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THE MULTIPLICITY OF A_α-EIGENVALUES OF GRAPHS*

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Abstract. For a graph G and real number $\alpha \in [0,1]$, the A_α-matrix of G is defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of the vertex degrees of G. In this paper, the largest multiplicity of the A_α-eigenvalues of a broom tree is considered, and all graphs with an A_α-eigenvalue of multiplicity at least $n-2$ are characterized.

Key words. A_α-matrix, Eigenvalue, Multiplicity.

AMS subject classifications. 05C50.

1. Introduction. Unless stated otherwise, we follow [2] for terminology and notations. All graphs considered here are simple and undirected. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G is denoted by $A(G)$. The (i,j)-entry of $A(G)$ is 1 if $v_iv_j \in E(G)$, and otherwise 0. Let $D(G)$ be the diagonal matrix of the vertex degrees of G. For real number $\alpha \in [0,1]$, Nikiforov [19] defined the A_α-matrix of G:

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G).$$

It is clear that $A_0(G)$ is the adjacency matrix, and $A_{\frac{1}{2}}(G)$ is essentially equivalent to the signless Laplacain matrix. The eigenvalues of $A_\alpha(G)$ are called A_α-eigenvalues of G. Clearly, the A_α-eigenvalues are the vertex degrees of G when $\alpha = 1$. Thus, unless otherwise specified, we only consider the case of $0 \le \alpha < 1$ throughout this paper. For more results about A_α-matrix, one can see [14, 15, 16, 17, 18, 20, 21, 22, 29].

The study of the eigenvalue multiplicity is a classical topic in spectral graph theory. Biggs [1] presented that for any symmetric graph with valency k, every adjacency eigenvalue λ ($\neq \pm k$) has multiplicity at least two. In [28], Terwilliger obtained a lower bound on the eigenvalue multiplicity for highly symmetric graphs. Yamazaki [30] proved that if a bipartite distance-regular graph has an eigenvalue with multiplicity equals its valency, then such graph is 2-homogeneous. Furthermore, the relationship between the eigenvalue multiplicity and the valency of triangle-free distance-regular graphs was investigated in [6, 11, 12]. The upper bounds on the eigenvalue multiplicity for cubic graphs and triangle-free graphs were considered in [24, 25]. The multiplicity of a specific eigenvalue was also studied in many papers. The multiplicity of the eigenvalue zero of a graph is called its nullity. There are many studies about the nullity of graphs (see, for example, [7, 8, 9, 10, 23]). In particular, Cheng and Liu [7] determined the graphs with the nullity $n-2$ or $n-3$. In [8], the bipartite graphs with nullity $n-4$ and the regular bipartite graphs with nullity $n-6$

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were characterized. For A_α-matrix, Gardoso, Pastén and Rojo [5] considered the multiplicity of α as an A_α-eigenvalue of a graph. Along this line, we study the multiplicity of the A_α-eigenvalue of a graph.

For a graph G, we use $M_\alpha(G)$ to denote the largest multiplicity for its A_α-eigenvalues, that is,

$$M_\alpha(G) = \max\{m(\lambda) : \lambda \text{ is an } A_\alpha\text{-eigenvalue of } G\},$$

where $m(\lambda)$ is the multiplicity of λ. Let $d(G)$ be the diameter of a connected graph G. Nikiforov presented that $A_\alpha(G)$ has at least $d(G)+1$ distinct eigenvalues (see Corollary 33 in [19]). Therefore, for a connected graph we obtain that

$$M_\alpha(G) \le n - d(G). \tag{1.1}$$

Note that the upper bound is sharp. Brualdi and Goldwasser [3] defined the *broom* $B_{n,k}$ as follows: it is a tree obtained from the path P_k by attaching $n-k$ pendent vertices to an end vertex of P_k. The broom tree has the extremal values with respect to some spectral parameters (see, for example, [13, 26, 27, 31]). Clearly, $B_{n,k} \cong K_{1,n-1}$ if $k = 1, 2$ and $B_{n,k} \cong P_n$ if $k = n-1, n$. It is easy to see that both $K_{1,n-1}$ and P_n achieve the upper bound in (1.1). In Section 2, we will determine the value of $M_\alpha(B_{n,k})$ for $3 \le k \le n-2$, and show that

$$M_\alpha(B_{n,k}) = n - d(B_{n,k}) - 1 \tag{1.2}$$

if $\alpha > 2/3$. Clearly, $M_\alpha(K_n) = n-1$ and $M_\alpha(nK_1) = n$. Hence, any integer from 1 to n is a possible value of $M_\alpha(G)$. It is natural to consider the following problem: *Characterize all graphs with* $M_\alpha(G) = i$ *for* $i = 1, \ldots, n$. We consider the problem for some special values. In Section 3, we determine all graphs with an A_α-eigenvalue of multiplicity at least $n-2$.

2. The largest multiplicity of the A_α-eigenvalues of a broom tree. Let S be a symmetric real matrix whose rows and columns are indexed by $X = \{1, 2, \ldots, n\}$. Let $\{X_1, \ldots, X_m\}$ be a partition of X. The matrix S may be represented as

$$S = \begin{bmatrix} S_{1,1} & \cdots & S_{1,m} \\ \vdots & \ddots & \vdots \\ S_{m,1} & \cdots & S_{m,m} \end{bmatrix},$$

where $S_{i,j}$ is a sub-matrix (block) of S with respect to rows in X_i and columns in X_j. Let R be a matrix of order m whose (i,j)-entry equals the average row sum of $S_{i,j}$. We say that R is a *quotient matrix* of S corresponding to this partition. If the row sum of each block $S_{i,j}$ is constant, then the partition is called *equitable*. The following lemma presents the relationship between the eigenvalues of S and R.

