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ORDERED MULTIPLICITY INVERSE EIGENVALUE PROBLEM FOR GRAPHS ON SIX VERTICES*

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Abstract. For a graph G, we associate a family of real symmetric matrices, $\mathcal{S}(G)$, where for any $M \in \mathcal{S}(G)$, the location of the nonzero off-diagonal entries of M is governed by the adjacency structure of G. The ordered multiplicity *Inverse Eigenvalue Problem of a Graph (IEPG)* is concerned with finding all attainable ordered lists of eigenvalue multiplicities for matrices in $\mathcal{S}(G)$. For connected graphs of order six, we offer significant progress on the IEPG, as well as a complete solution to the ordered multiplicity IEPG. We also show that while $K_{m,n}$ with $\min(m,n) \geq 3$ attains a particular ordered multiplicity list, it cannot do so with arbitrary spectrum.

Key words. Inverse eigenvalue problem, Ordered multiplicity, Strong spectral property, Cloning, Spectrally arbitrary.

AMS subject classification. 05C50.

1. Introduction.

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. Given G with vertices $v_1, \ldots, v_n$, a real symmetric matrix M is in $\mathcal{S}(G)$ if for all $i \neq j$, $M_{i,j} = 0$ if and only if $v_i v_j \notin E(G)$; there are no restrictions on the diagonal entries.

The *spectrum* of a matrix M is the set of eigenvalues of M. Let $\lambda_1 < \cdots < \lambda_k$ be the distinct eigenvalues of M in increasing order, and let γ_i be the multiplicity of λ_i as an eigenvalue of M. Then the *ordered multiplicity list* of M is $(\gamma_1, \ldots, \gamma_k)$. (With this convention, the spectrum of M is $\{\lambda_1^{(\gamma_1)}, \lambda_2^{(\gamma_2)}, \ldots, \lambda_k^{(\gamma_k)}\}$.)

The *Inverse Eigenvalue Problem of a Graph (IEPG)* is stated as follows: given G and a set of numbers $L = \{\ell_1, \ldots, \ell_n\}$, does there exist a matrix $M \in \mathcal{S}(G)$ with spectrum L?

This problem has been completely resolved through graphs on five vertices (see [5]). More information about the IEPG can be found in the survey of Hogben [12]. Because of the difficulty of the IEPG, many relaxations have been considered; previous works have examined inverse inertia (see [7]), minimum rank and maximum nullity (see [9]), and the minimum number of distinct eigenvalues (see [8]).

We consider the *ordered multiplicity inverse eigenvalue problem for graphs*, a slight relaxation of the IEPG: given a graph G and an ordered list of integers $\Gamma = (\gamma_1, \ldots, \gamma_k)$, does there exist a matrix $M \in \mathcal{S}(G)$ that attains Γ as its ordered multiplicity list?

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We label graphs using the *Atlas of Graphs* [13]; these graphs are also reproduced in the appendix for reference. In this paper, we solve the ordered multiplicity IEPG for all connected graphs of order six. The result is summarized in Figure 1: the graphs are in 26 different equivalence classes based on what ordered multiplicity lists are attainable. To determine what lists are attainable, locate the equivalence class it belongs in and then read off all ordered multiplicity lists (and reversals) on the edges of a directed path from that equivalence class to $\emptyset$. This diagram also gives all possible relationships between equivalence classes, namely the equivalence class containing G attains all ordered multiplicity lists as the equivalence class containing H if and only if there is a directed path from the equivalence class containing G to the one containing H.

A graph G is *spectrally arbitrary* for an ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$ if for any $\lambda_1 < \cdots < \lambda_k$, there is a matrix $M \in \mathcal{S}(G)$ with spectrum $\{\lambda_1^{(\gamma_1)}, \ldots, \lambda_k^{(\gamma_k)}\}$. Many of the techniques we use show that a graph is spectrally arbitrary for an ordered multiplicity list. In the appendix, for each ordered multiplicity list, we give all graphs which can attain that list, indicating those which are not known to be attainable with arbitrary spectrum.

We proceed as follows. In Section 2, we will review what is known for the IEPG for graphs on five or fewer vertices which we will build on for the case of six vertices. In Section 3, we will introduce a technique we call cloning and how it connects with ordered multiplicity lists of eigenvalues. In Section 4, we justify what ordered multiplicity lists are unattainable, while in Section 5, we justify what ordered multiplicity lists are attainable for graphs on six vertices. In Section 6, we show that $K_{m,n}$ with $\min(m, n) \geq 3$ is a graph for which the ordered multiplicity IEPG differs from the IEPG. Finally, in Section 7, we give concluding remarks.

Because we are working on graphs with six or fewer vertices, it will be unambiguous to write the ordered multiplicity list $(\gamma_1, \ldots, \gamma_k)$ as $\gamma_1 \ldots \gamma_k$. We will say a graph G attains $\gamma_1 \ldots \gamma_k$ if there is some $M \in \mathcal{S}(G)$ with multiplicity list $\gamma_1 \ldots \gamma_k$; similarly, G does not attain $\gamma_1 \ldots \gamma_k$ if there is no $M \in \mathcal{S}(G)$ with multiplicity list $\gamma_1 \ldots \gamma_k$. We note that a graph attains $\gamma_1 \ldots \gamma_k$ if and only if it attains $\gamma_k \ldots \gamma_1$, which follows by noting if $M \in \mathcal{S}(G)$ then so also is $-M$.

2. IEPG for graphs on five or fewer vertices.

The IEPG for all graphs of order at most five was solved by Barrett et al. [5]. They showed that for graphs with five or fewer vertices, the IEPG is equivalent to the ordered multiplicity IEPG. Thus, a graph on five or fewer vertices attains a given spectrum if and only if the corresponding multiplicity list is attainable. Two of the main tools used to solve this problem were the strong spectral property (SSP) and the strong multiplicity property (SMP), introduced in an earlier paper (see [6]).

Definition 2.1. *An $n \times n$ symmetric matrix A has the SSP if the only symmetric matrix X satisfying $A \circ X = I \circ X = AX - XA = O$ is $X = O$ (where '$\circ$' indicates the Hadamard, or entry-wise, product of matrices).*

Definition 2.2. *An $n \times n$ symmetric matrix A satisfies the SMP if the only symmetric matrix X satisfying $A \circ X = I \circ X = AX - XA = O$ and $tr(A^i X) = 0$ for $i = 2, \ldots, n-1$ is $X = O$.*

These properties are important for the IEPG because SSP (SMP) allows us to determine the attainability of certain spectra (ordered multiplicity lists) for many graphs simultaneously. It should be noted that testing if a matrix has SSP (SMP) reduces to showing if a large linear system has full rank. This has been implemented and is available online (see [11]); any matrix which is claimed to have SSP follows either from Barrett et al. [5] or by using the online implementation.

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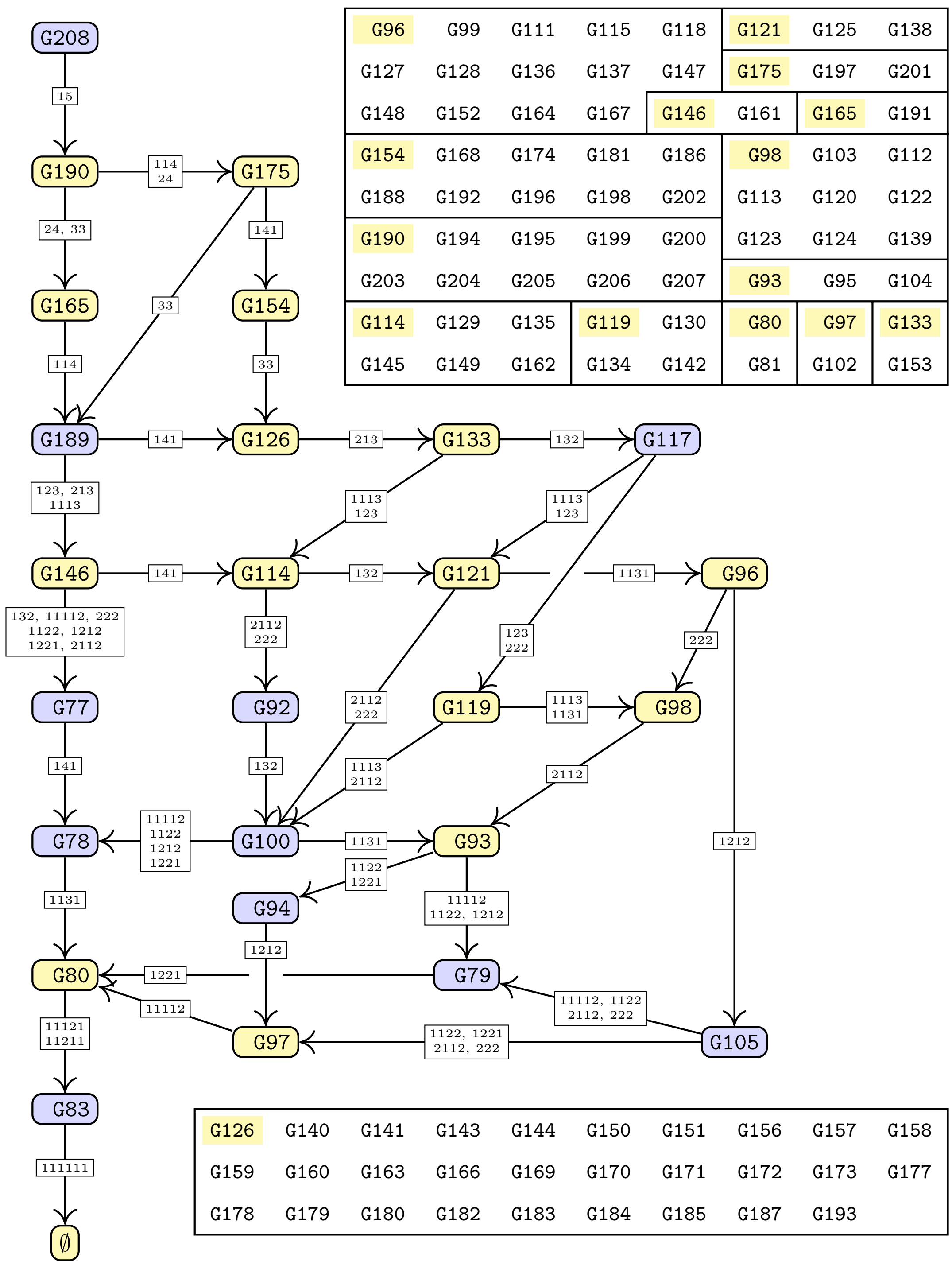

Figure 1. *There are* 26 *equivalence classes based on attainable ordered multiplicity lists; those in blue have one graph while those in yellow have their full membership given in the side boxes. To determine attainable ordered multiplicity lists for a graph, find its equivalence class in the diagram and take* any *path to* $\emptyset$*; the multiplicity lists (and reversals) that occur on the edges of the path are the only ones attainable. The graph* G *attains all multiplicity lists that* H *attains if and only if there is a directed path from the class containing* G *to the class containing* H*; the difference in what is attainable are the multiplicity lists which occur on* any *directed path between them.*

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THEOREM 2.3 (Barrett et al. [6] Theorems 2.10 and 2.20). *If $A \in \mathcal{S}(G)$ has SSP (SMP), then every* supergraph *of G with the same vertex set has an SSP (SMP) realization with the same spectrum (ordered multiplicity list).*

THEOREM 2.4 (Barrett et al. [6], Theorem 3.8). *If $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$ both have SSP (SMP) and* $\operatorname{spec}(A) \cap \operatorname{spec}(B) = \emptyset$, *then $A \oplus B \in \mathcal{S}(G \cup H)$ has SSP (SMP).*

Using the above theorems and several constructions, Barrett et al. [5] determined that the multiplicity lists given in Table 1 are attainable with SSP, and those in Table 2 are attainable but without SMP or SSP. Moreover, they established that the graphs can attain any spectrum compatible with the ordered multiplicity lists it attains. Thus, the IEPG and the ordered multiplicity IEPG are equivalent for graphs on five or fewer vertices.

TABLE 1
Realizable ordered multiplicity lists for connected graphs with five or fewer vertices and are attainable with SSP.

