensor space $\mathbb{R}^{2\times2\times2\times2}$ by f^{-1}, we have $\mathcal{B}^\dagger = f^{-1}\left(\mathbf{B}^\dagger\right)$, i.e.,

$$\mathcal{B}^\dagger_{ij11} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B}^\dagger_{ij21} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \mathcal{B}^\dagger_{ij12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{B}^\dagger_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Recall that if the index of a matrix A is one, then the Drazin of A is called the group inverse of A, denoted by A_g. A characterization of the outer inverse of a matrix through group inverse was given in [38].

LEMMA 2.8. *Let $A \in \mathbb{C}_r^{m\times n}$, let T be a subspace of $\mathbb{C}^n$ of dimension $t \le r$, and S be a subspace of $\mathbb{C}^m$ of dimension $m-t$. In addition, suppose $G \in \mathbb{C}^{n\times m}$ such that $\mathrm{R}(G) = T$ and $\mathrm{N}(G) = S$. If A has a $\{2\}$-inverse $A^{(2)}_{T,S}$, then*

$$\mathrm{Ind}(A \cdot G) = \mathrm{Ind}(G \cdot A) = 1. \tag{2.41}$$

Furthermore, we have

$$A^{(2)}_{T,S} = G \cdot (A \cdot G)_g = (G \cdot A)_g \cdot G. \tag{2.42}$$

We end up this section by extending Lemma 2.8 from the matrix spaces to the tensor spaces of even-order with the Einstein product. The Drazin inverse of a tensor $\mathcal{A}$ of index 1 is also called the group inverse and denoted by $\mathcal{A}_g$.

THEOREM 2.9. *Let $\mathcal{A} \in \mathbb{C}_r^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}$, $\mathcal{T}$ be a subspace of $\mathbb{C}^{J_1\times J_2\times\cdots\times J_k}$ of a dimension $t \le r$ and $\mathcal{S}$ a subspace of $\mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$ of dimension $I_1 I_2 \cdots I_k - t$. In addition, let $\mathcal{G} \in \mathbb{C}^{J_1\times\cdots\times J_k\times I_1\times\cdots\times I_k}$ such that $\mathrm{R}(\mathcal{G}) = \mathcal{T}$ and $\mathrm{N}(\mathcal{G}) = \mathcal{S}$. If $\mathcal{A}$ has a $\{2\}$-inverse $\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}$, then*

$$\mathrm{Ind}(\mathcal{A} * \mathcal{G}) = \mathrm{Ind}(\mathcal{G} * \mathcal{A}) = 1. \tag{2.43}$$

Furthermore,

$$\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}} = \mathcal{G} * (\mathcal{A} * \mathcal{G})_g = (\mathcal{G} * \mathcal{A})_g * \mathcal{G}. \tag{2.44}$$

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Proof. Consider $f(\mathcal{A}) \in \mathbb{C}^{I_1 I_2 \cdots I_k \times J_1 J_2 \cdots J_k}$ and $f(\mathcal{G}) \in \mathbb{C}^{J_1 J_2 \cdots J_k \times I_1 I_2 \cdots I_k}$. If $\mathcal{A}$ has a $\{2\}$-inverse $\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}$, then $[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}$ exists in view of Theorem 2.4. It is seen from (2.11) and (2.12) that $\mathrm{R}(f(\mathcal{G})) = f(\mathrm{R}(\mathcal{G})) = f(\mathcal{T})$ and $\mathrm{N}(f(\mathcal{G})) = f(\mathrm{N}(\mathcal{G})) = f(\mathcal{S})$. Applying Lemma 2.8 to $f(\mathcal{A})$ and $f(\mathcal{G})$, with the help of (2.6) and Theorem 2.6, we can write

$$\mathrm{Ind}(\mathcal{A} * \mathcal{G}) = \mathrm{Ind}(f(\mathcal{A} * \mathcal{G})) = \mathrm{Ind}(f(\mathcal{A}) \cdot f(\mathcal{G})) = 1,$$

$$\mathrm{Ind}(\mathcal{G} * \mathcal{A}) = \mathrm{Ind}(f(\mathcal{G} * \mathcal{A})) = \mathrm{Ind}(f(\mathcal{G}) \cdot f(\mathcal{A})) = 1,$$

and

$$[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} = [f(\mathcal{A})]^{(2)}_{\mathrm{R}(f(\mathcal{G})),\mathrm{N}(f(\mathcal{G}))} = f(\mathcal{G}) \cdot [f(\mathcal{A}) \cdot f(\mathcal{G})]_g = [f(\mathcal{G}) \cdot f(\mathcal{A})]_g \cdot f(\mathcal{G}). \tag{2.45}$$

The results of (2.44) are immediately from Theorem 2.4 and Theorem 2.6 with the help of (2.6). ∎

3. The outer inverse under a small perturbation. Let us recall a result on the perturbation for the outer inverse $A^{(2)}_{T,S}$ of matrix A in [39, 40].