Lemma 2.1. ([4]) *Let R be a quotient matrix of a symmetric real matrix S with respect to an equitable partition. If λ is an eigenvalue of R, then λ is also an eigenvalue of S.*

Lemma 2.2. ([14]) *Let G be a graph with an independent set of order s. If all vertices in this independent set have the same neighbours and the same degrees σ, then $\sigma\alpha$ is an A_α-eigenvalue of G with multiplicity at least $s-1$.*

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Let $\kappa_0, \kappa_1, \ldots, \kappa_i, \ldots$ be a sequence defined as follows:

$$\kappa_0 = 0, \quad \kappa_1 = \begin{vmatrix} -\alpha & \alpha-1 \\ \alpha-1 & 0 \end{vmatrix}, \quad \ldots, \quad \kappa_i = \begin{vmatrix} -\alpha & \alpha-1 & & & \\ \alpha-1 & -\alpha & \alpha-1 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha-1 & -\alpha & \alpha-1 \\ & & & \alpha-1 & 0 \end{vmatrix}, \quad \ldots \tag{2.3}$$

We remark that any entry in the above matrices is zero if it does not belong to the three middle diagonal lines. By computing the determinant, it is easy to see that

$$\kappa_i + \alpha\kappa_{i-1} + (\alpha-1)^2\kappa_{i-2} = 0$$

for every integer $i \geq 2$. Thus, the characteristic equation of the above recurrence formulas is

$$t^2 + \alpha t + (\alpha-1)^2 = 0.$$

If $1 > \alpha > 2/3$, then it follows that $\alpha^2 - 4(\alpha-1)^2 > 0$ and so the characteristic equation has two distinct real roots t_1 and t_2 such that

$$t_1 + t_2 = -\alpha, \quad t_1 t_2 = (\alpha-1)^2.$$

By the theory of linear recurrence equations, there exist two real numbers a and b such that $\kappa_i = at_1^i + bt_2^i$ for $i \geq 0$. Note that $0 = \kappa_0 = a + b$. It follows that $a = -b$. Thus, we obtain that

$$\kappa_i = a(t_1^i - t_2^i). \tag{2.4}$$

Since $t_1 + t_2 = -\alpha < 0$ and $t_1 t_2 = (\alpha-1)^2 > 0$, we see that $t_1^i - t_2^i \neq 0$. Hence, $\kappa_i \neq 0$ $(i \geq 1)$ if $a \neq 0$. Note that

$$\kappa_1 = \begin{vmatrix} -\alpha & \alpha-1 \\ \alpha-1 & 0 \end{vmatrix} = -(\alpha-1)^2 \neq 0.$$

Also, since $\kappa_1 = a(t_1 - t_2)$ (by equation (2.4)), it follows that $a \neq 0$. Thus, we obtain the following lemma.

Lemma 2.3. *Let $\kappa_0, \kappa_1, \ldots, \kappa_i, \ldots$ be a sequence defined as in (2.3). If $1 > \alpha > 2/3$, then $\kappa_i \neq 0$ for any nonnegative integer i.*

Now let us give the main result of this section.

Theorem 2.4. *Let $3 \leq k \leq n-2$. If $1 > \alpha > 2/3$, then $M_\alpha(B_{n,k}) = n-k-1$. If $0 \leq \alpha \leq 2/3$, then $M_\alpha(B_{n,k}) = n-k-1$ or $n-k$.*

Proof. Suppose that $1 > \alpha > 2/3$. We consider the partition $\{V_1, V_2, \ldots, V_{k+1}\}$ of $V(B_{n,k})$, where V_1 contains the $n-k$ attached pendent vertices and $|V_i| = 1$ for $2 \leq i \leq k+1$. According to this partition, we obtain the quotient matrix of $A_\alpha(B_{n,k})$:

$$R = \begin{bmatrix} \alpha & 1-\alpha & & & & \\ (n-k)(1-\alpha) & (n-k+1)\alpha & 1-\alpha & & & \\ & 1-\alpha & 2\alpha & 1-\alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-\alpha & 2\alpha & 1-\alpha \\ & & & & 1-\alpha & \alpha \end{bmatrix}.$$

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Hence,

$$|\alpha I - R| = \begin{vmatrix} 0 & \alpha-1 & & & & & \\ (n-k)(\alpha-1) & -(n-k)\alpha & \alpha-1 & & & & \\ & \alpha-1 & -\alpha & \alpha-1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \alpha-1 & -\alpha & \alpha-1 \\ & & & & \alpha-1 & 0 \end{vmatrix} = -(n-k)(\alpha-1)^2\kappa_{k-1}.$$

Using Lemma 2.3, it follows that $\kappa_{k-1} \neq 0$, and so $|\alpha I - R| \neq 0$. This implies that α cannot be an eigenvalue of R. Lemma 2.2 shows that α is an A_α-eigenvalue of $B_{n,k}$ with multiplicity at least $n-k-1$. Combining these observations and Lemma 2.1, one can see that the A_α-spectrum of $B_{n,k}$ contains all eigenvalues of R together with α of multiplicity $n-k-1$. Then we will show that R has $k+1$ distinct eigenvalues. Otherwise, assume that λ is an eigenvalues of R with multiplicity at least two. Hence, there exists a nonzero eigenvector $\nu = (\nu_1, \nu_2, \ldots, \nu_{k+1})^t$ of λ such that $\nu_1 = 0$. Since $R\nu = \lambda\nu$, it follows that $\lambda\nu_1 = \alpha\nu_1 + (1-\alpha)\nu_2$. Using $\nu_1 = 0$ in the above equation, we have $\nu_2 = 0$. Similarly, we obtain that $\nu_3 = \cdots = \nu_{k+1} = 0$, which contradicts the fact that ν is nonzero. Therefore,

$$M_\alpha(B_{n,k}) = \max\{n-k-1, 1\} = n-k-1,$$

as required. If $0 \leq \alpha \leq 2/3$, then Lemma 2.2 shows that $M_\alpha(B_{n,k}) \geq n-k-1$. Using (1.1), we have $M_\alpha(B_{n,k}) \leq n-k$, thus the result follows. ∎

3. Graphs containing an A_α-eigenvalue of multiplicity at least $n-2$. For any connected graph with at least two vertices, the Perron-Frobenius Theory shows that its largest A_α-eigenvalue is simple. Then we obtain the following result.

THEOREM 3.1. *Let G be a graph of order n. Then $M_\alpha(G) = n$ if and only if $G \cong nK_1$.*

THEOREM 3.2. *Let G be a graph of order $n \geq 2$. Then $M_\alpha(G) = n-1$ if and only if*
(i) $G \cong K_n$, or
(ii) $G \cong K_p \cup (n-p)K_1$ with $n-1 \geq p \geq 2$ and $\alpha = 1/p$.