Graphs	Attainable ordered multiplicity lists
G1	1
G3	11
G6	111
G7	111, 12, 21
G14	1111
G13	1111, 121
G15	1111, 121, 112, 211
G16, G17	1111, 121, 112, 211, 22
G18	1111, 121, 112, 211, 22, 13, 31
G31	11111
G29, G30	11111, 1121, 1211
G35, G36	11111, 1121, 1211, 1112, 2111
G34, G38	11111, 1121, 1211, 1112, 2111, 122, 221
G37, G40, G41 G42, G43, G47	11111, 1121, 1211, 1112, 2111, 122, 221, 212
G44, G46	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131
G45	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311
G48, G49 G50, G51	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G52	11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32, 14, 41

TABLE 2
Realizable ordered multiplicity lists for connected graphs with five or fewer vertices which cannot have SMP or SSP.

Graphs	Attainable ordered multiplicity lists
G29	131
G42	113, 131, 311

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3. Cloning vertices and ordered multiplicity lists. In this section, we introduce *cloning*, a graph operation that, given G and $v \in V(G)$, constructs a new graph H by adding a new vertex v' which is a clone (or twin) of v. This operation is sometimes referred to as duplicating or blow-ups.

DEFINITION 3.1. *Two vertices u and w are* twins *in H if $N_H(u) \setminus \{w\} = N_H(w) \setminus \{u\}$, where $N_H(v)$ is the set of neighbors of v (i.e., vertices which share an edge with v).*

Twins do not need to be adjacent. This leads to two variants of cloning: *cloning v with an edge* requires that $v \sim v'$ (i.e., v and v' are adjacent), while *cloning without an edge* requires $v \not\sim v'$ (i.e., v and v' are not adjacent).

THEOREM 3.2. *Let G be a graph with $M \in \mathcal{S}(G)$ with multiplicity list $(\gamma_1, \ldots, \gamma_k)$ where the eigenvalue 0 has multiplicity γ_i. Then the following two cases are possible:*

1. *If the diagonal entry of M corresponding to v_j is zero, then the graph H attained from G by cloning v_j* without *an edge has a matrix $N \in \mathcal{S}(H)$ that attains the multiplicity list $(\gamma_1, \ldots, \gamma_i + 1, \ldots \gamma_k)$.*
2. *If the diagonal entry of M corresponding to v_j is nonzero, then the graph H attained from G by cloning v_j* with *an edge has a matrix $N \in \mathcal{S}(H)$ that attains the multiplicity list $(\gamma_1, \ldots, \gamma_i+1, \ldots \gamma_k)$.*

Proof. Let $-\lambda_1 \leq \cdots \leq \lambda_{n-\gamma_i}$ be the nonzero eigenvalues of M where a is the last index such that $-\lambda_a < 0$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-\gamma_i}$ be the corresponding orthonormal eigenvectors. Then

$$\begin{aligned}
M &= \sum_{k=1}^{n-\gamma_i} \lambda_k \mathbf{x}_k \mathbf{x}_k^T \\
&= \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_{n-\gamma_i} \end{pmatrix} \begin{pmatrix} -\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_i} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_{n-\gamma_i}^T \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\lambda_1}\mathbf{x}_1 & \cdots & \sqrt{\lambda_{n-\gamma_i}}\mathbf{x}_{n-\gamma_i} \end{pmatrix} \underbrace{\begin{pmatrix} -I_a & O \\ O & I_{n-\gamma_i-a} \end{pmatrix}}_{=S} \begin{pmatrix} \sqrt{\lambda_1}\mathbf{x}_1^T \\ \vdots \\ \sqrt{\lambda_{n-\gamma_i}}\mathbf{x}_{n-\gamma_i}^T \end{pmatrix}.
\end{aligned}$$

The columns of

$$Y = \begin{pmatrix} \sqrt{\lambda_1}\mathbf{x}_1^T \\ \vdots \\ \sqrt{\lambda_{n-\gamma_i}}\mathbf{x}_{n-\gamma_i}^T \end{pmatrix},$$

are an orthogonal representation of G with respect to the indefinite inner product S. That is, if $\mathbf{y}_k$ denotes the k-th column, then $\mathbf{y}_k^T S \mathbf{y}_\ell = 0$ if and only if $v_k \not\sim v_\ell$ in G.

Let Z be a $(n - \gamma_i) \times (n + 1)$ matrix with columns as follows:

- $\mathbf{z}_k = \mathbf{y}_k$ for $1 \leq k < j$;
- $\mathbf{z}_j = \mathbf{z}_{j+1} = \frac{1}{\sqrt{2}}\mathbf{y}_j$;
- $\mathbf{z}_k = \mathbf{y}_{k-1}$ for $j + 1 < k \leq n + 1$.

Now consider the matrix $N = Z^T S Z$. Since N is real symmetric, there exists some graph H such that $N \in \mathcal{S}(H)$. The following two observations will now conclude the proof.

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First, use the columns of Z as an orthogonal representation for H with respect to S. This corresponds to the graph G with the vertex v_j cloned (i.e., columns still have the same orthogonality relationships as given by Y). This will have cloned with an edge if and only if $\mathbf{y}_j^T S \mathbf{y}_j \neq 0$. The latter holds if and only if the diagonal entry of M corresponding to v_j is nonzero.

Second, the inner product of any two rows of Z agrees with the inner product of the corresponding rows of Y. So the nonzero eigenvalues of N are

$$SZZ^T = \begin{pmatrix} -I_a & O \\ O & I_{n-\gamma_i-a} \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_i} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-\gamma_i} \end{pmatrix},$$

which are the same as those of M. Hence, N has the same spectrum of M with the addition of a single eigenvalue of 0, giving us the desired ordered multiplicity list. ∎

COROLLARY 3.3. *Let G be a graph without isolated vertices, and let $M \in \mathcal{S}(G)$ with multiplicity list $(m_1, \ldots, m_k)$. If a graph H is attained by cloning $v \in V(G)$ with an edge, then H attains the multiplicity list $(m_1 + 1, \ldots, m_k)$.*

Proof. By translation, we can assume M is positive semidefinite with nullity m_1. If any diagonal entry were 0, then this would force a row (and column) of zeroes which follows by noting otherwise there is a 2×2 submatrix with negative determinant which is impossible. However, this implies that G contains an isolated vertex, a contradiction. Thus, the entries of the diagonal are nonzero, and so we apply the previous theorem by cloning with an edge. ∎

4. Unattainable multiplicity lists for graphs.

In this section, we will determine which ordered multiplicity lists are unattainable for connected graphs on six vertices.

4.1. Using known graph parameters.

Since we can assign any particular eigenvalue to 0 by translation, we have the following observations.

OBSERVATION 1. *If $M(G)$ denotes the maximum nullity of a matrix in $\mathcal{S}(G)$, then all entries of a multiplicity list of a matrix in $\mathcal{S}(G)$ are bounded above by $M(G)$.*

OBSERVATION 2. *If $M_+(G)$ denotes the maximum nullity of a positive semidefinite matrix in $\mathcal{S}(G)$, then the first (and by reversal from negation, the last) entry of a multiplicity list of a matrix in $\mathcal{S}(G)$ is bounded above by $M_+(G)$.*

In general, the computation of $M(G)$ and $M_+(G)$ is an open problem [9]. However, for graphs on seven or fewer vertices, known techniques can find these values (in particular, the inertia tables—see [4]). It suffices to provide an upper bound for these parameters, which can be done through the combinatorial parameters $Z(G)$ and $Z_+(G)$, respectively, known as the zero-forcing number and semidefinite zero-forcing number of a graph. Because of their combinatorial nature, $Z(G)$ and $Z_+(G)$ can be easily computed for small graphs through exhaustive analysis. The definition of these parameters, as well as related extensions and results, can be found in the survey of Fallat and Hogben [9]. For our purposes, we will use the following result.

LEMMA 4.1 (AIM [1], Prop. 2.4). *For any graph G, we have $M(G) \leq Z(G)$.*

LEMMA 4.2 (Barioli et al. [2], Theorem 3.5). *For any graph G, we have $M_+(G) \leq Z_+(G)$.*

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Another useful parameter is $q(G)$, the minimum number of distinct eigenvalues.

Observation 3. *The length of any ordered multiplicity list for $M \in \mathcal{S}(G)$ is at least $q(G)$.*

This is a harder parameter to compute; for connected graphs of order at most six, $q(G)$ was determined (see [8]). For all connected graphs of order six, the parameters $Z(G)$, $Z_+(G)$, and $q(G)$ are given in Table 3 and rule out many ordered multiplicity lists.

4.2. Previous results to rule out ordered multiplicity lists. The following two results, both from [5], rule out several cases.

Lemma 4.3 (Barrett et al. [5], Lemma 3.3). *If G is a connected unicyclic graph with odd girth, then at least one of the first or last eigenvalues has multiplicity one.*

Table 3
The values for $Z(G)$, $Z_+(G)$, and $q(G)$ for connected graphs on six vertices.

G	Z	Z_+	q	G	Z	Z_+	q	G	Z	Z_+	q	G	Z	Z_+	q
G77	4	1	3	G121	3	2	3	G151	3	3	3	G181	3	3	2
G78	3	1	4	G122	2	2	4	G152	2	2	3	G182	3	3	3
G79	2	1	4	G123	2	2	4	G153	3	3	3	G183	3	3	3
G80	2	1	5	G124	2	2	4	G154	3	3	2	G184	3	3	3
G81	2	1	5	G125	3	2	3	G156	3	3	3	G185	3	3	3
G83	1	1	6	G126	3	3	3	G157	3	3	3	G186	3	3	2
G92	3	2	3	G127	2	2	3	G158	3	3	3	G187	3	3	3
G93	2	2	4	G128	2	2	3	G159	3	3	3	G188	3	3	2
G94	2	2	4	G129	3	2	3	G160	3	3	3	G189	4	3	3
G95	2	2	4	G130	3	3	4	G161	4	2	3	G190	4	4	2
G96	2	2	3	G133	3	3	3	G162	3	2	3	G191	4	4	3
G97	2	2	5	G134	3	3	4	G163	3	3	3	G192	3	3	2
G98	2	2	4	G135	3	2	3	G164	2	2	3	G193	3	3	3
G99	2	2	3	G136	2	2	3	G165	4	4	3	G194	4	4	2
G100	3	2	4	G137	2	2	3	G166	3	3	3	G195	4	4	2
G102	2	2	5	G138	3	2	3	G167	2	2	3	G196	3	3	2
G103	2	2	4	G139	2	2	4	G168	3	3	2	G197	4	3	2
G104	2	2	4	G140	3	3	3	G169	3	3	3	G198	3	3	2
G105	2	2	3	G141	3	3	3	G170	3	3	3	G199	4	4	2
G111	2	2	3	G142	3	3	4	G171	3	3	3	G200	4	4	2
G112	2	2	4	G143	3	3	3	G172	3	3	3	G201	4	3	2
G113	2	2	4	G144	3	3	3	G173	3	3	3	G202	3	3	2
G114	3	2	3	G145	3	2	3	G174	3	3	2	G203	4	4	2
G115	2	2	3	G146	4	2	3	G175	4	3	2	G204	4	4	2
G117	3	3	3	G147	2	2	3	G177	3	3	3	G205	4	4	2
G118	2	2	3	G148	2	2	3	G178	3	3	3	G206	4	4	2
G119	3	3	4	G149	3	2	3	G179	3	3	3	G207	4	4	2
G120	2	2	4	G150	3	3	3	G180	3	3	3	G208	5	5	2

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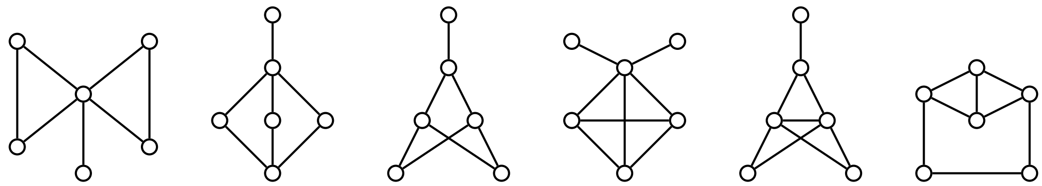

G117: 132, 231, 312, 213 — G121: 132, 231 — G125: 132, 231 — G133: 312, 213 — G138: 132, 231 — G153: 312, 213

FIGURE 2. *The remaining unattainable ordered multiplicity lists.*

This rules out 2112 and 222 for G92, G93, G94, G95, G100, and G104.

LEMMA 4.4 (Barrett et al. [5], Lemmas 2.3 and 5.2). *A generalized star or a generalized* 3*-sun does not allow an ordered multiplicity list with consecutive multiple eigenvalues.*

This rules out 1122, 1221, and 2211 for G77 and G78 (generalized stars) and G94 (3-sun).