LEMMA 3.1. *Let $B = A + E \in \mathbb{C}^{m\times n}$, and let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, such that $AT \oplus S = \mathbb{C}^m$. Suppose $\mathrm{R}(E) \subseteq AT$ and $\mathrm{R}(E^*) \subseteq A^* S^\perp$. If $\Delta_1 \equiv ||EA^{(2)}_{T,S}||_F < 1$, then $B^{(2)}_{T,S}$ exists and*

$$B^{(2)}_{T,S} = [I + A^{(2)}_{T,S} E]^{-1} A^{(2)}_{T,S} = A^{(2)}_{T,S} [I + EA^{(2)}_{T,S}]^{-1}.$$

In addition, we have

$$\frac{\Delta_1}{(1+\Delta_1)\kappa(A)} \le \frac{||B^{(2)}_{T,S} - A^{(2)}_{T,S}||_F}{||A^{(2)}_{T,S}||_F} \le \frac{\Delta_1}{1-\Delta_1}$$

where $\kappa(A) = ||A||_F ||A^{(2)}_{T,S}||_F$ is the condition number [40] of the generalized inverse $A^{(2)}_{T,S}$ and $||\cdot||_F$ is the Frobenius norm of matrices.

We comment that Lemma 3.1 was established in [39] for a general norm. Next, we will extend this result to the outer inverse of a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_k \times J_1 \times \cdots \times J_k}$ under the Frobenius norm which is defined as

$$||\mathcal{A}||_F = \left(\sum_{i_r \in [I_r], j_r \in [J_r], r \in [k]} |\mathcal{A}_{i_1 i_2 \cdots i_k j_1 j_2 \cdots j_k}|^2 \right)^{\frac{1}{2}}. \tag{3.46}$$

Since the matrix unfolding of a tensor is just the re-arrangement of its elements into a matrix, we obviously have

$$||\mathcal{A}||_F = ||f(\mathcal{A})||_F. \tag{3.47}$$

Define the inner product on $\mathbb{C}^{I_1 \times \cdots \times I_k}$

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{i_r \in [I_r], r \in [k]} \overline{\mathcal{X}}_{i_1 \ldots i_k} \mathcal{Y}_{i_1 \ldots i_k} \quad \text{for all } \ \mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_k}. \tag{3.48}$$

The orthogonal complement of a subspace $\mathcal{L}$ in $\mathbb{C}^{I_1 \times \cdots \times I_k}$ is defined by

$$\mathcal{L}^\perp = \left\{ \mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_k} : \ \langle \mathcal{X}, \mathcal{Y} \rangle = 0 \quad \text{for all } \ \mathcal{Y} \in \mathcal{L} \right\}.$$

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LEMMA 3.2. *For any subspace* $\mathcal{L} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$, *we have* $[f(\mathcal{L})]^\perp = f(\mathcal{L}^\perp)$.

Proof. Observe that $z \in [f(\mathcal{L})]^\perp$ is equivalent to $\langle z, f(\mathcal{X})\rangle = 0$ for all $\mathcal{X} \in \mathcal{L}$. Define $\mathcal{Z} \equiv f^{-1}(z) \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$. In view of (3.48), we can write

$$\langle z, f(\mathcal{X})\rangle = \sum_{i=1}^{I_1 I_2\cdots I_k} \bar{z}_i\ [f(\mathcal{X})]_i = \sum_{i_r\in[I_r], r\in[k]} \bar{\mathcal{Z}}_{i_1 i_2\cdots i_k}\mathcal{X}_{i_1 i_2\cdots i_k} = \langle \mathcal{Z}, \mathcal{X}\rangle.$$

Therefore, $z \in [f(\mathcal{L})]^\perp$ is equivalent to $f^{-1}(z) \in \mathcal{L}^\perp$, i.e., $z \in f(\mathcal{L}^\perp)$. ∎

Define $\kappa(\mathcal{A}) = ||\mathcal{A}||_F||\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F$ for the generalized inverse $\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}$ of tensor $\mathcal{A}$.