Proof. Suppose that $M_\alpha(G) = n-1$. If G is a connected graph, then (1.1) implies that G is a complete graph. If G is disconnected, then clearly all but one components are isolated vertices. Thus, $G \cong K_p \cup (n-p)K_1$ with $n-1 \geq p \geq 2$. The A_α-eigenvalues of K_p are $p-1, p\alpha-1, \ldots, p\alpha-1$. Since 0 is the A_α-eigenvalue of G with multiplicity $n-1$, it follows that $\alpha = 1/p$. Thus, we complete the proof. ∎

In the following, we will determine the graphs with $M_\alpha(G) = n-2$. We first consider connected graphs. Let $\mathcal{G}$ be a set of connected graphs on n vertices: $\mathcal{G} = \{K_1 \vee 2K_{\frac{n-1}{2}}, K_1 \vee (K_1 \cup K_{n-2}), K_s \vee (K_1 \cup K_{n-s-1}), sK_1 \vee (K_1 \cup K_{n-s-1}), K_1 \vee K_{\frac{n-1}{2},\frac{n-1}{2}}, K_s \vee (n-s)K_1, K_{s,n-s}\}$. For conciseness, we use J and $\mathbf{0}$ to denote the all ones matrix and all zeros matrix of appropriate size, respectively.

LEMMA 3.3. *Let G be a connected graph of order $n \geq 3$. If $M_\alpha(G) = n-2$, then $G \in \mathcal{G}$.*

Proof. Suppose that λ is an A_α-eigenvalue of G with multiplicity $n-2$. Thus, $rank(A_\alpha(G) - \lambda I) = 2$. Let us consider the matrix $\tilde{A} = \frac{A_\alpha(G)-\lambda I}{1-\alpha}$. Suppose that the degrees of G are denoted by $d_1, d_2, \ldots, d_n$. Let $\varepsilon_i = \frac{d_i\alpha-\lambda}{1-\alpha}$ for $i = 1, \ldots, n$. Hence, $\tilde{A} = diag(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) + A(G)$. The following fact is direct.

Fact 1. If two diagonal entries of $\tilde{A}$ are equal, then the corresponding two vertices have the same degree.

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Clearly, $rank(\tilde{A})=2$. Assume that its first two rows (say v_1-row and v_2-row) are linear independent. It follows that:

Fact 2. Any other row of $\tilde{A}$ must be a linear combination of v_1-row and v_2-row.

Let $V^*=V(G)\backslash\{v_1,v_2\}$. We infer that any vertex of V^* is adjacent to v_1 or v_2. Otherwise, assume that w is a vertex of V^* which is nonadjacent to v_1 and v_2, the (v_1,w)-entry and (v_2,w)-entry equal zero. Hence, Fact 2 shows that the (u,w)-entry of $\tilde{A}$ is zero for any vertex $u\in V(G)\backslash\{w\}$, and so w is an isolated vertex. But this contradicts the connectivity of G. We denote by $N(v_i)$ the set of neighbours of v_i in G. Let $N_1=N(v_1)\backslash\{v_2\}$ and $N_2=N(v_2)\backslash\{v_1\}$. Thus, $V^*=N_1\cup N_2$. We divide our proof into five cases:

(I) $N_1\nsubseteq N_2$, $N_2\nsubseteq N_1$ and $N_1\cap N_2\neq\emptyset$;
(II) $N_2\neq\emptyset$, $N_2\neq N_1$ and $N_2\subseteq N_1$;
(III) $N_1\neq\emptyset$, $N_2\neq\emptyset$ and $N_1\cap N_2=\emptyset$;
(IV) $N_1=N_2\neq\emptyset$;
(V) $N_1=\emptyset$ and $N_2\neq\emptyset$.

Therefore, the corresponding possible structures of $\tilde{A}$ should be as follows:

$$\underset{(I)}{\begin{array}{c} v_1\\ v_2\\ V_1\\ V_2\\ V_3\end{array}\begin{bmatrix}\varepsilon_1 & b & J & J & \mathbf{0}\\ b & \varepsilon_2 & J & \mathbf{0} & J\\ J & J & M_{1,1} & M_{1,2} & M_{1,3}\\ J & \mathbf{0} & M_{2,1} & M_{2,2} & M_{2,3}\\ \mathbf{0} & J & M_{3,1} & M_{3,2} & M_{3,3}\end{bmatrix}},\quad \underset{(II)}{\begin{array}{c} v_1\\ v_2\\ V_1\\ V_2\end{array}\begin{bmatrix}\varepsilon_1 & b & J & J\\ b & \varepsilon_2 & J & \mathbf{0}\\ J & J & M_{1,1} & M_{1,2}\\ J & \mathbf{0} & M_{2,1} & M_{2,2}\end{bmatrix}},$$

$$\underset{(III)}{\begin{array}{c} v_1\\ v_2\\ V_1\\ V_2\end{array}\begin{bmatrix}\varepsilon_1 & b & J & \mathbf{0}\\ b & \varepsilon_2 & \mathbf{0} & J\\ J & \mathbf{0} & M_{1,1} & M_{1,2}\\ \mathbf{0} & J & M_{2,1} & M_{2,2}\end{bmatrix}},\quad \underset{(IV)}{\begin{array}{c} v_1\\ v_2\\ V_1\end{array}\begin{bmatrix}\varepsilon_1 & b & J\\ b & \varepsilon_2 & J\\ J & J & M_{1,1}\end{bmatrix}},\quad \underset{(V)}{\begin{array}{c} v_1\\ v_2\\ V_1\end{array}\begin{bmatrix}\varepsilon_1 & b & \mathbf{0}\\ b & \varepsilon_2 & J\\ \mathbf{0} & J & M_{1,1}\end{bmatrix}},$$

where $b\in\{0,1\}$ and $M_{i,j}$ denotes a block sub-matrix. We next show the properties of the block sub-matrices of $\tilde{A}$.

Fact 3. If $i\neq j$, then $M_{i,j}=\mathbf{0}$ or J.

Proof of Fact 3. Note that each entry of $M_{i,j}$ is 1 or 0. According to Fact 2, it is clear that the entries of each row of $M_{i,j}$ are either all ones or all zeros. The same property should also apply to $M_{j,i}$. Hence, we see that $M_{i,j}=\mathbf{0}$ or J.

Fact 4. If the size of $M_{i,i}$ is at least 2, then $M_{i,i}=\mathbf{0}$ or J.

Proof of Fact 4. Since the off-diagonal entry of $M_{i,i}$ is 1 or 0, this claim follows immediately from Fact 2 and the symmetry of $M_{i,i}$.

In other words, Fact 4 also implies that V_i is either an independent set or a clique.

Fact 5. G dose not contain an induced P_4.