The inverse eigenvalue problem for cycles has been determined by Fernandes and Fonseca [10], and in particular it follows that 1212 and 2121 are not attainable for the graph C_6 (G105).

4.3. Remaining cases. After applying the graph parameters, Lemma 4.3, and Lemma 4.4, the remaining unattainable cases are shown in Figure 2.

Proving that these cases are unattainable will be done by contradiction, namely by assuming the existence of a matrix that achieves the specified ordered multiplicity list on a graph. An examination of the corresponding orthogonal representation allows us to argue that two of the vectors are scalar multiples. This pair of vectors allows us to 'declone' the original graph to a smaller graph with a corresponding matrix that has an impossible ordered multiplicity list.

Recall that cloning works by taking a single vertex and creating a pair of vertices where the vector corresponding with each vertex is a (nonzero) scalar multiple of the original. Decloning does the opposite: if we can show that the vectors corresponding to a pair of vertices must be (nonzero) scalar multiples, then the two vertices must be twins. We can then delete one of the twins and decrease an entry in the ordered multiplicity list.

LEMMA 4.5 (Decloning). *Let G be a graph with $M = Q^TSQ \in \mathcal{S}(G)$ having multiplicity list $(\gamma_1, \dots, \gamma_k)$ where the eigenvalue 0 has multiplicity γ_i. Further assume that S is symmetric and has dimension $n - \gamma_i$, where n is the order of G. If the columns of Q corresponding to vertices a and b are scalar multiples, then there exists $N \in \mathcal{S}(H)$ where H is attained from G by deleting vertex a and N has ordered multiplicity list $(\gamma_1, \dots, \gamma_i - 1, \dots, \gamma_k)$.*

Proof. Let $Q = (\mathbf{x}_1\, \mathbf{x}_2\, \cdots\, \mathbf{x}_n)$, and without loss of generality assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are scalar multiples, i.e., $\mathbf{x}_2 = \alpha\mathbf{x}_1$. Let $\widehat{Q} = (\sqrt{1+\alpha^2}\mathbf{x}_2\, \cdots\, \mathbf{x}_n)$; we claim $N = \widehat{Q}^T S\widehat{Q}$.

Note that two vertices u and v in the graph are not adjacent if and only if $\mathbf{x}_u^T S\mathbf{x}_v = 0$. Since scaling by a nonzero value does not change this, we have that the adjacencies are preserved and $N \in \mathcal{S}(H)$. It remains to check the spectrum of N, but for this we note that $QQ^T = \widehat{Q}\widehat{Q}^T$ since the dot products of any pair of corresponding rows are identical. Since the nonzero portion of the spectrum comes from $SQQ^T = S\widehat{Q}\widehat{Q}^T$ (since the nonzero eigenvalues of AB and BA agree), the result follows. ∎

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PROPOSITION 4.6. *If a symmetric matrix M has nullity three and ordered multiplicity* 312*, then we have* $M = Q^T IQ$ *where I is the* 3×3 *identity matrix.*

Proof. The matrix M has spectrum $\{0^{(3)}, \lambda^{(1)}, \mu^{(2)}\}$. Let $\mathbf{x}$ be a unit eigenvector for λ and $\mathbf{y}, \mathbf{z}$ be orthonormal eigenvectors for μ. We have that

$$M = \begin{pmatrix}\mathbf{x}\ \mathbf{y}\ \mathbf{z}\end{pmatrix} \begin{pmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{pmatrix} \begin{pmatrix}\mathbf{x}^T\\ \mathbf{y}^T\\ \mathbf{z}^T\end{pmatrix} = \begin{pmatrix}\sqrt{\lambda}\mathbf{x}\ \sqrt{\mu}\mathbf{y}\ \sqrt{\mu}\mathbf{z}\end{pmatrix} \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \underbrace{\begin{pmatrix}\sqrt{\lambda}\mathbf{x}^T\\ \sqrt{\mu}\mathbf{y}^T\\ \sqrt{\mu}\mathbf{z}^T\end{pmatrix}}_{=Q}.$$

□

THEOREM 4.7. *The graphs* `G117` *and* `G133` *cannot attain ordered multiplicity lists* 312 *or* 213.

Proof. Label the vertices of `G117` as shown in Figure 3. Suppose $M \in \mathcal{S}(\texttt{G117})$ attains ordered multiplicity list 312 with nullity three. Applying Proposition 4.6, we can write $M = Q^T IQ$, where Q is a matrix which forms an orthogonal representation for `G117`. Let $\mathbf{v}_i$ denote the ith column of Q.

Since v_1 and v_3 are not adjacent, the corresponding vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ are orthogonal and thus form a plane in $\mathbb{R}^3$. Both v_5 and v_6 are not adjacent to v_1 and v_3, so $\mathbf{v}_5$ and $\mathbf{v}_6$ are orthogonal to this plane and must be scalar multiples.

The decloning lemma implies that the graph with vertex v_6 deleted has a matrix that attains 212. However, this graph is odd unicyclic and thus cannot attain 212 by Lemma 4.3.

A similar argument establishes the result for `G133`. □

THEOREM 4.8. *The graph* `G153` *cannot attain ordered multiplicity lists* 213 *or* 312.

Proof. Label the vertices of `G153` as shown in Figure 4. Suppose that $M \in \mathcal{S}(\texttt{G153})$ attains ordered multiplicity list 312 with nullity three. Applying Proposition 4.6 we can write $M = Q^T IQ$, where Q forms an orthogonal representation for `G153`. Let $\mathbf{v}_i$ denote the ith column of Q.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are scalar multiples, the decloning lemma implies a matrix for C_5 (the five-cycle) with multiplicity list 212, which is impossible by Lemma 4.3.

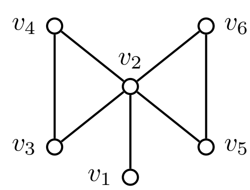

FIGURE 3. *The graph G117.*

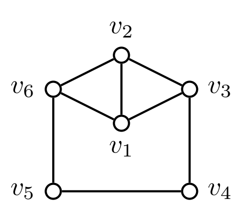

FIGURE 4. *The graph G153.*

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If $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples, then $\mathbf{v}_1$ and $\mathbf{v}_2$ form a plane in $\mathbb{R}^3$. Since vertices v_4 and v_5 are not adjacent to v_1 and v_2, the vectors $\mathbf{v}_4$ and $\mathbf{v}_5$ are orthogonal to the aforementioned plane. Thus, $\mathbf{v}_4$ and $\mathbf{v}_5$ are scalar multiples, an impossibility given that v_4 and v_5 have distinct neighbors.

In either case, we get a contradiction; thus, `G153` cannot attain 312. ∎

For the remaining cases, we first establish an analogous result to Proposition 4.6.

Proposition 4.9. *If a symmetric matrix M has nullity three and ordered multiplicity* 132*, then we have $M = Q^T SQ$ where $S = \operatorname{diag}(-1, 1, 1)$.*

Proof. The matrix M has spectrum $\{-\lambda^{(1)}, 0^{(3)}, \mu^{(2)}\}$. Let $\mathbf{x}$ be a unit eigenvector for $-\lambda$ and $\mathbf{y}, \mathbf{z}$ be orthonormal eigenvectors for μ. We have

$$M = \begin{pmatrix}\mathbf{x} & \mathbf{y} & \mathbf{z}\end{pmatrix} \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{pmatrix} = \begin{pmatrix}\sqrt{\lambda}\mathbf{x} & \sqrt{\mu}\mathbf{y} & \sqrt{\mu}\mathbf{z}\end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{\lambda}\mathbf{x}^T \\ \sqrt{\mu}\mathbf{y}^T \\ \sqrt{\mu}\mathbf{z}^T \end{pmatrix}}_{=Q}.$$

∎

Theorem 4.10. *The graph `G117` cannot attain ordered multiplicity lists* 132 *or* 231*.*

Proof. Label the vertices of `G117` as shown in Figure 3. Suppose that $M \in \mathcal{S}(\texttt{G117})$ attains ordered multiplicity list 132 with nullity three. Applying Proposition 4.9, $M = Q^T SQ$ where $S = \operatorname{diag}(-1, 1, 1)$. Let $\mathbf{v}_i$ denote the ith column of Q.

Since v_1, v_3, and v_5 have distinct sets of neighbors, no two of $\mathbf{v}_1$, $\mathbf{v}_3$, and $\mathbf{v}_5$ are scalar multiples of each other. We will use $N(v)$ to denote the set of neighbors of v.

Because S is invertible, $R = \binom{\mathbf{v}_1^T S}{\mathbf{v}_3^T S}$ has rank two and nullity one. Since $v_1, v_3 \notin N(v_5)$, we have $R\mathbf{v}_5 = \binom{0}{0}$; similarly, $v_1, v_3 \notin N(v_6)$, so $R\mathbf{v}_6 = \binom{0}{0}$. Thus, $\mathbf{v}_5$ and $\mathbf{v}_6$ are scalar multiples. A similar argument allows us to conclude that $\mathbf{v}_3$ and $\mathbf{v}_4$ are scalar multiples.

Now we can apply the decloning lemma *twice* (i.e., once for each scalar multiple pair) to produce a matrix for the graph $K_{1,3}$ which attains 112. But $Z_+(K_{1,3}) = 1$, so the end terms of any multiplicity list of $K_{1,3}$ must be 1 (by Lemma 4.2 and Observation 2), a contradiction. ∎

In the following proposition, we introduce a tool that shows two vectors are scalar multiples, a technique similar to the one used in the previous results.

Proposition 4.11. *If S is a 3×3 symmetric invertible matrix and $\mathbf{x}, \mathbf{y}$ are vectors that satisfy the relationships $\mathbf{x}^T S\mathbf{x} = \mathbf{y}^T S\mathbf{y} = \mathbf{x}^T S\mathbf{y} = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are scalar multiples of each other.*

Proof. Assume $\mathbf{x}$ and $\mathbf{y}$ are not scalar multiples; then $S\mathbf{x}$ and $S\mathbf{y}$ are also not scalar multiples. This implies the matrix

$$R = \begin{pmatrix} \mathbf{x}^T S \\ \mathbf{y}^T S \end{pmatrix},$$

has rank two and nullity one. On the other hand, by our hypothesis we have

$$R\mathbf{x} = \begin{pmatrix} \mathbf{x}^T S\mathbf{x} \\ \mathbf{y}^T S\mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}^T S\mathbf{y} \\ \mathbf{y}^T S\mathbf{y} \end{pmatrix} = R\mathbf{y},$$

which shows that R has nullity at least two, a contradiction. ∎

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(a) Labeled graph.

(b) A clone of the graph with an edge and marked zero-forcing set.

FIGURE 5. *The graph* `G121`.

THEOREM 4.12. *The graph* `G121` *cannot attain ordered multiplicity lists* 132 *or* 231.

Proof. Label the vertices of `G121` as shown in Figure 5(a). Suppose that $M \in \mathcal{S}(\texttt{G121})$ has an ordered multiplicity list 132 with nullity three.

We claim that $M_{3,3} = M_{5,5} = M_{6,6} = 0$. To see this, suppose that $M_{6,6} \neq 0$; cloning v_6 with an edge produces the graph shown in Figure 5(b) which attains an ordered multiplicity list 142. The set marked in Figure 5(b) is a zero-forcing set of order three. Thus, the cloned graph cannot attain 142 (by Lemma 4.1 and Observation 1), a contradiction. Hence, $M_{6,6} = 0$, and $M_{3,3} = M_{5,5} = 0$ by symmetry.

We apply Proposition 4.9 to write $M = Q^T SQ$ where $S = \mathrm{diag}(-1, 1, 1)$. Let $\mathbf{v}_i$ denote the ith column of Q. Since v_3, v_5, and v_6 form an independent set, we have $\mathbf{v}_3^T S\mathbf{v}_6 = \mathbf{v}_3^T S\mathbf{v}_5 = \mathbf{v}_5^T S\mathbf{v}_6 = 0$. Moreover, because $M_{3,3} = M_{5,5} = M_{6,6} = 0$, we have $\mathbf{v}_3^T S\mathbf{v}_3 = \mathbf{v}_5^T S\mathbf{v}_5 = \mathbf{v}_6^T S\mathbf{v}_6 = 0$. From Proposition 4.11, it follows that $\mathbf{v}_3$, $\mathbf{v}_5$, and $\mathbf{v}_6$ are pairwise scalar multiples.