THEOREM 3.3. *Let* $\mathcal{B} = \mathcal{A} + \mathcal{E} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$, *and let* $\mathcal{T}$ *and* $\mathcal{S}$ *be subspaces of* $\mathbb{C}^{J_1\times J_2\times\cdots\times J_k}$ *and* $\mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$, *respectively, such that* $\mathcal{A} * \mathcal{T} \oplus \mathcal{S} = \mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$. *Suppose* $\mathrm{R}(\mathcal{E}) \subseteq \mathcal{A} * \mathcal{T}$ *and* $\mathrm{R}(\mathcal{E}^*) \subseteq \mathcal{A}^* * \mathcal{S}^\perp$. *If* $\Delta \equiv ||\mathcal{E} * \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F < 1$, *then* $\mathcal{B}^{(2)}_{\mathcal{T},\mathcal{S}}$ *exists and*

$$\mathcal{B}^{(2)}_{\mathcal{T},\mathcal{S}} = [\mathcal{I} + \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}} * \mathcal{E}]^{-1} * \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}} = \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}} * [\mathcal{I} + \mathcal{E} * \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}]^{-1}. \tag{3.49}$$

In addition, we have

$$\frac{\Delta}{(1+\Delta)\kappa(\mathcal{A})} \le \frac{||\mathcal{B}^{(2)}_{\mathcal{T},\mathcal{S}} - \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F}{||\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F} \le \frac{\Delta}{1-\Delta}. \tag{3.50}$$

Proof. Note that $f(\mathcal{B}) = f(\mathcal{A}) + f(\mathcal{E}) \in \mathbb{C}^{I_1 I_2\cdots I_k\times J_1 J_2\cdots J_k}$. It is seen from the proof of Theorem 2.4 that $f(\mathcal{T})$ and $f(\mathcal{S})$ are subspaces of $\mathbb{C}^{J_1 J_2\cdots J_k}$ and $\mathbb{C}^{I_1 I_2\cdots I_k}$, respectively, and in addition, we have $f(\mathcal{A})f(\mathcal{T}) \oplus f(\mathcal{S}) = \mathbb{C}^{I_1 I_2\cdots I_k}$. With the help of (2.6), (2.8), (2.11), (3.47), Theorem 2.4, and Lemma 3.2, we can write

$$\mathrm{R}(f(\mathcal{E})) = f(\mathrm{R}(\mathcal{E})) \subseteq f(\mathcal{A} * \mathcal{T}) = f(\mathcal{A})f(\mathcal{T}),$$
$$\mathrm{R}\left([f(\mathcal{E})]^*\right) = \mathrm{R}\left(f(\mathcal{E}^*)\right) = f\left(\mathrm{R}(\mathcal{E}^*)\right) \subseteq f\left(\mathcal{A}^* * \mathcal{S}^\perp\right) = f(\mathcal{A}^*)f\left(\mathcal{S}^\perp\right) = [f(\mathcal{A})]^*\,[f(\mathcal{S})]^\perp,$$

and

$$\Delta_2 \equiv ||f(\mathcal{E})\,[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\,||_F = ||f(\mathcal{E})f\left(\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}\right)||_F = ||f\left(\mathcal{E} * \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}\right)||_F = ||\mathcal{E} * \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F = \Delta < 1.$$

Being applied to $f(\mathcal{A})$ and $f(\mathcal{B})$, Lemma 3.2 indicates that $[f(\mathcal{B})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}$ exists,

$$[f(\mathcal{B})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} = \left(I + [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}f(\mathcal{E})\right)^{-1}[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} \tag{3.51}$$
$$= [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\left(I + f(\mathcal{E})\,[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\right)^{-1}, \tag{3.52}$$

and

$$\frac{\Delta_2}{(1+\Delta_2)\kappa(f(\mathcal{A}))} \le \frac{||\,[f(\mathcal{B})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} - [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\,||_F}{||\,[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\,||_F} \le \frac{\Delta_2}{1-\Delta_2}. \tag{3.53}$$

In view of (2.6), (2.7), (2.8), (3.51), and Theorem 2.4, we have

$$\begin{aligned}\mathcal{B}^{(2)}_{\mathcal{T},\mathcal{S}} &= f^{-1}\left([f(\mathcal{B})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\right) = f^{-1}\left(\left(I + [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}f(\mathcal{E})\right)^{-1}[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\right)\\ &= f^{-1}\left(\left[I + [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}f(\mathcal{E})\right]^{-1}\right) * f^{-1}\left([f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})}\right)\\ &= [\mathcal{I} + \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}} * \mathcal{E}]^{-1} * \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}.\end{aligned}$$

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Thus, the first equality of (3.49) is derived from (3.51) and the second equality of (3.49) can be derived similarly from (3.52).