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Proof of Fact 5. If not, we can obtain a sub-matrix of $\tilde{A}$ with respect to P_4:

$$\begin{bmatrix} * & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ 0 & 1 & * & 1 \\ 0 & 0 & 1 & * \end{bmatrix}.$$

The first two rows of the above matrix are linear independent. But clearly its third row cannot be represented as a linear combination of the first two rows due to the last column, contradicting $rank(\tilde{A}) = 2$. Therefore, P_4 cannot be an induced subgraph of G.

(I) Suppose that $\{v_1, v_2, V_1, V_2, V_3\}$ is the partition of $V(G)$ corresponding to the partition of $\tilde{A}$. Let $\theta_1 = [\varepsilon_1, b, J, J, \mathbf{0}]$ and $\theta_2 = [b, \varepsilon_2, J, \mathbf{0}, J]$ be the first two rows of $\tilde{A}$.

Case I-1. $M_{1,2} = \mathbf{0}$ and $M_{1,3} = \mathbf{0}$.

According to Fact 2, each row of $[J, J, M_{1,1}, M_{1,2}, M_{1,3}]$ should be represented as a linear combination of θ_1 and θ_2. Assume that the first row of $[J, J, M_{1,1}, M_{1,2}, M_{1,3}]$ is equal to $k_1\theta_1 + k_2\theta_2$. Since $M_{1,2} = \mathbf{0}$ and $M_{1,3} = \mathbf{0}$, it follows that $k_1 \cdot 1 + k_2 \cdot 0 = 0$, $k_1 \cdot 0 + k_2 \cdot 1 = 0$ and $k_1\varepsilon_1 + k_2 b = 1$, but clearly these three equations cannot be simultaneously true.

Case I-2. $M_{1,2} = J$ and $M_{1,3} = J$.

Consider the sub-matrix $[J, J, M_{1,1}, M_{1,2}, M_{1,3}]$, and suppose that one of its rows is

$$k_1\theta_1 + k_2\theta_2 = [k_1\varepsilon_1 + k_2 b, k_1 b + k_2\varepsilon_2, (k_1 + k_2)J, k_1 J, k_2 J].$$

Since $M_{1,2} = J$ and $M_{1,3} = J$, it follows that $k_1 = k_2 = 1$. Hence, $M_{1,1} = 2J$. According to Fact 4, we see that the size of $M_{1,1}$ is one, and so $M_{1,1} = 2$. Moreover, since $k_1\varepsilon_1 + k_2 b = 1$ and $k_1 b + k_2\varepsilon_2 = 1$, we have $\varepsilon_1 = \varepsilon_2$ and $b + \varepsilon_1 = 1$.

Subcase I-2.1. $b = 1$.

Thus, $\varepsilon_1 = \varepsilon_2 = 0$. In this case, $\theta_1 = [0, 1, J, J, \mathbf{0}]$ and $\theta_2 = [1, 0, J, \mathbf{0}, J]$. According to Fact 2, one can easily obtain that each row of $[J, \mathbf{0}, M_{2,1}, M_{2,2}, M_{2,3}]$ is equal to θ_2 (since its second column is $\mathbf{0}$), yielding $M_{2,2} = \mathbf{0}$ and $M_{2,3} = J$. Similarly, since each row of $[\mathbf{0}, J, M_{3,1}, M_{3,2}, M_{3,3}]$ is equal to θ_1, we have $M_{3,2} = J$ and $M_{3,3} = \mathbf{0}$. To summarize what we have obtained:

- $b = 1$ implies that v_1 is adjacent to v_2;
- $\varepsilon_1 = \varepsilon_2$ and Fact 1 imply $|V_2| = |V_3|$;
- $M_{1,1} = 2$ implies $|V_1| = 1$;
- $M_{2,2} = \mathbf{0}$ and $M_{3,3} = \mathbf{0}$ imply that V_2 and V_3 are independent sets;
- $M_{1,2} = J$, $M_{1,3} = J$ and $M_{2,3} = J$ imply that two vertices are adjacent if they belong to two different sets of V_1, V_2, V_3.

The structure of G is depicted in Figure 1-i. Therefore, $G \cong K_1 \vee K_{|V_2|+1,|V_2|+1} \in \mathcal{G}$.

Subcase I-2.2. $b = 0$.

Hence, $\varepsilon_1 = \varepsilon_2 = 1$. It follows from Fact 1 that v_1 and v_2 have the same degree, which implies $|V_2| = |V_3|$. By an argument similar to above, we obtain $M_{2,2} = J$, $M_{3,3} = J$ and $M_{2,3} = \mathbf{0}$. Figure 1-ii shows the structure of G, that is, $G \cong K_1 \vee 2K_{|V_2|+1} \in \mathcal{G}$.

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Case I-3. Without loss of generality, suppose that $M_{1,2} = J$ and $M_{1,3} = \mathbf{0}$.

Consider the sub-matrix $[J, J, M_{1,1}, M_{1,2}, M_{1,3}] = [J, J, M_{1,1}, J, \mathbf{0}]$; by Fact 2, we may assume that one of its rows equals $k_1\theta_1 + k_2\theta_2$. Recall that $\theta_1 = [\varepsilon_1, b, J, J, \mathbf{0}]$ and $\theta_2 = [b, \varepsilon_2, J, \mathbf{0}, J]$. A simple calculation shows that $k_1 = 1$ and $k_2 = 0$. It follows that $\varepsilon_1 = 1$, $b = 1$ and $M_{1,1} = J$. If two vertices $u \in V_2$ and $u' \in V_3$ are nonadjacent, then G contains an induced subgraph $P_4 = uv_1v_2u'$, contradicting Fact 5. Hence, any vertex of V_2 is adjacent to all vertices of V_3, and so $M_{2,3} = J$. Consider the sub-matrix $[J, \mathbf{0}, M_{2,1}, M_{2,2}, M_{2,3}] = [J, \mathbf{0}, J, M_{2,2}, J]$. By Fact 2, we may assume that one of its rows is $k_1'\theta_1 + k_2'\theta_2 = [k_1' + k_2', k_1' + k_2'\varepsilon_2, (k_1' + k_2')J, k_1'J, k_2'J]$. Thus, $k_1' = 0$ and $k_2' = 1$, and so $\varepsilon_2 = 0$ and $M_{2,2} = \mathbf{0}$. Hence, $\theta_1 = [1, 1, J, J, \mathbf{0}]$ and $\theta_2 = [1, 0, J, \mathbf{0}, J]$. Clearly, we see that each row of $[\mathbf{0}, J, M_{3,1}, M_{3,2}, M_{3,3}] = [\mathbf{0}, J, \mathbf{0}, J, M_{3,3}]$ is equal to $\theta_1 - \theta_2$, yielding $M_{3,3} = -J$. Using Fact 4, we obtain that the size of $M_{3,3}$ is one and $M_{3,3} = -1$. It follows that

$$\tilde{A} = \begin{array}{c} v_1 \\ v_2 \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{bmatrix} 1 & 1 & J & J & 0 \\ 1 & 0 & J & \mathbf{0} & 1 \\ J & J & J & J & \mathbf{0} \\ J & \mathbf{0} & J & \mathbf{0} & J \\ 0 & 1 & \mathbf{0} & J & -1 \end{bmatrix}.$$

Figure 1-iii shows the structure of G. Therefore, $G \cong (|V_2| + 1)K_1 \vee (K_{|V_1|+1} \cup K_1) \in \mathcal{G}$.