Applying the decloning lemma *twice* produces a matrix for P_4 which attains 112. However P_4, the path on four vertices, can only attain 1111 (see `G14` in Table 1), a contradiction. ∎

THEOREM 4.13. *The graphs* `G125` *and* `G138` *cannot attain ordered multiplicity lists* 132 *or* 231.

Proof. Label the vertices of `G125` as shown in Figure 6(a). Suppose $M \in \mathcal{S}(\texttt{G125})$ has spectrum $\{-\lambda, 0^{(3)}, 2^{(2)}\}$ with $\lambda > 0$ (by scale and shift, this holds without loss of generality).

We claim that $M_{5,5} = M_{6,6} = 0$. To see this, suppose that $M_{5,5} \neq 0$; cloning v_5 with an edge produces the graph shown in Figure 6(b) which attains an ordered multiplicity list 142. The set marked in Figure 6(b) is a

(a) Labeled graph.

(b) A clone of the graph with an edge and marked zero-forcing set.

FIGURE 6. *The graph* `G125`.

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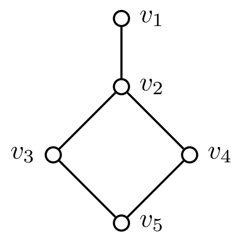

FIGURE 7. *The banner graph.*

zero-forcing set of order three. Thus, the cloned graph cannot attain 142 (by Lemma 4.1 and Observation 1), a contradiction. Hence, $M_{5,5} = 0$, and $M_{6,6} = 0$ by symmetry.

Applying Proposition 4.9 gives $M = Q^T SQ$, where $S = \operatorname{diag}(-1, 1, 1)$. Let $\mathbf{v}_i$ denote the ith column of Q. Note $v_5 \not\sim v_6$, so $\mathbf{v}_5^T S\mathbf{v}_6 = 0$. Moreover, $M_{5,5} = M_{6,6} = 0$, so $\mathbf{v}_5^T S\mathbf{v}_5 = \mathbf{v}_6^T S\mathbf{v}_6 = 0$. From Proposition 4.11, it follows that $\mathbf{v}_5$ and $\mathbf{v}_6$ are scalar multiples.

Applying the decloning lemma, we attain a matrix N for the banner graph as labeled in Figure 7, where N has eigenvalues $\{-\lambda, 0^{(2)}, 2^{(1)}\}$ and $N_{5,5} = 0$; particularly, N has the following form, where c_i are nonzero and d_i are arbitrary:

$$N = \begin{pmatrix} d_1 & c_1 & 0 & 0 & 0 \\ c_1 & d_2 & c_2 & c_3 & 0 \\ 0 & c_2 & d_3 & 0 & c_4 \\ 0 & c_3 & 0 & d_4 & c_5 \\ 0 & 0 & c_4 & c_5 & 0 \end{pmatrix}.$$

The matrix N has rank three; moreover, the first three rows are linearly independent. Therefore, the fifth row is a linear combination of the first three rows, which implies that $c_4 = \alpha c_2$ and $c_5 = \alpha c_3$ for some $\alpha \neq 0$.

Let $R = (N - I)^2$ which has spectrum $\{1^{(4)}, (1 + \lambda)^2\}$. The only graphs that attain the ordered multiplicity list 41 are unions of complete graphs with isolated vertices. Since $R_{1,3} = c_1c_2 \neq 0$ and $R_{1,5} = 0$, we must have that v_1 is in the clique and v_5 is isolated. In particular,

$$0 = R_{2,5} = c_2c_4 + c_3c_5 = \alpha\big(c_2^2 + c_3^2\big) \neq 0,$$

which is impossible.

A similar argument establishes the result for `G138`. □

5. Attainable multiplicity lists for graphs. In this section, we establish which ordered multiplicity lists are attainable for connected graphs on six vertices. We first utilize techniques which will also achieve arbitrary spectrum, and then describe those that fail to do so.

Before we begin, we note that a few special cases have already been done in the literature; we refer the reader elsewhere for details on the following cases.

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- The inverse eigenvalue problem for cycles has been determined by Fernandes and Fonseca.

  THEOREM 5.1 (Fernandes and Fonseca [10], Theorem 3.3). *The numbers* $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ *are the spectrum for some matrix* $M \in \mathcal{S}(C_n)$ *if and only if*

  $$\lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \lambda_5 \leq \lambda_6 < \cdots ,$$

  or

  $$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \lambda_6 < \cdots .$$

  (In particular, `G105` attains 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1221, and 2112 spectrally arbitrary.)
- All trees on six vertices are generalized stars or double stars for which the IEPG has been solved (see [3]). (In particular, `G77` attains 1131 and `G79` attains 1221 with arbitrary spectra.)
- The graph `G129` attains 222; a construction can be found in [8].

5.1. Using SSP for connected graphs of order at most five. Let G be a disconnected graph on six vertices. From each of its connected components, select an attainable ordered multiplicity list (see Table 1). Let $\gamma_1 \dots \gamma_k$ be the ordered multiplicity list built by interlacing in some way these lists (in particular, it is assumed that no two components share a common eigenvalue). By Theorem 2.4, if each component attains its multiplicity list with SSP, G attains $\gamma_1 \dots \gamma_k$ with SSP. By Theorem 2.3, any supergraph of G—namely, any connected supergraph H of order six—attains $\gamma_1 \dots \gamma_k$ (with SSP as well).

Since the components are graphs on at most five vertices, all of the attainable multiplicity lists are more generally spectrally arbitrary in choice of the corresponding eigenvalues [5]. Since SSP preserves spectrum, any ordered multiplicity list constructed in this fashion must be spectrally arbitrary as well. Exhaustively performing the process detailed above gives the results listed in Table 4.

5.2. Using cloning for connected graphs of order five. Let G be a connected graph on five vertices, and let H be the graph constructed by cloning $v \in v(G)$ with an edge. If G attains the ordered multiplicity list $(\gamma_1, \dots, \gamma_k)$, by Corollary 3.3 it follows that H attains both $(\gamma_1 + 1, \dots, \gamma_k)$ and $(\gamma_1, \dots, \gamma_k + 1)$.

The method above produces the results listed in Table 5 (note we exclude information that follows from previous results).

The full application of Theorem 3.2 generally requires a constructed matrix; then, confirming the value of the appropriate diagonal entry allows for the manipulation of the interior entries of the corresponding ordered multiplicity list. However, an explicit construction is not always necessary. Given a prescribed multiplicity list, for some graphs G we can sometimes guarantee that a particular diagonal entry of any $M \in \mathcal{S}(G)$ must be zero (or nonzero), as we now demonstrate.

PROPOSITION 5.2. *Let* $M \in \mathcal{S}(\texttt{G40})$ *have nullity two, then the diagonal entry of* M *corresponding to the leaf is nonzero. In particular, for any ordered multiplicity list of* `G40` *with a* 2 *we can clone the leaf vertex with an edge to get* `G144` *and change the* 2 *to a* 3*.*

Proof. In Figure 8, we have `G40` and the two possible graphs that result from cloning without an edge (`G111`) and with an edge (`G144`). In addition, we have marked minimal zero-forcing sets for the two clones.

The graph `G111` has a zero-forcing number of two, which would imply that the maximum nullity (and hence also maximum multiplicity of an eigenvalue) is at most two. Therefore, no matrix associated with

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TABLE 4

Using SSP properties for graphs of order at most five to attain (spectrally arbitrary) ordered multiplicity lists for graphs of order six.

Graph(s)	Supergraphs	Multiplicity lists
G54 ∪ G1	G191, G200, G205, G207, G208	111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1113, 1131, 1311, 3111, 123, 132, 213, 231, 312, 321, 114, 141, 411
G48 ∪ G1	G140, G141, G143, G156, G157, G158, G159, G160, G166, G168, G170, G172, G173, G177, G178, G179, G180, G181, G182, G183, G184, G185, G186, G188, G189, G190, G192, G193, G194, G195, G196, G197, G198, G199, G201, G202, G203, G204, G206	111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1113, 1131, 1311, 3111, 123, 132, 213, 231, 312, 321
G18 ∪ 2G1	G133, G134, G142, G165, G169	111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1113, 1131, 1311, 3111
G44 ∪ G1	G121, G125, G135, G138, G146, G149, G154, G161, G162, G175	111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211, 1131, 1311
G7 ∪ G7 G42 ∪ G1 G16 ∪ 2G1	G96, G98, G99, G103, G111, G112, G113, G114, G115, G117, G118, G119, G120, G122, G123, G124, G126, G128, G129, G130, G136, G137, G139, G144, G145, G147, G148, G150, G151, G152, G153, G163, G164, G167, G171, G174, G187	111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2112, 2121, 2211
G34 ∪ G1 G38 ∪ G1	G92, G93, G95, G104, G127	111111, 11112, 11121, 11211, 12111, 21111, 1122, 1212, 1221, 2121, 2211
G7 ∪ 3G1	G94, G97, G100, G102	111111, 11112, 11121, 11211, 12111, 21111
G13 ∪ 2G1	G77, G78, G79, G80, G81	111111, 11121, 11211, 12111
6G1	G83, G105	111111

G111 can have an ordered multiplicity list with an entry which is ≥ 3. On the other hand that is possible for G144.

We can run cloning with the matrix M which has nullity two and produce a matrix that has nullity three. This can only be possible if we cloned to G144 and not G111 which means that the diagonal entry corresponding to the leaf vertex is nonzero.

Finally, since we can always translate any particular eigenvalue to 0 then for any matrix in $\mathcal{S}(\mathtt{G40})$ with an entry of two in the ordered multiplicity list, first we translate so that the entry corresponds to an eigenvalue of 0, apply the preceding argument changing the two to a three, and then translate back. ∎

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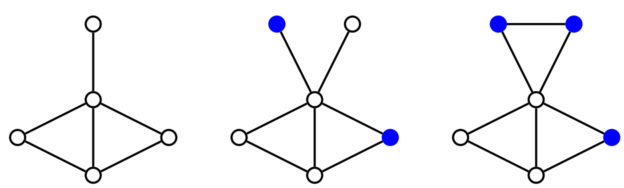
TABLE 5
Using cloning for graphs on five vertices to attain ordered multiplicity lists for graphs on six vertices.

Graph	Cloned graph(s)	Multiplicity lists
G29	G92, G161	132, 231
G30	G100	1122, 1212, 2121, 2211
G34	G117, G133, G179	1113, 123, 222, 3111, 321
G35	G119	1113, 3111
G36	G130, G150	1113, 3111
G37	G126, G141, G143, G168	1113, 123, 213, 222, 3111, 312, 321
G38	G153	1113, 123, 222, 3111, 321
G40	G144, G156, G177, G192	1113, 123, 213, 222, 3111, 312, 321
G41	G150, G160, G178, G181	1113, 123, 213, 222, 3111, 312, 321
G42	G165, G195	114, 123, 132, 213, 222, 231, 312, 321, 411
G43	G169, G172, G185	123, 213, 222, 312, 321
G44	G170, G189	222
G45	G165, G191, G200	114, 123, 132, 213, 222, 231, 312, 321, 411
G46	G179, G201	222
G47	G183, G193, G202	222
G48	G190, G194, G199	114, 222, 24, 33, 411, 42
G49	G195, G200, G205	114, 222, 24, 33, 411, 42
G50	G203, G206	114, 222, 24, 33, 411, 42
G51	G205, G207	222, 24, 33, 42
G52	G208	15, 222, 24, 33, 42, 51



(a) G40 (b) G111 (c) G144

FIGURE 8. *G40 and its two clones via the leaf. Minimal zero-forcing sets have been marked for the clones.*

A similar argument works for several other cases which are listed in Table 6.

Because we know that graphs on five vertices attain their ordered multiplicity lists spectrally arbitrarily and cloning preserves the eigenvalues, we can conclude that these lists found by cloning are also spectrally arbitrary.

5.3. Constructions of graphs which are spectrally arbitrary. Several cases were handled by finding matrices, usually through the aid of orthogonal representations. Constructions with SSP allowed multiple cases to be handled simultaneously.

PROPOSITION 5.3. *For* G96*, we can attain* 222 *spectrally arbitrary using SSP matrices.*

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TABLE 6
Using cloning for graphs on five vertices to attain ordered multiplicity lists for graphs on six vertices where a middle entry is increased.