We have observed $\Delta_2 = \Delta$. Moreover, in view of (2.8), (3.47), and Theorem 2.4, we can write

$$|| \, [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} \, ||_F = ||f\left(\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}\right)||_F = ||\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F,$$

$$\kappa(f(\mathcal{A})) = ||f(\mathcal{A})||_F|| \, [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} \, ||_F = ||\mathcal{A}||_F||\mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F = \kappa(\mathcal{A}),$$

and

$$|| \, [f(\mathcal{B})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} - [f(\mathcal{A})]^{(2)}_{f(\mathcal{T}),f(\mathcal{S})} \, ||_F = ||f\left(\mathcal{B}^{(2)}_{\mathcal{T},\mathcal{S}} - \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}\right)||_F = ||\mathcal{B}^{(2)}_{\mathcal{T},\mathcal{S}} - \mathcal{A}^{(2)}_{\mathcal{T},\mathcal{S}}||_F$$

which, together with (3.53), immediately implies (3.50). □

4. The generalized inverses and full-rank factorizations. In a recently published paper [1], the authors asked about whether or not there exists a full rank factorization of a tensor and how to compute the Moore-Penrose inverse from its full rank factorization if there is one. As we know, the answer to this question is affirmative in the matrix case [2]. It is well-known that for $A \in \mathbb{C}^{m\times n}_r$ there exist $B \in \mathbb{C}^{m\times r}_r$ and $C \in \mathbb{C}^{r\times n}_r$ such that $A = BC$ and $A^\dagger = C^*\left(B^*AC^*\right)^{-1}B^*$. This classical result was extended to the $\{2\}$-inverse of a matrix A with prescribed range and nullspace in ([27, Theorem 3.1], [29, 37]).

Lemma 4.1. *Let $A \in \mathbb{C}^{m\times n}_r$, let T be a subspace of $\mathbb{C}^n$ of dimension $t \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - t$. In addition, suppose $G \in \mathbb{C}^{n\times m}$ such that $\mathrm{R}(G) = T$ and $\mathrm{N}(G) = S$.If A has a $\{2\}$-inverse $A^{(2)}_{T,S}$, then QAP is invertible and $A^{(2)}_{T,S} = P(QAP)^{-1}Q$ for any full-rank decomposition $G = PQ$ of G.*

Definition 4.2. For a given tensor $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}_r$, the factorization $\mathcal{A} = \mathcal{B}*\mathcal{C}$ is called a full rank factorization of $\mathcal{A}$ if $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_k\times R_1\times\cdots\times R_k}_r$ and $\mathcal{C} \in \mathbb{C}^{R_1\times\cdots\times R_k\times J_1\times\cdots\times J_k}_r$ for positive integers $R_1, R_2, \cdots, R_k$ such that $R_1R_2\cdots R_k = r$.

With the help of the matrix unfolding f and its inverse f^{-1}, the existence of a full rank factorization of a tensor can be easily established and Lemma 4.1 can be extended to the $\{2\}$-inverse of a tensor $\mathcal{A}$ with prescribed range and nullspace.

Theorem 4.3. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}_r$. Then there exist two $2k$-order tensors $\mathcal{B} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times r\times 1\times\cdots\times 1}_r$ and $\mathcal{C} \in \mathbb{C}^{r\times 1\times\cdots\times 1\times J_1\times J_2\times\cdots\times J_k}_r$ such that $\mathcal{A} = \mathcal{B}*\mathcal{C}$.*

Proof. Consider the matrix unfolding $f(\mathcal{A})$ of the tensor $\mathcal{A}$. It is seen from Theorem 2.3 that $\mathrm{rank}(f(\mathcal{A})) = r$ and thus, there exist $B \in \mathbb{C}^{I_1I_2\cdots I_k\times r}_r$ and $C \in \mathbb{C}^{r\times J_1J_2\cdots J_k}_r$ such that $f(\mathcal{A}) = B\cdot C$. Define

$$\mathcal{B} \equiv f^{-1}(B) \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times r\times 1\times\cdots\times 1} \quad \text{and} \quad \mathcal{C} \equiv f^{-1}(C) \in \mathbb{C}^{r\times 1\times\cdots\times 1\times J_1\times J_2\times\cdots\times J_k}.$$