(II) Suppose that $\{v_1, v_2, V_1, V_2\}$ is the partition of $V(G)$ corresponding to the partition of $\tilde{A}$. Let $\theta_1 = [\varepsilon_1, b, J, J]$ and $\theta_2 = [b, \varepsilon_2, J, 0]$ be the first two rows of $\tilde{A}$.

Case II-1. $b = 0$.

Thus, $\theta_1 = [\varepsilon_1, 0, J, J]$ and $\theta_2 = [0, \varepsilon_2, J, \mathbf{0}]$. See Figure 1-iv. We first claim that any vertex of V_1 is adjacent to all vertices of V_2. Otherwise, if two vertices $u \in V_1$ and $u' \in V_2$ are nonadjacent, then G contains an induced subgraph $P_4 = v_2uv_1u'$, this contradicting Fact 5. Hence, $M_{1,2} = J$. Consider the sub-matrix $[J, \mathbf{0}, M_{2,1}, M_{2,2}] = [J, \mathbf{0}, J, M_{2,2}]$. Assume that one of its rows is $k_1\theta_1 + k_2\theta_2 = [k_1\varepsilon_1, k_2\varepsilon_2, (k_1 + k_2)J, k_1J]$. It follows that $k_1\varepsilon_1 = 1$ and $M_{2,2} = k_1J$. If $M_{2,2} = \mathbf{0}$, then $k_1 = 0$. But this is impossible since $k_1\varepsilon_1 = 1$. Therefore, $M_{2,2} \neq \mathbf{0}$. From Fact 4, we see that V_2 is a clique, and also V_1 is either a clique or an independent set. It follows that $G \cong K_{|V_1|} \vee (K_1 \cup K_{|V_2|+1})$ or $|V_1|K_1 \vee (K_1 \cup K_{|V_2|+1})$, and thus, $G \in \mathcal{G}$.

Case II-2. $b = 1$.

Thus, $\theta_1 = [\varepsilon_1, 1, J, J]$ and $\theta_2 = [1, \varepsilon_2, J, \mathbf{0}]$. Fact 3 shows that $M_{1,2}$ is either $\mathbf{0}$ or J. Suppose first that $M_{1,2} = \mathbf{0}$. Consider the sub-matrix $[J, J, M_{1,1}, M_{1,2}] = [J, J, M_{1,1}, \mathbf{0}]$. Suppose that a row of the above sub-matrix is equal to $k_1\theta_1 + k_2\theta_2 = [k_1\varepsilon_1 + k_2, k_1 + k_2\varepsilon_2, (k_1 + k_2)J, k_1J]$. Since $k_1\varepsilon_1 + k_2 = 1$ and $k_1J = \mathbf{0}$, it follows that $k_1 = 0$ and $k_2 = 1$, yielding $M_{1,1} = J$. Moreover, we see that $M_{2,2} \neq \mathbf{0}$. If not, $[J, \mathbf{0}, M_{2,1}, M_{2,2}] = [J, \mathbf{0}, \mathbf{0}, \mathbf{0}]$ and its rows cannot be the linear combination of θ_1 and θ_2, a contradiction. Thus, V_1 and V_2 are cliques. The structure of G is depicted in Figure 1-v. If the size of $M_{2,2}$ is one (i.e., $|V_2| = 1$), then $G \cong K_1 \vee (K_1 \cup K_{|V_1|+1}) \in \mathcal{G}$. If the size of $M_{2,2}$ is at least two, then Fact 4 shows that $M_{2,2} = J$. Since $M_{1,1} = J$ and $M_{2,2} = J$, it follows from Fact 1 that the vertices in V_1 and V_2 have the same degree, that is, $|V_2| = |V_1| + 1$. Thus, $G \cong K_1 \vee (K_{|V_1|+1} \cup K_{|V_1|+1}) \in \mathcal{G}$.

Now suppose that $M_{1,2} = J$. If $M_{1,1} \neq \mathbf{0}$, then Fact 4 implies that V_1 is a clique. Again, using fact 4, it follows that V_2 is either a clique or an independent set. Figure 1-vi shows the structure of G; it follows that $G \cong K_{|V_1|+1} \vee (K_{|V_2|} \cup K_1)$ or $K_{|V_1|+1} \vee (|V_2| + 1)K_1$, and so $G \in \mathcal{G}$. If $M_{1,1} = \mathbf{0}$, then

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$[J, J, M_{1,1}, M_{1,2}] = [J, J, \mathbf{0}, J]$. Assume that one of the rows of the above sub-matrix is represented as

$$k_1\theta_1 + k_2\theta_2 = [k_1\varepsilon_1 + k_2, k_1 + k_2\varepsilon_2, (k_1 + k_2)J, k_1J].$$

Thus, $k_1\varepsilon_1 + k_2 = 1$, $k_1 + k_2\varepsilon_2 = 1$, $k_1 + k_2 = 0$ and $k_1 = 1$. Therefore, we have $\varepsilon_1 = 2$ and $\varepsilon_2 = 0$, yielding that $\theta_1 = [2, 1, J, J]$ and $\theta_2 = [1, 0, J, \mathbf{0}]$. Consider the sub-matrix $[J, \mathbf{0}, M_{2,1}, M_{2,2}]$. According to Fact 2, it is easy to see that each row of $[J, \mathbf{0}, M_{2,1}, M_{2,2}]$ is equal to θ_2, implying $M_{2,2} = \mathbf{0}$. Since $M_{1,1} = \mathbf{0}$ and $M_{2,2} = \mathbf{0}$, both V_1 and V_2 are independent sets. Moreover, Fact 1 implies that all vertices in $V_1 \cup V_2$ have the same degree. Note that the degree of any vertex in V_1 is $|V_2| + 2$ and the degree of any vertex in V_2 is $|V_1|+1$. Hence, $|V_1| = |V_2|+1$. The structure of G is depicted in Figure 1-vii, and so $G \cong K_1 \vee K_{|V_1|,|V_1|} \in \mathcal{G}$.