Graph	Cloned graph	Multiplicity lists
G29	G77	141
G30	G78	1131, 1311
G30	G100	1131, 1311
G30	G114	1131, 1311
G34	G133	132, 231
G35	G119	1131, 1311
G37	G126	1131, 1311, 132, 231
G40	G144	1131, 1311, 132, 231
G42	G165	141
G44	G146	141
G46	G161	141
G48	G190	141
G48	G194	141
G49	G195	141

Proof. For $b > 0$, the following matrix for G96 attains $\{0^{(2)}, 1^{(2)}, (1+4b^2)^{(2)}\}$ with SSP.

$$\begin{pmatrix} 4b^2 & \sqrt{2}b & -\sqrt{2}b & 0 & 0 & 0 \\ \sqrt{2}b & 1 & 0 & b & 0 & 0 \\ -\sqrt{2}b & 0 & 1 & b & 0 & 0 \\ 0 & b & b & 4b^2 & b & b \\ 0 & 0 & 0 & b & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \end{pmatrix}$$

By choosing b, we can make the ratio of the two gaps between the eigenvalues arbitrarily large. By scaling and shifting, this establishes attainability with arbitrary spectrum. ∎

By Theorem 2.3, we have that all supergraphs which contain G96 can attain the ordered multiplicity list 222 with arbitrary spectrum. Thus, the graphs G111, G114, G118, G121, G135, G136, G137, G140, G145, G146, G147, G148, G149, G157, G158, G159, G161, G162, G163, G164, G166, G167, G171, G173, G180, G182, G184, G186, G187, G188, G196, G197, G198, and G204 are spectrally arbitrary for 222.

PROPOSITION 5.4. *For* G105*, we can attain* 222 *spectrally arbitrary using SSP matrices.*

Proof. The matrix

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & a \\ -1 & 0 & a & 0 & 0 & 0 \\ 0 & a & a^2 & a & 0 & 0 \\ 0 & 0 & a & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & a \\ a & 0 & 0 & 0 & a & a^2 \end{pmatrix} \in \mathcal{S}(\texttt{G105}),$$

has SSP and has spectrum

$$\left\{(\tfrac{1}{2}a^2 - \tfrac{1}{2}\sqrt{a^4+10a^2+5} - \tfrac{1}{2})^{(2)}, 0^{(2)}, (\tfrac{1}{2}a^2 + \tfrac{1}{2}\sqrt{a^4+10a^2+5} - \tfrac{1}{2})^{(2)}\right\}.$$

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Since we can scale the spectrum, it suffices to show that the following ratio of the absolute value of the two nonzero eigenvalues contains the interval $[1,\infty)$:

$$\frac{a^2+\sqrt{a^4+10a^2+5}-1}{-a^2+\sqrt{a^4+10a^2+5}+1}.$$

This is continuous for $a \geq 1$ and if $a = 1$ we get 1. Furthermore, the numerator has growth a^2 and the denominator approaches 6 so the ratio is unbounded. ◻

By Theorem 2.3, we have that all supergraphs which contain $\mathtt{G105}$ can attain the ordered multiplicity list 222 with arbitrary spectrum. Thus, the graphs $\mathtt{G127}$, $\mathtt{G128}$, $\mathtt{G151}$, $\mathtt{G152}$, $\mathtt{G154}$, $\mathtt{G174}$, and $\mathtt{G175}$ are spectrally arbitrary for 222.

Proposition 5.5. *For* $\mathtt{G99}$*, we can attain* 222 *spectrally arbitrary using non-SSP matrices.*

Proof. For $a > 0$, the following matrix for $\mathtt{G99}$ attains $\{0^{(2)}, 1^{(2)}, (1+3a^2)^{(2)}\}$.

$$\begin{pmatrix} 1 & a & 0 & 0 & 0 & 0 \\ a & 3\,a^2 & a & a & 0 & 0 \\ 0 & a & 1 & 0 & -a & 0 \\ 0 & a & 0 & 1 & a & 0 \\ 0 & 0 & -a & a & 3\,a^2 & a \\ 0 & 0 & 0 & 0 & a & 1 \end{pmatrix}.$$

By choosing a, we can make the ratios of the two gaps arbitrarily big. By scaling and shifting, this establishes attainability with arbitrary spectrum.

It is impossible for any matrix to have 222 and SSP for this graph as $\mathtt{G112}$ is a supergraph which cannot attain 222 because $q(\mathtt{G112}) = 4$. ◻

Proposition 5.6. *For* $\mathtt{G189}$*, we can attain* 141 *spectrally arbitrary using SSP matrices.*

Proof. The matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & a \\ 0 & 0 & 0 & 1 & 1 & a \\ 0 & 0 & 0 & 1 & 1 & a \\ 1 & 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 & 0 \\ a & a & a & 0 & 0 & a^2 \end{pmatrix} \in \mathcal{S}(\mathtt{G189}),$$

has SSP and has characteristic polynomial $p(x) = x^4(x^2-(a^2-2)x-(5a^2+6))$. The spectrum is

$$\left\{\tfrac{1}{2}\big((a^2-2)-\sqrt{a^4+16a^2+28}\big), 0^{(4)}, \tfrac{1}{2}\big((a^2-2)+\sqrt{a^4+16a^2+28}\big)\right\}.$$

Since we can scale the spectrum, it suffices to show that the following ratio of the absolute value of the two nonzero eigenvalues contains the interval $[1,\infty)$:

$$\frac{(a^2-2)+\sqrt{a^4+16a^2+28}}{-(a^2-2)+\sqrt{a^4+16a^2+28}}.$$

This is continuous for $a \geq \sqrt{2}$ and if $a = \sqrt{2}$ we get 1. Furthermore, the numerator has growth a^2 and the denominator approaches 10 so the ratio is unbounded. ◻

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By Theorem 2.3, all supergraphs which contain G189 can attain the ordered multiplicity list 141 with arbitrary spectrum. Thus, the graphs G197, G199, G201, G203, and G206 are spectrally arbitrary for 141.

The remaining case for 141 is G204. Consider

$$\begin{pmatrix} a^2 & 0 & a & a & a & a \\ 0 & -1 & -1 & -1 & 1 & 1 \\ a & -1 & 0 & 0 & 2 & 2 \\ a & -1 & 0 & 0 & 2 & 2 \\ a & 1 & 2 & 2 & 0 & 0 \\ a & 1 & 2 & 2 & 0 & 0 \end{pmatrix} \in \mathcal{S}(\mathtt{G204}),$$

which has eigenvalues $\{-5, 0^{(4)}, 4+a^2\}$. Arbitrary spectrums are attained through appropriate choices of $a \geq 1$, scaling, and shifting.

Proposition 5.7. *For* G151*, we can attain* 213*,* 312*,* 1113*, and* 3111 *spectrally arbitrary using SSP matrices.*

Proof. For $a, b > 0$, the following matrix for G151 has spectrum $\{0^{(3)}, 2a^2, 2a^2+2, a^2+b^2+2\}$ and satisfies SSP:

$$\begin{pmatrix} a^2 & a & 0 & 0 & a^2 & 0 \\ a & a^2+2 & a & 0 & 0 & ab \\ 0 & a & a^2 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & a^2+1 & 1 & -b \\ a^2 & 0 & 0 & 1 & a^2+1 & -b \\ 0 & ab & 0 & -b & -b & b^2 \end{pmatrix}.$$

Setting $a = b$ gives 312 with a fixed gap between the last two eigenvalues and an arbitrary gap between the first two; the attainability of an arbitrary spectrum follows.

For 3111, by scale and shift we can assume a non-negative spectrum such that 0 is the eigenvalue of multiplicity 3, and the gap between the first and second positive eigenvalues is 2. Set $2a^2$ and a^2+b^2+2 to the first and third positive eigenvalues, respectively, and solve for a and b. Thus, an arbitrary spectrum is attainable. ∎

By Theorem 2.3, we have that all supergraphs of G151 can attain the ordered multiplicity lists 213, 312, 1113, and 3111 with arbitrary spectrum. Thus, the graphs G171 and G187 are spectrally arbitrary for 213, 312, 1113, and 3111.

Proposition 5.8. *For* G163*, we can attain* 213*,* 312*,* 1113*, and* 3111 *spectrally arbitrary using SSP matrices.*

Proof. For $a, b > 0$, the following matrix for G163 has spectrum $\{0^{(3)}, 1, 1+3a^2, 1+a^2+2b^2\}$ and satisfies SSP:

$$\begin{pmatrix} a^2+b^2 & b^2 & b & 0 & a^2 & a \\ b^2 & a^2+b^2 & b & a & -a^2 & 0 \\ b & b & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & -a & 0 \\ a^2 & -a^2 & 0 & -a & 2a^2 & a \\ a & 0 & 0 & 0 & a & 1 \end{pmatrix}.$$

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By appropriate choices of a, b, scaling, and translating, this attains 312, 213, 3111, and 1113 spectrally arbitrary. ∎

5.4. Two distinct eigenvalues. For the graphs G154, G168, G174, G175, G181, G186, G188, G192, G196, G197, G198, G201, and G202, $q(G) = 2$ and $Z_+(G) = 3$ (see Table 3), which implies that all these graphs attain 33. (For more on matrix realizations for these graphs, see [8].) For the graph G204, this attains 33 by the matrix below on the left, and 42 (24) by the matrix below on the right.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & -1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 3 & 1 \\ -1 & 1 & 0 & 2 & -1 & 3 \\ 2 & 1 & 3 & -1 & 5 & 0 \\ -1 & 2 & 1 & 3 & 0 & 5 \end{pmatrix},$$

When a matrix for a graph has two distinct eigenvalues, we can modify the matrix to get additional attainable ordered multiplicity lists. The following will suffice for our purposes (generalizations are possible for graphs with larger order).

LEMMA 5.9. *Let $G \neq K_6$ be a connected graph on six vertices. If G attains ordered multiplicity list* 33*, then with arbitrary spectrum it attains multiplicity lists* 1113*,* 123*,* 213*,* 312*,* 321*, and* 3111*. Similarly, if G attains* 42 *or* 24*, then with arbitrary spectrum it attains* 411 *and* 114*.*

Proof. Since G is not the complete graph, there are two nonadjacent vertices, which we assume to be v_1 and v_2.

Let $M \in \mathcal{S}(G)$ attain the ordered multiplicity list 33 with spectrum $\{0^{(3)}, 1^{(3)}\}$. We can write $M = Q^TQ$, where Q is a 3×6 matrix whose rows are *any* orthonormal basis of the eigenspace of 1 (in particular, M is the projection matrix onto the eigenspace associated with eigenvalue 1).

We claim we can choose our orthonormal basis so that for some $x, y \neq 0$,

$$Q = \left(\begin{array}{cc|cccc|} x & 0 & * & * & * & * \\ 0 & y & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array}\right).$$

To see this, first consider the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, which form a basis for our eigenspace.

- Note that no fixed entry can be 0 for all three of $\mathbf{a}, \mathbf{b}, \mathbf{c}$; otherwise this would imply that the corresponding vertex is isolated, a contradiction given that our graph is connected. Thus, at least one vector has a nonzero first entry. By taking linear combinations, we can assume that the first entry is 0 for $\mathbf{a}$ and $\mathbf{b}$ and nonzero for $\mathbf{c}$.
- Run Gram-Schmidt on $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (in this order) to get an orthonormal set $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, where the first entries of $\mathbf{a}'$, $\mathbf{b}'$ are 0 and the first entry of $\mathbf{c}'$ is nonzero. Note the second entry of $\mathbf{c}'$ must be zero; otherwise, using this set as an orthonormal basis for Q would force $M_{1,2} \neq 0$, a contradiction given that $v_1 \not\sim v_2$.
- Now repeat the argument for $\mathbf{a}', \mathbf{b}'$ by taking a linear combination so that the second entry of $\mathbf{a}'$ is zero. Run Gram-Schmidt again to produce $\mathbf{a}'', \mathbf{b}''$.
- Thus, the rows of Q are (from top to bottom) $\mathbf{c}', \mathbf{b}'', \mathbf{a}''$.

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Note we can now introduce parameters $\lambda, \mu > 0$ to give

$$\widehat{Q} = \left(\begin{array}{cc|cccc|} \lambda x & 0 & * & * & * & * \\ 0 & \mu y & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array} \right).$$

Since the orthogonality of the columns of $\widehat{Q}$ agrees with the orthogonality of the columns of Q, the matrix $\widehat{M} = \widehat{Q}^T\widehat{Q} \in \mathcal{S}(G)$. Because the nonzero eigenvalues of $\widehat{M}$ are the norms of the rows, $\widehat{M}$ has spectrum $\{0^{(3)}, 1, 1+(\lambda^2-1)x^2, 1+(\mu^2-1)y^2\}$. Appropriate choices of λ and μ, combined with scaling and translation, arbitrarily attain any spectrum that starts or ends with an eigenvalue of multiplicity 3.