Again, in view of Theorem 2.3, we have $\mathrm{rank}(\mathcal{B}) = \mathrm{rank}(f(\mathcal{B})) = \mathrm{rank}(B) = r$ and $\mathrm{rank}(\mathcal{C}) = \mathrm{rank}(f(\mathcal{C})) = \mathrm{rank}(C) = r$. In addition, in view of (2.6), we have

$$\mathcal{A} = f^{-1}\left(f(\mathcal{A})\right) = f^{-1}\left(B\cdot C\right) = f^{-1}\left(B\right)*f^{-1}\left(C\right) = \mathcal{B}*\mathcal{C}. \qquad \square$$

Theorem 4.4. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_k\times J_1\times\cdots\times J_k}_r$, $\mathcal{T}$ be a subspace of $\mathbb{C}^{J_1\times J_2\times\cdots\times J_k}$ of a dimension $t \le r$ and $\mathcal{S}$ a subspace of $\mathbb{C}^{I_1\times I_2\times\cdots\times I_k}$ of dimension $I_1I_2\cdots I_k - t$. In addition, let $\mathcal{G} \in \mathbb{C}^{J_1\times\cdots\times J_k\times I_1\times\cdots\times I_k}$ such that*

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$\mathrm{R}(\mathcal{G}) = T$ *and* $\mathrm{N}(\mathcal{G}) = S$ *and let* $\mathcal{G} = \mathcal{P} * \mathcal{Q}$ *be any full-rank decomposition of* $\mathcal{G}$*. If* A *has a* $\{2\}$*-inverse* $A^{(2)}_{T,S}$*, then* $\mathcal{Q} * \mathcal{A} * \mathcal{P}$ *is invertible and*

$$\mathcal{A}^{(2)}_{T,S} = \mathcal{P} * (\mathcal{Q} * \mathcal{A} * \mathcal{P})^{-1} * \mathcal{Q}. \tag{4.54}$$

Proof. Under the assumptions of the theorem, $f(\mathcal{A}) \in \mathbb{C}_r^{I_1 \cdots I_k \times J_1 \cdots J_k}$, $f(\mathcal{T})$ is a subspace of $\mathbb{C}^{J_1 J_2 \cdots J_k}$ of a dimension $t \leq r$, and $f(\mathcal{S})$ is a subspace of $\mathbb{C}^{I_1 I_2 \cdots I_k}$ of dimension $I_1 I_2 \cdots I_k - t$. In view of (2.11) and (2.12), we also have $f(T) = f(\mathrm{R}(\mathcal{G})) = \mathrm{R}(f(\mathcal{G}))$ and $f(S) = f(\mathrm{N}(\mathcal{G})) = \mathrm{N}(f(\mathcal{G}))$. In addition, $f(\mathcal{G}) = f(\mathcal{P}) \cdot f(\mathcal{Q})$ is a full-rank factorization of $f(\mathcal{G})$. In view of Theorem 2.4, $[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}), f(\mathcal{S})}$ exists. It is seen from Lemma 4.1 that $f(\mathcal{Q}) \cdot f(\mathcal{A}) \cdot f(\mathcal{P})$ is invertible and

$$[f(\mathcal{A})]^{(2)}_{f(\mathcal{T}), f(\mathcal{S})} = f(\mathcal{P}) \cdot (f(\mathcal{Q}) \cdot f(\mathcal{A}) \cdot f(\mathcal{P}))^{-1} \cdot f(\mathcal{Q}). \tag{4.55}$$

In view of (2.6), (2.7), (2.15), and (4.55), we have that $\mathcal{Q} * \mathcal{A} * \mathcal{P}$ is invertible since $f(\mathcal{Q}) \cdot f(\mathcal{A}) \cdot f(\mathcal{P})$ is invertible and we can write

$$\begin{aligned}
\mathcal{A}^{(2)}_{T,S} &= f^{-1}\left([f(\mathcal{A})]^{(2)}_{f(\mathcal{T}), f(\mathcal{S})}\right) \\
&= f^{-1}\left(f(\mathcal{P}) \cdot f\left([\mathcal{Q} * \mathcal{A} * \mathcal{P}]^{-1}\right) \cdot f(\mathcal{Q})\right) \\
&= \mathcal{P} * (\mathcal{Q} * \mathcal{A} * \mathcal{P})^{-1} * \mathcal{Q}. \qquad \square
\end{aligned}$$

Let $\mathcal{A} \in \mathbb{C}_r^{I_1 \times \cdots \times I_k \times J_1 \times \cdots \times J_k}$, and $\mathcal{A} = \mathcal{B} * \mathcal{C}$ be a full rank factorization of $\mathcal{A}$. Then it follows from Theorem 2.3 and Definition 4.2 that $\mathcal{A}^* = \mathcal{C}^* * \mathcal{B}^*$ is a full rank factorization of $\mathcal{A}^*$. In view of (2.31), we may take $\mathcal{G} = \mathcal{A}^*$ in (4.54) for the Moore-Penrose inverse, resulting in

$$\mathcal{A}^{\dagger} = \mathcal{C}^* * (\mathcal{B}^* * \mathcal{A} * \mathcal{C}^*)^{-1} * \mathcal{B}^*. \tag{4.56}$$