(III) Suppose that $\{v_1, v_2, V_1, V_2\}$ is the partition of $V(G)$ corresponding to the partition of $\tilde{A}$. Let $\theta_1 = [\varepsilon_1, b, J, \mathbf{0}]$ and $\theta_2 = [b, \varepsilon_2, \mathbf{0}, J]$ be the first two rows of $\tilde{A}$. See Figure 1-viii. Since P_4 is not an induced subgraph of the connected graph G, one can easily obtain that v_1 is adjacent to v_2, and any vertex of V_1 is adjacent to all vertices of V_2. This implies that $M_{1,2} = J$ and $b = 1$. There are three possible cases for V_1 and V_2: two cliques; two independent sets; one clique and one independent set. Suppose that V_1 and V_2 are two cliques. If $|V_1| \geq 2$ and $|V_2| \geq 2$, then Fact 4 shows that $M_{1,1} = J$ and $M_{2,2} = J$. Note that $\theta_1 = [\varepsilon_1, 1, J, \mathbf{0}]$ and $\theta_2 = [1, \varepsilon_2, \mathbf{0}, J]$. Consider the sub-matrix $[J, \mathbf{0}, M_{1,1}, M_{1,2}] = [J, \mathbf{0}, J, J]$. According to Fact 2, it follows that each row of the above sub-matrix is equal to $\theta_1 + \theta_2 = [\varepsilon_1 + 1, \varepsilon_2 + 1, J, J]$, yielding $\varepsilon_1 + 1 = 1$. Also, since $[\mathbf{0}, J, M_{2,1}, M_{2,2}] = [\mathbf{0}, J, J, J]$, its rows are equal to $\theta_1 + \theta_2 = [\varepsilon_1 + 1, \varepsilon_2 + 1, J, J]$. It follows that $\varepsilon_1+1 = 0$, contradicting $\varepsilon_1+1 = 1$. So, we may assume, without loss of generality, that $|V_1| = 1$. Thus, $G \cong 2K_1 \vee (K_1 \cup K_{|V_2|}) \in \mathcal{G}$. If A_1 and A_2 are two independent sets, then $G \cong K_{|V_1|+1,|V_2|+1} \in \mathcal{G}$. Finally, without loss of generality, suppose that V_1 is a clique and V_2 is an independent set, and thus, $G \cong (|V_2| + 1)K_1 \vee (K_1 \cup K_{|V_1|}) \in \mathcal{G}$.

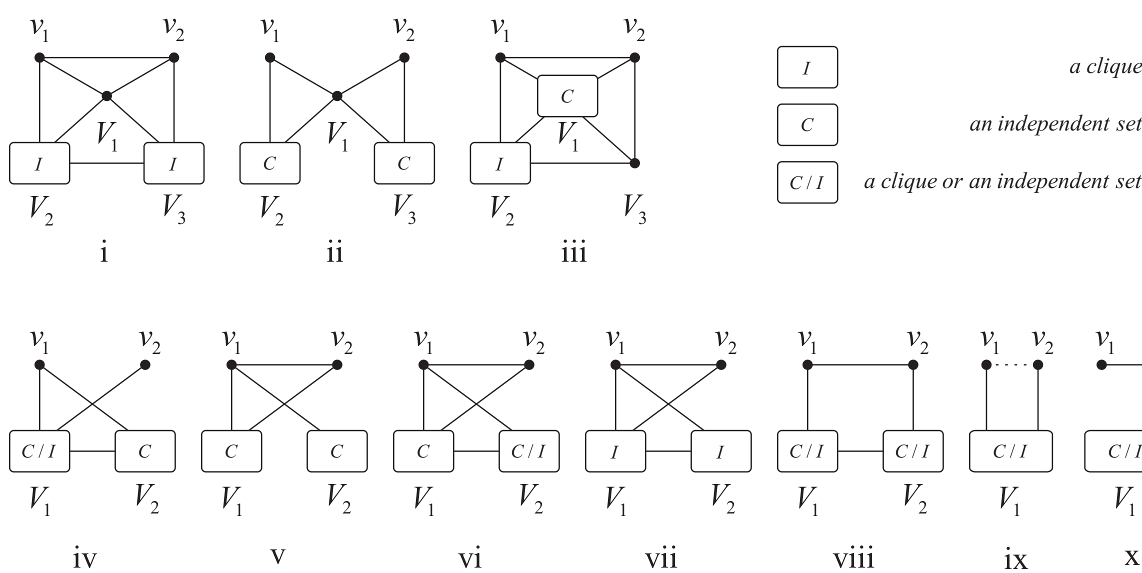

FIGURE 1. *Proof of Lemma 3.3.*

(IV) See Figure 1-ix. According to Fact 4, V_1 is either a clique or an independent set. Note that G cannot be a complete graph, and thus, it is easy to see that $G \cong K_2 \vee |V_1|K_1$, $K_{2,|V_1|}$ or $2K_1 \vee K_{|V_1|}$, yielding $G \in \mathcal{G}$.

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(V) See Figure 1-x. Since G is connected, we obtain that v_1 is adjacent to v_2. By Fact 4, it follows that $G \cong K_1 \vee (K_1 \cup K_{|V_1|})$ or $K_{1,|V_1|+1}$, yielding $G \in \mathcal{G}$. $\square$

From Lemma 3.3, we only need to consider the A_α-eigenvalue multiplicity of graphs belonging to $\mathcal{G}$. The next lemma is needed.