A similar argument handles the 42 case. □

This lemma establishes that 1113, 123, 213, 312, 321, and 3111 are all attainable with arbitrary spectrum for graphs `G154`, `G174`, and `G175`; similarly, 411 and 114 are attainable with arbitrary spectrum for `G204`.

5.5. Graph minor results. We now turn to results for graphs which attain certain ordered multiplicity lists, but which are not enough to prove we do so arbitrarily. We start with the following result which connects SSP and graph minors. This result follows immediately from Theorem 6.12 in [5].

THEOREM 5.10. *Let G be attained from H by contraction of a single edge, and let $M \in \mathcal{S}(G)$ have SSP and ordered multiplicity list $(\gamma_1, \dots, \gamma_k)$. Then there is $N \in \mathcal{S}(H)$ with ordered multiplicity list $(\gamma_1, \dots, \gamma_k, 1)$.*

Applying Theorem 5.10 by looking for minors on graphs of order six gives the results listed in Table 7 (we exclude information that follows from previous results).

Note Theorem 5.10 does not guarantee spectrally arbitrary results; the newly appended one on the ordered multiplicity list might need to be large (see [5] for more information).

5.6. Graphs with SSP/SMP. Table 8 lists matrices that attain the corresponding ordered multiplicity list for that graph. All these matrices have either SSP or SMP, which gives the results listed in Table 7.

- Because `G127` is a supergraph of `G105`, `G127` attains 2112.
- Because `G138` is a supergraph of `G125`, `G138` attains 222.
- Because `G145`, `G149`, and `G162` are supergraphs of `G129`, they attain 132 and 231.

TABLE 7
Using graph minors to attain ordered multiplicity lists for graphs of order six.

Graph(s)	Minor	Multiplicity lists
`G100`	`G34`	1221
`G129`, `G145`, `G151`, `G153`, `G171`, `G174`, `G187`	`G44`	1311, 1131
`G151`, `G153`, `G154`, `G169`, `G171`, `G174`, `G175`, `G187`	`G48`	321, 123, 231, 132

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TABLE 8
Matrices with SSP or SMP.

Graph	Multiplicity list(s)	Matrix
G105	2112	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -2 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ -2 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
G125	222	$\begin{pmatrix} -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
G129	132, 231	$\begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & -1 & 0 & 0 & 2 & 1 \end{pmatrix}$

5.7. Remaining cases. Table 9 gives the remaining attainable cases. The construction of these matrices included exhaustive searches for matrices with simple entries (i.e., 0, ± 1), as well as the use of orthogonal representations with respect to some (possibly indefinite) inner product.

6. Ordered multiplicity IEPG differs from IEPG. Through five vertices, the ordered multiplicity IEPG and the IEPG are equivalent: a graph attains an ordered multiplicity list if and only if it attains that multiplicity list with arbitrary spectrum. However, for graphs of order at least six, this relationship no longer holds.

THEOREM 6.1. *The complete bipartite graph $K_{m,n}$ where $\min(m,n) \geq 3$ attains the ordered multiplicity list $(1, m+n-2, 1)$, but not spectrally arbitrary.*

Proof. The spectrum of the adjacency matrix of $K_{m,n}$ is $\{-\sqrt{mn}, 0^{(m+n-2)}, \sqrt{mn}\}$. Thus, $K_{m,n}$ attains $(1, m+n-2, 1)$. Note the gaps between consecutive eigenvalues are equal. We claim that any $M \in \mathcal{S}(K_{m,n})$ that attains $(1, m+n-2, 1)$ preserves this relationship. Thus, $K_{m,n}$ cannot attain $(1, m+n-2, 1)$ with arbitrary spectrum.

Let $M \in \mathcal{S}(K_{m,n})$ attain multiplicity list $(1, m+n-2, 1)$, where after translation we may assume 0 is the eigenvalue of multiplicity $m+n-2$. Let $-\lambda_1 < \lambda_2$ be the nonzero eigenvalues of M with the corresponding orthogonal eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Note,

$$M = -\lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1}\mathbf{x}_1 & \sqrt{\lambda_2}\mathbf{x}_2 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{=S} \underbrace{\begin{pmatrix} \sqrt{\lambda_1}\mathbf{x}_1^T \\ \sqrt{\lambda_2}\mathbf{x}_2^T \end{pmatrix}}_{=Y}.$$

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TABLE 9
Remaining cases.

$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G92})$ 1131 1311	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G94})$ 2121 1212
$\begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G114})$ 132 231	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G115})$ 222
$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G117})$ 1131 1311	$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G130})$ 1131 1311
$\begin{pmatrix} 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & 0 & 3 & 0 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G135})$ 132 231	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 & -1 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G146})$ 132 231
$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G150})$ 1131 1311	$\begin{pmatrix} \gamma^2 & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & 0 & 2 & -1 & 0 & 0 \\ \gamma & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & \gamma & \gamma \\ 0 & 0 & 0 & \gamma & \gamma^2 & \gamma^2 \\ 0 & 0 & 0 & \gamma & \gamma^2 & \gamma^2 \end{pmatrix}$ $\gamma^2 = \sqrt{10} - 2$ $\in \mathcal{S}(\texttt{G150})$ 132 231
$\begin{pmatrix} 1 & 3 & 1 & 0 & 1 & 2 \\ 3 & 5 & 2 & 1 & 0 & 3 \\ 1 & 2 & 7 & 4 & -1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 2 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G163})$ 1131 1311	$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G163})$ 123 321
$\begin{pmatrix} 5 & \sqrt{5} & 5 & 0 & 0 & 0 \\ \sqrt{5} & 2 & \sqrt{5} & \sqrt{5} & \sqrt{5} & 0 \\ 5 & \sqrt{5} & 0 & 0 & -5 & -5 \\ 0 & \sqrt{5} & 0 & 5 & 5 & 0 \\ 0 & \sqrt{5} & -5 & 5 & 0 & -5 \\ 0 & 0 & -5 & 0 & -5 & -5 \end{pmatrix}$ $\in \mathcal{S}(\texttt{G163})$ 132 231	

Let $\mathbf{y}_i$ be the ith column of Y; thus, $\mathbf{y}_i \in \mathbb{R}^2$. Note the association between $\mathbf{y}_i$ and v_i of $K_{m,n}$: two vertices $v_i \not\sim v_j$ if and only if $\mathbf{y}_i^T S \mathbf{y}_j = 0$. Moreover, $K_{m,n}$ is connected, so $\mathbf{y}_i \neq 0$.

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Let $a, b \in K_{m,n}$ be nonadjacent. Since $\min(m, n) \geq 3$, there exists c such that a, b, and c are pairwise nonadjacent. If the corresponding vectors $\mathbf{a}$ and $\mathbf{b}$ are not scalar multiples, then the matrix

$$\begin{pmatrix} -a_1 & a_2 \\ -b_1 & b_2 \end{pmatrix},$$

has rank two. However, the adjacency structure of $K_{m,n}$ requires that

$$\begin{pmatrix} -a_1 & a_2 \\ -b_1 & b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which shows the matrix lacks full rank, a contradiction. Thus, $\mathbf{a}$ and $\mathbf{b}$ are scalar multiples. By symmetry, the vectors associated with pairwise nonadjacent vertices must be scalar multiples. These vectors must also satisfy $\mathbf{a}^T S \mathbf{b} = 0$ (since a and b are nonadjacent) implying $\mathbf{a}^T S \mathbf{a} = 0$ (which differs from the previous expression by a nonzero scaling factor), and thus must have the form $\binom{x}{\pm x}$. Hence, Y is of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m & \beta_1 & \beta_2 & \cdots & \beta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m & -\beta_1 & -\beta_2 & \cdots & -\beta_n \end{pmatrix},$$

for appropriate choice of α_i and β_j, so M has spectrum

$$\left\{0^{(m+n-2)}, \pm 2\sqrt{\left(\sum \alpha_i^2\right)\left(\sum \beta_j^2\right)}\right\}.$$

Therefore, $K_{m,n}$ cannot attain $(1, m+n-2, 1)$ with arbitrary spectrum. □

Corollary 6.2. *The graph $K_{3,3}$ (G175) attains 141, but not spectrally arbitrary.*

7. Conclusion. We have given a complete solution for the ordered multiplicity IEPG for connected graphs on six vertices. Moreover, many of the techniques used for attainability also allow for an arbitrary spectrum, allowing for significant progress on the IEPG for connected graphs on six vertices. In particular, there are 1326 cases of attainability. Among these, 1285 are known to be done with arbitrary spectrum, one does so without, and 40 remain undetermined (see Table 10). Finishing these cases, and thus solving the IEPG for graphs on six vertices, is an open problem.

We also showed that $K_{m,n}$ with $\min(m, n) \geq 3$ has at least one attainable multiplicity list which cannot be attained spectrally arbitrary. This shows that the ordered multiplicity IEPG and the IEPG differ for graphs on six or more vertices.

Table 10
The remaining cases for the IEPG for connected graphs on six vertices.

Multiplicity list(s)	Remaining cases
1212, 2121	G94
1221	G100
2112	G127
222	G115, G125, G129, G138
1131, 1311	G92, G117, G129, G130, G145, G150, G151, G153, G163, G171, G174, G187
123, 321	G151, G163, G171, G187
132, 231	G114, G129, G135, G145, G146, G149, G150, G151, G153, G154, G162, G163, G169, G171, G174, G175, G187

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A natural next problem to consider is the ordered multiplicity IEPG for connected graphs on seven or more vertices. While many of the techniques implemented in this work can be utilized in that setting (and indeed solves 'most' of the cases), the IEPG becomes dramatically harder. Thus, new tools and more automation will likely be needed to continue to make progress.[1]

The difficulty lies in part with the number of cases involved, both in terms of the number of graphs and the number of potential ordered multiplicity lists. In addition, one of the most useful tools we had was SMP and SSP which allowed for simultaneous handling of many cases by establishing a result for a graph and all its supergraphs. This does have some limitations.

Observation 4. *If $M \in \mathcal{S}(G)$ attains $\gamma_1 \ldots \gamma_k$ and H is a supergraph which does not attain $\gamma_1 \ldots \gamma_k$, then M does not have SSP or SMP.*

This can be used to explain why the cases given in Table 2 do not have SSP, and for graphs on six vertices can be used to show that there are over 40 occurrences where a graph attains an ordered multiplicity list but does so without SSP or SMP. These cases required either cloning or finding some appropriate orthogonal constructions. As the number of vertices increases, the number of cases needing individual attention (and hence difficulty) will rise as well.

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Appendix A. Connected graphs on six vertices realizing given ordered multiplicities. For the following ordered multiplicities, we list all connected graphs which can achieve a given multiplicity list. Any graph which is underlined is a graph which has not yet been determined to be spectrally arbitrary for that ordered multiplicity list. Any graph which is boxed is a graph which has been shown to *not* be spectrally arbitrary for that multiplicity list.