In order to get an expression for the weighted Moore-Penrose inverse, we take $\mathcal{G} = \mathcal{A}^{\#}$ due to (2.30), which has a full rank factorization

$$\mathcal{A}^{\#} = \mathcal{N}^{-1} * \mathcal{A}^* * \mathcal{M} = \left(\mathcal{N}^{-1} * \mathcal{C}^*\right) * (\mathcal{B}^* * \mathcal{M}).$$

Thus, the expression in (4.54) becomes

$$\begin{aligned}
\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} &= \mathcal{N}^{-1} * \mathcal{C}^* * \left(\mathcal{B}^* * \mathcal{M} * \mathcal{A} * \mathcal{N}^{-1} * \mathcal{C}^*\right)^{-1} * \mathcal{B}^* * \mathcal{M} \\
&= \mathcal{N}^{-1} * \mathcal{C}^* * \left[(\mathcal{B}^* * \mathcal{M} * \mathcal{B}) * \left(\mathcal{C} * \mathcal{N}^{-1} * \mathcal{C}^*\right)\right]^{-1} * \mathcal{B}^* * \mathcal{M}.
\end{aligned} \tag{4.57}$$

Finally, let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_k \times I_1 \times \cdots \times I_k}$ with $\mathrm{Ind}(\mathcal{A}) = i$. Notice that we can take $\mathcal{G} = \mathcal{A}^i$ for the Drazin inverse of $\mathcal{A}$ as indicated in (2.39). For any full rank factorization $\mathcal{A}^i = \mathcal{P} * \mathcal{Q}$ of $\mathcal{A}^i$, which always exists as indicated by Theorem 4.3, we have

$$\mathcal{A}_d = \mathcal{P} * (\mathcal{Q} * \mathcal{A} * \mathcal{P})^{-1} * \mathcal{Q}, \tag{4.58}$$

for the Drazin inverses of tensors as a special case of (4.54).

5. The reverse order rule. Next, we will turn our attention to the reverse order rules for the Moore-Penrose inverse and weighted Moore-Penrose inverse of tensors with the Einstein product. Recall that the two term reverse order law $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$ for two matrices A and B was studied by Greville [11] and Arghiriade about a half century ago. Their results are collected in the following lemma (see [2, pages 160-161]).

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LEMMA 5.1. *Let A and B be two matrices such that AB exists. Then the following statements are equivalent:*

(i) $(AB)^\dagger = B^\dagger A^\dagger$;
(ii) $\mathrm{R}(A^*ABB^*) = \mathrm{R}(BB^*A^*A)$;
(iii) $\mathrm{R}(A^*AB) \subseteq \mathrm{R}(B)$ *and* $\mathrm{R}(BB^*\mathcal{A}^*) \subseteq \mathrm{R}(A^*)$.

There exist many results in the literature on the reverse order for the Moore-Penrose inverses of matrices. As an application of the expression (2.31) for the Moore-Penrose inverses of tensors, the one in Lemma 5.1 will be extended in the next theorem to the tensor space of even-order with the Einstein product. In a similar way, we can actually extend many other results from matrix space to tensor space, including the ones in [22].

THEOREM 5.2. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k\times L_1\times L_2\times\cdots\times L_k}$. Then, the following statements are equivalent:*

(i) $(\mathcal{A} * \mathcal{B})^\dagger = \mathcal{B}^\dagger * \mathcal{A}^\dagger$;
(ii) $\mathrm{R}(\mathcal{A}^* * \mathcal{A} * \mathcal{B} * \mathcal{B}^*) = \mathrm{R}(\mathcal{B} * \mathcal{B}^* * \mathcal{A}^* * \mathcal{A})$;
(iii) $\mathrm{R}(\mathcal{A}^* * \mathcal{A} * \mathcal{B}) \subseteq \mathrm{R}(\mathcal{B})$ *and* $\mathrm{R}\left(\mathcal{B} * \mathcal{B}^* * \mathcal{A}^*\right) \subseteq \mathrm{R}(\mathcal{A}^*)$.