Lemma 3.4. *Let G be a connected graph of order $n \geq 4$ and $M_\alpha(G) = n-2$. If λ is a multiple eigenvalue of $A_\alpha(G)$, then its multiplicity is $n-2$ and the other two A_α-eigenvalues are simple.*

Proof. Clearly, $A_\alpha(G)$ is irreducible, so its spectral radius is simple. If there is another A_α-eigenvalue (not λ) of multiplicity at least two, then $M_\alpha(G) \leq n-3$, a contradiction. Therefore, λ is the only one multiple A_α-eigenvalue, thus this lemma follows. $\square$

Let $P_\alpha^G(x) = |xI - A_\alpha(G)|$ denote the A_α-characteristic polynomial of a graph G. In the following, we consider the A_α-eigenvalues of the graphs in $\mathcal{G}$ by using the A_α-characteristic polynomials. With the help of Matlab and the properties of the A_α-matrices, we obtain the A_α-characteristic polynomials for the graphs in $\mathcal{G}$. For conciseness, we present here the A_α-characteristic polynomials without proofs:

$$\begin{aligned} P_\alpha^{K_s\vee(K_1\cup K_t)}(x) =&(x-(s+t+1)\alpha+1)^{s-1}(x-(t+s)\alpha+1)^{t-1} \\ &\big(x^3+(2-s-t-2\alpha s-\alpha t-\alpha)x^2+(\alpha^2s^2+\alpha^2st+\alpha^2s+2\alpha s^2+3\alpha st \\ &-\alpha s+\alpha t^2-\alpha-2s-t+1)x-\alpha^2s^3-2\alpha^2s^2t-\alpha^2s^2-\alpha^2st^2+\alpha^2st \\ &+2\alpha s^2-\alpha st+\alpha s+st-s\big), \end{aligned} \tag{3.5}$$

$$\begin{aligned} P_\alpha^{sK_1\vee(K_1\cup K_t)}(x) =&(x-(t+1)\alpha)^{s-1}(x-(s+t)\alpha+1)^{t-1} \\ &\big(x^3+(1-t-2\alpha s-\alpha t-\alpha)x^2+(\alpha^2s^2+\alpha^2st+\alpha^2s+3\alpha st+\alpha s \\ &+\alpha t^2-\alpha-st-s)x-2\alpha^2s^2t-2\alpha^2s^2-\alpha^2st^2+\alpha^2st+\alpha s^2t+\alpha s^2 \\ &-2\alpha st+2\alpha s+st-s\big), \end{aligned} \tag{3.6}$$

$$P_\alpha^{K_1\vee 2K_s}(x) =(x-(s+1)\alpha+1)^{2s-2}(x-s-\alpha+1)\big(x^2+(1-s-2\alpha s-\alpha)x+2\alpha s-2s+2\alpha s^2\big), \tag{3.7}$$

$$P_\alpha^{K_1\vee K_{s,s}}(x) = (x-(s+1)\alpha)^{2s-2}(x-(2s+1)\alpha+s)\big(x^2-(\alpha+s+2\alpha s)x+4\alpha s-2s+2\alpha s^2\big), \tag{3.8}$$

$$P_\alpha^{K_s\vee tK_1}(x) = (x-(s+t)\alpha+1)^{s-1}(x-s\alpha)^{t-1}\big(x^2+(1-(s+t)\alpha-s)x+2st\alpha+s^2\alpha-s\alpha-st\big), \tag{3.9}$$

$$P_\alpha^{K_{s,t}}(x) = (x-s\alpha)^{t-1}(x-t\alpha)^{s-1}\big(x^2+\alpha(s+t)x+2\alpha st-st\big). \tag{3.10}$$

Lemma 3.5. *Let $G \cong K_1 \vee 2K_{\frac{n-1}{2}}$ with $n \geq 5$. Then $M_\alpha(G) = n-2$ if and only if $\alpha = \frac{2}{n+1}$.*

Proof. If $\alpha = \frac{2}{n+1}$, then clearly $M_\alpha(G) = n-2$ (see Table 1). Suppose that $M_\alpha(G) = n-2$. Let $s = \frac{n-1}{2} \geq 2$. By (3.7), the A_α-characteristic polynomial of G is as follows:

$$P_\alpha^G(x) = (x-(s+1)\alpha+1)^{2s-2}(x-s-\alpha+1)\big(x^2+(1-s-2\alpha s-\alpha)x+2\alpha s-2s+2\alpha s^2\big)$$

Since $2s-2 \geq 2$, we have $m((s+1)\alpha-1) \geq 2$. Moreover, Lemma 3.4 implies that $m((s+1)\alpha-1) = n-2$, and so $(s+1)\alpha-1$ is a root of the equation

$$(x-s-\alpha+1)\big(x^2+(1-s-2\alpha s-\alpha)x+2\alpha s-2s+2\alpha s^2\big) = 0.$$

Using $x = (s+1)\alpha-1$ in the above equation, we have $-s^2(\alpha-1)^2(\alpha+\alpha s-1) = 0$, yielding $\alpha(s+1) = 1$, as required. $\square$

Lemma 3.6. *Let $G \cong K_1 \vee (K_1 \cup K_{n-2})$ with $n \geq 4$. Then $M_\alpha(G) = n-3$.*

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Proof. According to (3.5), it follows that the A_α-characteristic polynomial of G is as follows:

$$P_\alpha^G(x) = (x-(n-1)\alpha+1)^{n-3}f(x),$$

where

$$f(x) = x^3 + (3-n-(n+1)\alpha)x^2 + (n\alpha^2 + (n^2-n-2)\alpha - n+1)x - (n^2-3n+4)\alpha^2 - (n-5)\alpha + n - 3.$$

By calculation, we have $f((n-1)\alpha-1) = (n-2)(1-\alpha)((n-1)\alpha^2-3\alpha+1)$ and $f(n-2) = (\alpha-1)((n-4)\alpha+1)$. Since $n \geq 4$ and $0 \leq \alpha < 1$, we have $f((n-1)\alpha-1) > 0$ and $f(n-2) < 0$. This implies that $f(x) = 0$ has three distinct roots which belong to intervals $(-\infty, (n-1)\alpha-1)$, $((n-1)\alpha-1, n-2)$ and $(n-2, +\infty)$. Therefore, $M_\alpha(G) = n-3$. ∎

Lemma 3.7. *Let $G \cong K_s \vee (K_1 \cup K_{n-s-1})$ with $2 \leq s \leq n-3$. Then $M_\alpha(G) \leq n-3$.*

Proof. From (1.1), we have $M_\alpha(G) \leq n-2$. By contradiction, assume that $M_\alpha(G) = n-2$. Let $t = n-s-1 \geq 2$. Thus, by (3.5), the A_α-characteristic polynomial of G is as follows:

$$P_\alpha^G(x) = (x-(s+t+1)\alpha+1)^{s-1}(x-(t+s)\alpha+1)^{t-1}f(x), \tag{3.11}$$

where

$$\begin{aligned}f(x) = {} & x^3 + (2-s-t-2\alpha s-\alpha t-\alpha)x^2 + (\alpha^2 s^2 + \alpha^2 st + \alpha^2 s + 2\alpha s^2 + 3\alpha st - \alpha s + \alpha t^2 - \alpha - 2s \\ & - t + 1)x - \alpha^2 s^3 - 2\alpha^2 s^2 t - \alpha^2 s^2 - \alpha^2 st^2 + \alpha^2 st + 2\alpha s^2 - \alpha st + \alpha s + st - s.\end{aligned}$$