111111

G77	G78	G79	G80	G81	G83	G92	G93	G94	G95	G96	G97
G98	G99	G100	G102	G103	G104	G105	G111	G112	G113	G114	G115
G117	G118	G119	G120	G121	G122	G123	G124	G125	G126	G127	G128
G129	G130	G133	G134	G135	G136	G137	G138	G139	G140	G141	G142
G143	G144	G145	G146	G147	G148	G149	G150	G151	G152	G153	G154
G156	G157	G158	G159	G160	G161	G162	G163	G164	G165	G166	G167
G168	G169	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180
G181	G182	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192
G193	G194	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204
G205	G206	G207	G208								

[1] Even for graphs on six vertices it is not immediately clear that our claim of all cases being established holds. A SAGE worksheet which implements and runs through all cases is available online and can be used for simplifying the verification: https://sage.math.iastate.edu/home/pub/120/

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11112 and 21111

G92	G93	G94	G95	G96	G97	G98	G99	G100	G102	G103	G104
G105	G111	G112	G113	G114	G115	G117	G118	G119	G120	G121	G122
G123	G124	G125	G126	G127	G128	G129	G130	G133	G134	G135	G136
G137	G138	G139	G140	G141	G142	G143	G144	G145	G146	G147	G148
G149	G150	G151	G152	G153	G154	G156	G157	G158	G159	G160	G161
G162	G163	G164	G165	G166	G167	G168	G169	G170	G171	G172	G173
G174	G175	G177	G178	G179	G180	G181	G182	G183	G184	G185	G186
G187	G188	G189	G190	G191	G192	G193	G194	G195	G196	G197	G198
G199	G200	G201	G202	G203	G204	G205	G206	G207	G208		

11121 and 12111

G77	G78	G79	G80	G81	G92	G93	G94	G95	G96	G97	G98
G99	G100	G102	G103	G104	G105	G111	G112	G113	G114	G115	G117
G118	G119	G120	G121	G122	G123	G124	G125	G126	G127	G128	G129
G130	G133	G134	G135	G136	G137	G138	G139	G140	G141	G142	G143
G144	G145	G146	G147	G148	G149	G150	G151	G152	G153	G154	G156
G157	G158	G159	G160	G161	G162	G163	G164	G165	G166	G167	G168
G169	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181
G182	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193
G194	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205
G206	G207	G208									

11211

G77	G78	G79	G80	G81	G92	G93	G94	G95	G96	G97	G98
G99	G100	G102	G103	G104	G105	G111	G112	G113	G114	G115	G117
G118	G119	G120	G121	G122	G123	G124	G125	G126	G127	G128	G129
G130	G133	G134	G135	G136	G137	G138	G139	G140	G141	G142	G143
G144	G145	G146	G147	G148	G149	G150	G151	G152	G153	G154	G156
G157	G158	G159	G160	G161	G162	G163	G164	G165	G166	G167	G168
G169	G170	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181
G182	G183	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193
G194	G195	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205
G206	G207	G208									

1122 and 2211

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G92	G93	G95	G96	G98	G99	G100	G103	G104	G105	G111	G112
G113	G114	G115	G117	G118	G119	G120	G121	G122	G123	G124	G125
G126	G127	G128	G129	G130	G133	G134	G135	G136	G137	G138	G139
G140	G141	G142	G143	G144	G145	G146	G147	G148	G149	G150	G151
G152	G153	G154	G156	G157	G158	G159	G160	G161	G162	G163	G164
G165	G166	G167	G168	G169	G170	G171	G172	G173	G174	G175	G177
G178	G179	G180	G181	G182	G183	G184	G185	G186	G187	G188	G189
G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200	G201
G202	G203	G204	G205	G206	G207	G208					

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G79	G92	G93	G95	G96	G98	G99	G100	G103	G104	G105	G111
G112	G113	G114	G115	G117	G118	G119	G120	G121	G122	G123	G124
G125	G126	G127	G128	G129	G130	G133	G134	G135	G136	G137	G138
G139	G140	G141	G142	G143	G144	G145	G146	G147	G148	G149	G150
G151	G152	G153	G154	G156	G157	G158	G159	G160	G161	G162	G163
G164	G165	G166	G167	G168	G169	G170	G171	G172	G173	G174	G175
G177	G178	G179	G180	G181	G182	G183	G184	G185	G186	G187	G188
G189	G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200
G201	G202	G203	G204	G205	G206	G207	G208				

1212 and 2121

G92	G93	G94	G95	G96	G98	G99	G100	G103	G104	G111	G112
G113	G114	G115	G117	G118	G119	G120	G121	G122	G123	G124	G125
G126	G127	G128	G129	G130	G133	G134	G135	G136	G137	G138	G139
G140	G141	G142	G143	G144	G145	G146	G147	G148	G149	G150	G151
G152	G153	G154	G156	G157	G158	G159	G160	G161	G162	G163	G164
G165	G166	G167	G168	G169	G170	G171	G172	G173	G174	G175	G177
G178	G179	G180	G181	G182	G183	G184	G185	G186	G187	G188	G189
G190	G191	G192	G193	G194	G195	G196	G197	G198	G199	G200	G201
G202	G203	G204	G205	G206	G207	G208					

2112

G96	G98	G99	G103	G105	G111	G112	G113	G114	G115	G117	G118
G119	G120	G121	G122	G123	G124	G125	G126	G127	G128	G129	G130
G133	G134	G135	G136	G137	G138	G139	G140	G141	G142	G143	G144
G145	G146	G147	G148	G149	G150	G151	G152	G153	G154	G156	G157
G158	G159	G160	G161	G162	G163	G164	G165	G166	G167	G168	G169
G170	G171	G172	G173	G174	G175	G177	G178	G179	G180	G181	G182
G183	G184	G185	G186	G187	G188	G189	G190	G191	G192	G193	G194
G195	G196	G197	G198	G199	G200	G201	G202	G203	G204	G205	G206
G207	G208										

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G96 G99 G105 G111 G114 $\underline{\text{G115}}$ G117 G118 G121 $\underline{\text{G125}}$ G126 G127
G128 $\underline{\text{G129}}$ G133 G135 G136 G137 $\underline{\text{G138}}$ G140 G141 G143 G144 G145
G146 G147 G148 G149 G150 G151 G152 G153 G154 G156 G157 G158
G159 G160 G161 G162 G163 G164 G165 G166 G167 G168 G169 G170
G171 G172 G173 G174 G175 G177 G178 G179 G180 G181 G182 G183
G184 G185 G186 G187 G188 G189 G190 G191 G192 G193 G194 G195
G196 G197 G198 G199 G200 G201 G202 G203 G204 G205 G206 G207
G208

1113 and 3111

G117 G119 G126 G130 G133 G134 G140 G141 G142 G143 G144 G150
G151 G153 G154 G156 G157 G158 G159 G160 G163 G165 G166 G168
G169 G170 G171 G172 G173 G174 G175 G177 G178 G179 G180 G181
G182 G183 G184 G185 G186 G187 G188 G189 G190 G191 G192 G193
G194 G195 G196 G197 G198 G199 G200 G201 G202 G203 G204 G205
G206 G207 G208

1131 and 1311

G77 G78 $\underline{\text{G92}}$ G100 G114 $\underline{\text{G117}}$ G119 G121 G125 G126 $\underline{\text{G129}}$ $\underline{\text{G130}}$
G133 G134 G135 G138 G140 G141 G142 G143 G144 $\underline{\text{G145}}$ G146 G149
$\underline{\text{G150}}$ $\underline{\text{G151}}$ $\underline{\text{G153}}$ G154 G156 G157 G158 G159 G160 G161 G162 $\underline{\text{G163}}$
G165 G166 G168 G169 G170 $\underline{\text{G171}}$ G172 G173 $\underline{\text{G174}}$ G175 G177 G178
G179 G180 G181 G182 G183 G184 G185 G186 $\underline{\text{G187}}$ G188 G189 G190
G191 G192 G193 G194 G195 G196 G197 G198 G199 G200 G201 G202
G203 G204 G205 G206 G207 G208

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123 and 321

G117 G126 G133 G140 G141 G143 G144 G150 $\underline{\text{G151}}$ G153 G154 G156
G157 G158 G159 G160 $\underline{\text{G163}}$ G165 G166 G168 G169 G170 $\underline{\text{G171}}$ G172
G173 G174 G175 G177 G178 G179 G180 G181 G182 G183 G184 G185
G186 $\underline{\text{G187}}$ G188 G189 G190 G191 G192 G193 G194 G195 G196 G197
G198 G199 G200 G201 G202 G203 G204 G205 G206 G207 G208

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132 and 231

G92 $\underline{\text{G114}}$ G126 $\underline{\text{G129}}$ G133 $\underline{\text{G135}}$ G140 G141 G143 G144 $\underline{\text{G145}}$ $\underline{\text{G146}}$
$\underline{\text{G149}}$ $\underline{\text{G150}}$ $\underline{\text{G151}}$ $\underline{\text{G153}}$ $\underline{\text{G154}}$ G156 G157 G158 G159 G160 G161 $\underline{\text{G162}}$
$\underline{\text{G163}}$ G165 G166 G168 $\underline{\text{G169}}$ G170 $\underline{\text{G171}}$ G172 G173 $\underline{\text{G174}}$ $\underline{\text{G175}}$ G177
G178 G179 G180 G181 G182 G183 G184 G185 G186 $\underline{\text{G187}}$ G188 G189
G190 G191 G192 G193 G194 G195 G196 G197 G198 G199 G200 G201
G202 G203 G204 G205 G206 G207 G208

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213 and 312

G126 G140 G141 G143 G144 G150 G151 G154 G156 G157 G158 G159
G160 G163 G165 G166 G168 G169 G170 G171 G172 G173 G174 G175
G177 G178 G179 G180 G181 G182 G183 G184 G185 G186 G187 G188
G189 G190 G191 G192 G193 G194 G195 G196 G197 G198 G199 G200
G201 G202 G203 G204 G205 G206 G207 G208

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G154 G168 G174 G175 G181 G186 G188 G190 G192 G194 G195 G196
G197 G198 G199 G200 G201 G202 G203 G204 G205 G206 G207 G208

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114 and 411

G165 G190 G191 G194 G195 G199 G200 G203 G204 G205 G206 G207
G208

141

G77 G146 G161 G165 $\boxed{\texttt{G175}}$ G189 G190 G191 G194 G195 G197 G199
G200 G201 G203 G204 G205 G206 G207 G208

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24 and 42

G190 G194 G195 G199 G200 G203 G204 G205 G206 G207 G208

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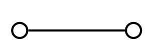
15 and 51

G208

Appendix B. Attainable ordered multiplicity lists. For each connected graph on six or fewer vertices, we give the Atlas of Graph numbering, draw the graph, and list all attainable ordered multiplicity lists.

G1		1

G3		11

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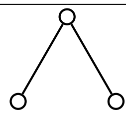
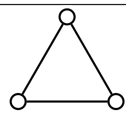
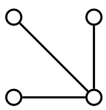
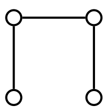
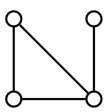
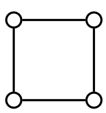
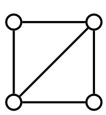
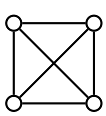
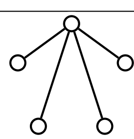
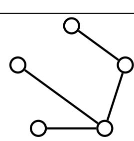
G6		111
G7		111, 12, 21
G13		1111, 121
G14		1111
G15		1111, 121, 112, 211
G16		1111, 121, 112, 211, 22
G17		1111, 121, 112, 211, 22
G18		1111, 121, 112, 211, 22, 13, 31
G29		11111, 1121, 1211, 131
G30		11111, 1121, 1211

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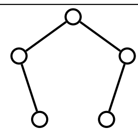
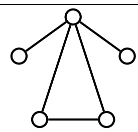
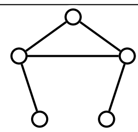
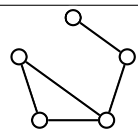
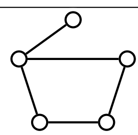
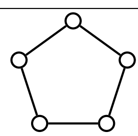
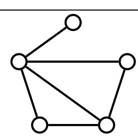
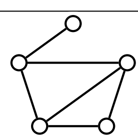
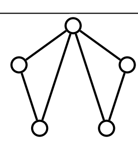
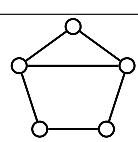
G31		11111
G34		11111, 1121, 1211, 1112, 2111, 122, 221
G35		11111, 1121, 1211, 1112, 2111
G36		11111, 1121, 1211, 1112, 2111
G37		11111, 1121, 1211, 1112, 2111, 122, 221, 212
G38		11111, 1121, 1211, 1112, 2111, 122, 221
G40		11111, 1121, 1211, 1112, 2111, 122, 221, 212
G41		11111, 1121, 1211, 1112, 2111, 122, 221, 212
G42		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 113, 131, 311
G43		11111, 1121, 1211, 1112, 2111, 122, 221, 212

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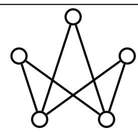
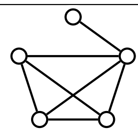
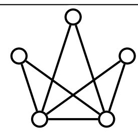
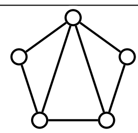
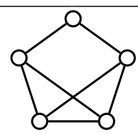
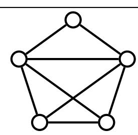
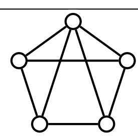
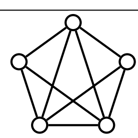
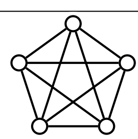
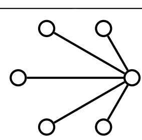
G44		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131
G45		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311
G46		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131
G47		11111, 1121, 1211, 1112, 2111, 122, 221, 212
G48		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G49		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G50		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G51		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32
G52		11111, 1121, 1211, 1112, 2111, 122, 221, 212, 131, 113, 311, 23, 32, 14, 41
G77		111111, 11121, 12111, 11211, 1131, 1311, 141