Proof. Notice that $(\mathcal{A} * \mathcal{B})^\dagger = \mathcal{B}^\dagger * \mathcal{A}^\dagger$ is equivalent to $f((\mathcal{A} * \mathcal{B})^\dagger) = f(\mathcal{B}^\dagger * \mathcal{A}^\dagger)$ which, together with (2.6) and (2.31), is also equivalent to

$$[f(\mathcal{A}) \cdot f(\mathcal{B})]^\dagger = [f(\mathcal{B})]^\dagger \cdot [f(\mathcal{A})]^\dagger. \tag{5.59}$$

For matrices $f(\mathcal{A})$ and $f(\mathcal{B})$, it is seen from Lemma 5.1 that (5.59) holds if and only if

$$\mathrm{R}\left([f(\mathcal{A})]^* \cdot f(\mathcal{A}) \cdot f(\mathcal{B}) \cdot [f(\mathcal{B})]^*\right) = \mathrm{R}\left(f(\mathcal{B}) \cdot [f(\mathcal{B})]^* \cdot [f(\mathcal{A})]^* \cdot f(\mathcal{A})\right) \tag{5.60}$$

holds. With the help of (2.6) and (2.8), and (2.11), the condition (5.60) is equivalent to

$$f\left(\mathrm{R}(\mathcal{A}^* * \mathcal{A} * \mathcal{B} * \mathcal{B}^*)\right) = f\left(\mathrm{R}(\mathcal{B} * \mathcal{B}^* * \mathcal{A}^* * \mathcal{A})\right), \tag{5.61}$$

which is obviously equivalent to $\mathrm{R}(\mathcal{A}^* * \mathcal{A} * \mathcal{B} * \mathcal{B}^*) = \mathrm{R}(\mathcal{B} * \mathcal{B}^* * \mathcal{A}^* * \mathcal{A})$ due to (2.10). Therefore, part (i) is equivalent to part (ii).

Similarly, part (i) holds if and only if (5.59) holds which, in view of Lemma 5.1, is equivalent to

$$\mathrm{R}(f(A)^* \cdot f(A) \cdot f(B)) \subseteq \mathrm{R}(f(B)) \quad \text{and} \quad \mathrm{R}(f(B) \cdot f(B)^* \cdot f(\mathcal{A})^*) \subseteq \mathrm{R}(f(A)^*),$$

or equivalently,

$$\mathrm{R}\left(\mathcal{A}^* * \mathcal{A} * \mathcal{B}\right) \subseteq \mathrm{R}\left(\mathcal{B}\right) \quad \text{and} \quad \mathrm{R}\left(\mathcal{B} * \mathcal{B}^* * \mathcal{A}^*\right) \subseteq \mathrm{R}\left(\mathcal{A}^*\right).$$

Thus, part (i) is equivalent to part (iii). ◻

We end this paper up with the following extension of Theorem 5.2 to the weighted Moore-Penrose inverse of even-order tensors. Our approach closely follows the lines of [31].

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THEOREM 5.3. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times J_1\times J_2\times\cdots\times J_k}$ *and* $\mathcal{B} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k\times L_1\times L_2\times\cdots\times L_k}$. *Let* $\mathcal{M} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_k\times I_1\times I_2\times\cdots\times I_k}, \mathcal{L} \in \mathbb{C}^{L_1\times L_2\times\cdots\times L_k\times L_1\times L_2\times\cdots\times L_k}$, *and* $\mathcal{N} \in \mathbb{C}^{J_1\times J_2\times\cdots\times J_k\times J_1\times\cdots\times J_k}$ *be Hermitian positive tensors. And let* $\mathcal{A}^{\#}$ *be given in (2.23) and define* $\mathcal{B}^{\#} = \mathcal{L}^{-1} * B^* * \mathcal{N}$. *Then the following statements are equivalent:*

(i) $(\mathcal{A}*\mathcal{B})^{\dagger}_{\mathcal{M},\mathcal{L}} = \mathcal{B}^{\dagger}_{\mathcal{N},\mathcal{L}} * \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$;
(ii) $\mathrm{R}(\mathcal{A}^{\#}*\mathcal{A}*\mathcal{B}*\mathcal{B}^{\#}) = \mathrm{R}(\mathcal{B}*\mathcal{B}^{\#}*\mathcal{A}^{\#}*\mathcal{A})$;
(iii) $\mathrm{R}(\mathcal{A}^{\#}*\mathcal{A}*\mathcal{B}) \subseteq \mathrm{R}(\mathcal{B})$ *and* $\mathrm{R}(\mathcal{B}*\mathcal{B}^{\#}*\mathcal{A}^{\#}) \subseteq \mathrm{R}(\mathcal{A}^{\#})$.

Proof. It is seen from (2.11) of [13] that

$$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{N}^{-1/2} * \left(\mathcal{M}^{1/2}*\mathcal{A}*\mathcal{N}^{-1/2}\right)^{\dagger} * \mathcal{M}^{1/2},$$
$$\mathcal{B}^{\dagger}_{\mathcal{N},\mathcal{L}} = \mathcal{L}^{-1/2} * \left(\mathcal{N}^{1/2}*\mathcal{B}*\mathcal{L}^{-1/2}\right)^{\dagger} * \mathcal{N}^{1/2},$$
$$(\mathcal{A}*\mathcal{B})^{\dagger}_{\mathcal{M},\mathcal{L}} = \mathcal{L}^{-1/2} * \left(\mathcal{M}^{1/2}*(\mathcal{A}*\mathcal{B})*\mathcal{L}^{-1/2}\right)^{\dagger} * \mathcal{M}^{1/2}.$$

Therefore, $(\mathcal{A}*\mathcal{B})^{\dagger}_{\mathcal{M},\mathcal{L}} = \mathcal{B}^{\dagger}_{\mathcal{N},\mathcal{L}} * \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ is equivalent to

$$\left(\mathcal{M}^{1/2}*(\mathcal{A}*\mathcal{B})*\mathcal{L}^{-1/2}\right)^{\dagger} = \left(\mathcal{N}^{1/2}*\mathcal{B}*\mathcal{L}^{-1/2}\right)^{\dagger} * \left(\mathcal{M}^{1/2}*\mathcal{A}*\mathcal{N}^{-1/2}\right)^{\dagger}$$

or

$$\left(\tilde{\mathcal{A}}*\tilde{\mathcal{B}}\right)^{\dagger} = \tilde{\mathcal{B}}^{\dagger}*\tilde{\mathcal{A}}^{\dagger}, \tag{5.62}$$

where $\tilde{\mathcal{A}} = \mathcal{M}^{1/2}*\mathcal{A}*\mathcal{N}^{-1/2}$ and $\tilde{\mathcal{B}} = \mathcal{N}^{1/2}*\mathcal{B}*\mathcal{L}^{-1/2}$. Now, in view of Theorem 5.2, the reverse order (5.62) of the Moore-Penrose inverse of $\tilde{\mathcal{A}}*\tilde{\mathcal{B}}$ holds if and only if

$$\mathrm{R}\left(\tilde{\mathcal{A}}^**\tilde{\mathcal{A}}*\tilde{\mathcal{B}}*\tilde{\mathcal{B}}^*\right) = \mathrm{R}\left(\tilde{\mathcal{B}}*\tilde{\mathcal{B}}^**\tilde{\mathcal{A}}^**\tilde{\mathcal{A}}\right)$$

or

$$\mathrm{R}\left(\mathcal{N}^{-1/2}*\mathcal{A}^**\mathcal{M}*\mathcal{A}*\mathcal{B}*\mathcal{L}^{-1}*\mathcal{B}^**\mathcal{N}^{1/2}\right) = \mathrm{R}\left(\mathcal{N}^{1/2}*\mathcal{B}*\mathcal{L}^{-1}*\mathcal{B}^**\mathcal{A}^**\mathcal{M}*\mathcal{A}*\mathcal{N}^{-1/2}\right),$$

which can be readily re-written as $\mathrm{R}(\mathcal{A}^{\#}*\mathcal{A}*\mathcal{B}*\mathcal{B}^{\#}) = \mathrm{R}(\mathcal{B}*\mathcal{B}^{\#}*\mathcal{A}^{\#}*\mathcal{A})$. Thus, part (i) is equivalent to part (ii).

Similarly, (5.62) is also equivalent to

$$\mathrm{R}(\tilde{\mathcal{A}}^**\tilde{\mathcal{A}}*\tilde{\mathcal{B}}) \subseteq \mathrm{R}(\tilde{\mathcal{B}}) \quad \text{and} \quad \mathrm{R}\left(\tilde{\mathcal{B}}*\tilde{\mathcal{B}}^**\tilde{\mathcal{A}}^*\right) \subseteq \mathrm{R}(\tilde{\mathcal{A}}^*)$$

from which $\mathrm{R}(\mathcal{A}^{\#}*\mathcal{A}*\mathcal{B}) \subseteq \mathrm{R}(\mathcal{B})$ and $\mathrm{R}(\mathcal{B}*\mathcal{B}^{\#}*\mathcal{A}^{\#}) \subseteq \mathrm{R}(\mathcal{A}^{\#})$ can be easily deduced. Thus, part (i) is equivalent to part (iii). ∎

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