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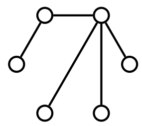
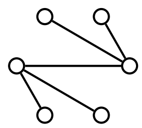
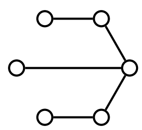
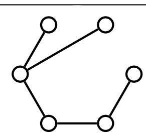
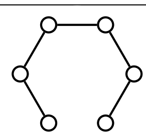
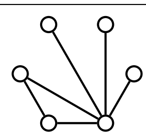
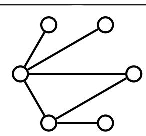
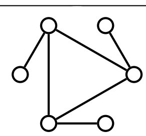
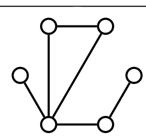
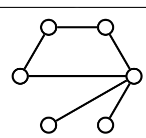
G78		111111, 11121, 12111, 11211, 1131, 1311
G79		111111, 11121, 12111, 11211, 1221
G80		111111, 11121, 12111, 11211
G81		111111, 11121, 12111, 11211
G83		111111
G92		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 1131, 1311, 132, 231
G93		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221
G94		111111, 11112, 21111, 11121, 12111, 11211, 1212, 2121
G95		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221
G96		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222

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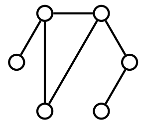
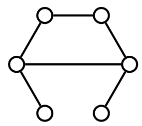
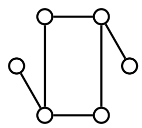
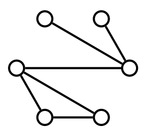
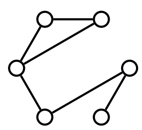
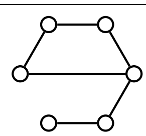
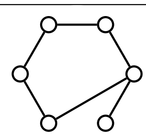
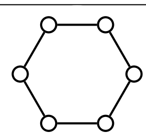
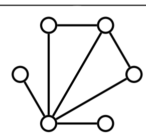
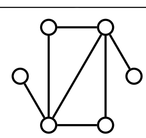
G97		111111, 11112, 21111, 11121, 12111, 11211
G98		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G99		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G100		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 1131, 1311
G102		111111, 11112, 21111, 11121, 12111, 11211
G103		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G104		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221
G105		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1221, 2112, 222
G111		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G112		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112

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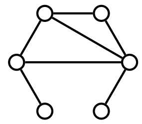
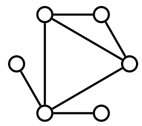
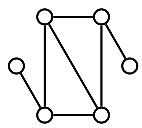
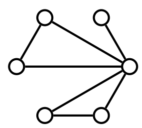
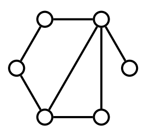
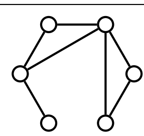
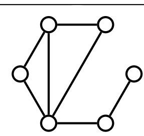
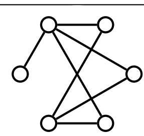
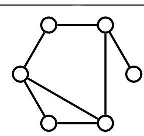
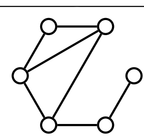
G113		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G114		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231
G115		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G117		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321
G118		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G119		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, 1131, 1311
G120		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G121		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311
G122		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G123		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112

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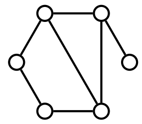
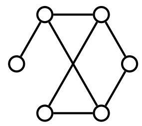
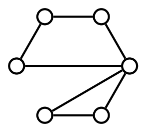
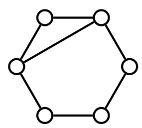
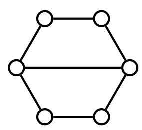
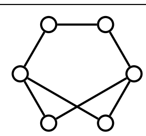
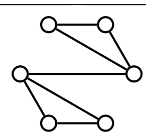
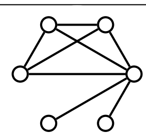
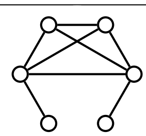
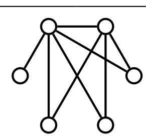
G124		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G125		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, $\underline{222}$, 1131, 1311
G126		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G127		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, $\underline{2112}$, 222
G128		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G129		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, $\underline{222}$, $\underline{1131}$, $\underline{1311}$, $\underline{132}$, $\underline{231}$
G130		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, $\underline{1131}$, $\underline{1311}$
G133		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231
G134		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, 1131, 1311
G135		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, $\underline{132}$, $\underline{231}$

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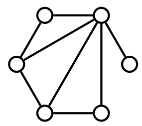
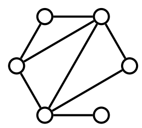
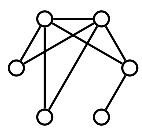
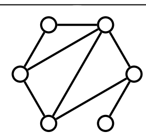
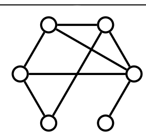
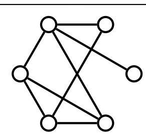
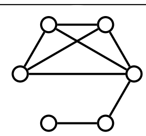
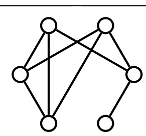
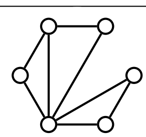
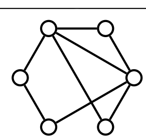
G136		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G137		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G138		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, $\underline{222}$, 1131, 1311
G139		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112
G140		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G141		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G142		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 1113, 3111, 1131, 1311
G143		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G144		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G145		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, $\underline{1131}$, $\underline{1311}$, $\underline{132}$, $\underline{231}$

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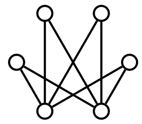
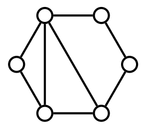
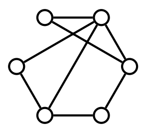
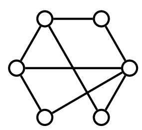
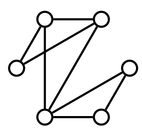
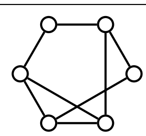
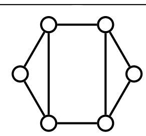
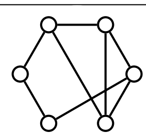
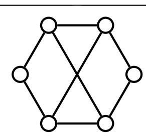
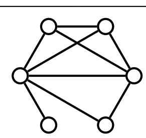
G146		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, $\underline{132}$, $\underline{231}$, 141
G147		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G148		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G149		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, $\underline{132}$, $\underline{231}$
G150		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, $\underline{1131}$, $\underline{1311}$, 123, 321, $\underline{132}$, $\underline{231}$, 213, 312
G151		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, $\underline{1131}$, $\underline{1311}$, $\underline{123}$, $\underline{321}$, $\underline{132}$, $\underline{231}$, 213, 312
G152		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G153		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, $\underline{1131}$, $\underline{1311}$, 123, 321, $\underline{132}$, $\underline{231}$
G154		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, $\underline{132}$, $\underline{231}$, 213, 312, 33
G156		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312

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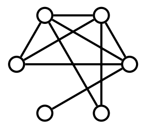
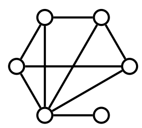
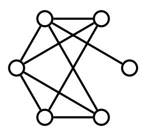
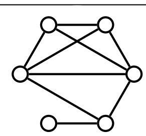
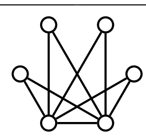
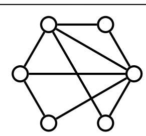
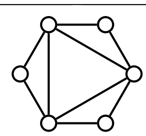
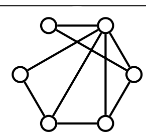
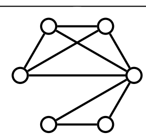
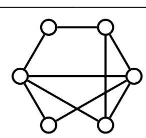
G157		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G158		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G159		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G160		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G161		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, 132, 231, 141
G162		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1131, 1311, <u>132</u>, <u>231</u>
G163		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, <u>1131</u>, <u>1311</u>, <u>123</u>, <u>321</u>, <u>132</u>, <u>231</u>, 213, 312
G164		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G165		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 114, 411, 141
G166		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312

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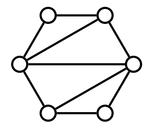
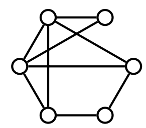
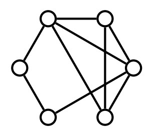
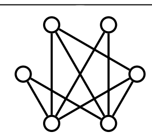
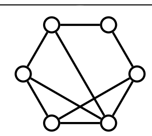
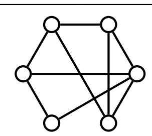
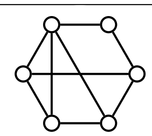
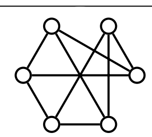
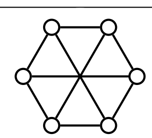
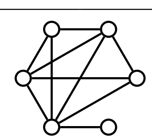
G167		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222
G168		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G169		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, $\underline{132}$, $\underline{231}$, 213, 312
G170		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G171		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, $\underline{1131}$, $\underline{1311}$, $\underline{123}$, $\underline{321}$, $\underline{132}$, $\underline{231}$, 213, 312
G172		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G173		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G174		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, $\underline{1131}$, $\underline{1311}$, 123, 321, $\underline{132}$, $\underline{231}$, 213, 312, 33
G175		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, $\underline{132}$, $\underline{231}$, 213, 312, 33, $\boxed{141}$
G177		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312

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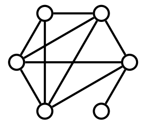
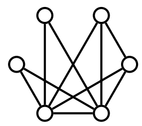
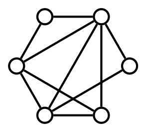
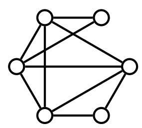
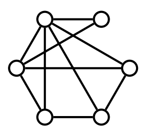
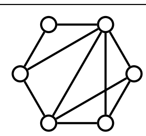
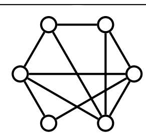
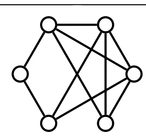
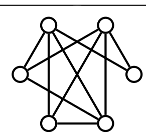
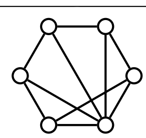
G178		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G179		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G180		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G181		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G182		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G183		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G184		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G185		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G186		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G187		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312

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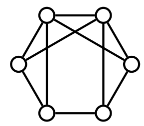
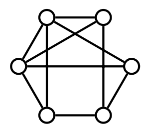
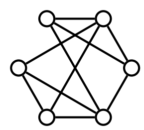
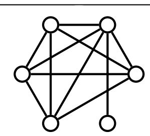
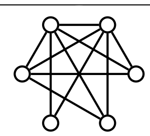
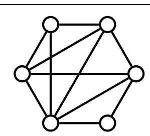
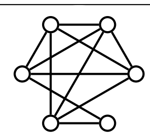
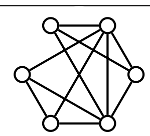
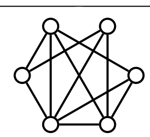
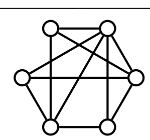
G188		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G189		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 141
G190		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G191		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 114, 411, 141
G192		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G193		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312
G194		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G195		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G196		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G197		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 141

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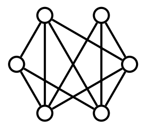
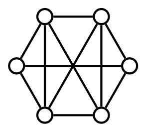
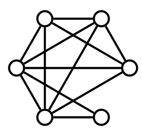
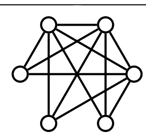
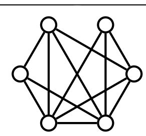
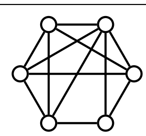
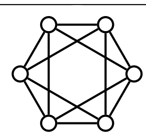
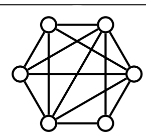
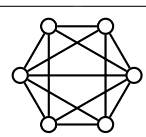
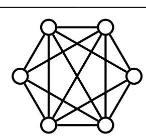
G198		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G199		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G200		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G201		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 141
G202		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33
G203		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G204		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G205		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G206		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42
G207		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42

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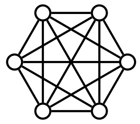
G208		111111, 11112, 21111, 11121, 12111, 11211, 1122, 2211, 1212, 2121, 1221, 2112, 222, 1113, 3111, 1131, 1311, 123, 321, 132, 231, 213, 312, 33, 114, 411, 141, 24, 42, 15, 51

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