If $t \geq 3$, then $(s+t)\alpha-1$ is a multiple A_α-eigenvalue. According to Lemma 3.4, we see that $m((s+t)\alpha-1) = n-2$, and so it is a root of $f(x) = 0$. But

$$f((s+t)\alpha-1) = t(1-\alpha)((t+s)\alpha^2-(2s+1)\alpha+s) \neq 0$$

since $(2s+1)^2 - 4s(s+t) < 0$. Thus, we infer that $t = 2$, and so

$$f(x) = x^3 + (-3\alpha-s-2\alpha s)x^2 + (\alpha^2 s^2 + 3\alpha^2 s + 2\alpha s^2 + 5\alpha s + 3\alpha - 2s - 1)x - \alpha^2 s^3 - 5\alpha^2 s^2 - 2\alpha^2 s + 2\alpha s^2 - \alpha s + s.$$

By calculation, it follows that $f(s\alpha) = s(1-\alpha)^2 > 0$ and $f(s\alpha+1) = -2s(1-\alpha)^2 < 0$. This implies that $f(x) = 0$ has three distinct roots. Note that $(s+t)\alpha-1$ is not a root of $f(x) = 0$. Hence, G has at least four distinct A_α-eigenvalues, contradicting the assumption $M_\alpha(G) = n-2$. Thus, we complete the proof. ∎

Lemma 3.8. *Let $G \cong sK_1 \vee (K_1 \cup K_{n-s-1})$ with $2 \leq s \leq n-3$. Then $M_\alpha(G) = n-2$ if and only if $n = 3s-2$ and $\alpha = \frac{3}{n-1}$.*

Proof. Suppose that $M_\alpha(G) = n-2$. Let $t = n-s-1 \geq 2$. It follows from (3.6) that the A_α-characteristic polynomial of G is as follows:

$$P_\alpha^G(x) = (x-(t+1)\alpha)^{s-1}(x-(s+t)\alpha+1)^{t-1}f(x),$$

where

$$\begin{aligned}f(x) = {} & x^3 + (1-t-2\alpha s-\alpha t-\alpha)x^2 + (\alpha^2 s^2 + \alpha^2 st + \alpha^2 s + 3\alpha st + \alpha s + \alpha t^2 - \alpha - st - s)x \\ & - 2\alpha^2 s^2 t - 2\alpha^2 s^2 - \alpha^2 st^2 + \alpha^2 st + \alpha s^2 t + \alpha s^2 - 2\alpha st + 2\alpha s + st - s.\end{aligned}$$

By calculation, we obtain that $f(s\alpha) = s(t-1)(1-\alpha)^2 > 0$ and $f(s\alpha+t-1) = -st(t-1)(1-\alpha)^2 < 0$. This implies that $f(x) = 0$ has three distinct roots. Note that $(t+1)\alpha$ and $(s+t)\alpha-1$ are A_α-eigenvalues of G.

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If $(t+1)\alpha \neq (s+t)\alpha - 1$, then $M_\alpha(G) \leq n-3$, a contradiction. Therefore, we have $(t+1)\alpha = (s+t)\alpha - 1$, that is,

$$\alpha = \frac{1}{s-1}. \tag{3.12}$$

Moreover, since $M_\alpha(G) = n-2$, by Lemma 3.4, we see that $(t+1)\alpha$ must be a root of $f(x) = 0$. Thus,

$$f((t+1)\alpha) = -s(\alpha-1)^2(\alpha - t - \alpha s + 2\alpha t + \alpha t^2 - \alpha st + 1) = 0,$$

yielding

$$\alpha - t - \alpha s + 2\alpha t + \alpha t^2 - \alpha st + 1 = 0. \tag{3.13}$$

Combining (3.12) and (3.13), it follows that $t = 2s-3$. Since $t = n-s-1$, we obtain that $n = 3s-2$ and $\alpha = \frac{3}{n-1}$, as required. Conversely, if $n = 3s-2$ and $\alpha = \frac{3}{n-1}$, then Table 1 shows that $M_\alpha(G) = n-2$. ∎

Lemma 3.9. *Let $G \cong K_1 \vee K_{\frac{n-1}{2},\frac{n-1}{2}}$ where $n \geq 5$ is odd. Then $M_\alpha(G) = n-2$ if and only if and $\alpha = \frac{4}{n+1}$.*

Proof. If $\alpha = \frac{4}{n+1}$, then Table 1 shows that $M_\alpha(G) = n-2$. Conversely, suppose that $M_\alpha(G) = n-2$. Let $s = \frac{n-1}{2} \geq 2$. By (3.8), the A_α-characteristic polynomial of G is as follows:

$$P_\alpha^G(x) = (x-(s+1)\alpha)^{2s-2}(x-(2s+1)\alpha+s)\big(x^2 - (\alpha+s+2\alpha s)x + 4\alpha s - 2s + 2\alpha s^2\big).$$

Since $s \geq 2$, we obtain that $(s+1)\alpha$ is a multiple eigenvalue. According to Lemma 3.4, it follows that $m((s+1)\alpha) = n-2$, and so it must be a root of

$$(x-(2s+1)\alpha+s)\big(x^2 - (\alpha+s+2\alpha s)x + 4\alpha s - 2s + 2\alpha s^2\big) = 0.$$

Using $x = (s+1)\alpha$ in the above equation, it follows that $s^2(1-\alpha)^2(\alpha+s\alpha-2) = 0$, yielding $\alpha = \frac{2}{s+1} = \frac{4}{n+1}$, which completes the proof. ∎

Lemma 3.10. *Let $G \cong K_s \vee (n-s)K_1$ with $n-2 \geq s \geq 2$. Then $M_\alpha(G) = n-2$ if and only if and $\alpha = \frac{1}{n-s}$.*

Proof. Suppose that $M_\alpha(G) = n-2$. From (3.9), the A_α-characteristic polynomial of G is as follows:

$$P_\alpha^G(x) = (x-n\alpha+1)^{s-1}(x-s\alpha)^{n-s-1}\big(x^2 + (1-\alpha n - s)x + s^2 - ns - \alpha s^2 - \alpha s + 2\alpha ns\big).$$

Let $f(x) = x^2 + (1-\alpha n - s)x + s^2 - ns - \alpha s^2 - \alpha s + 2\alpha ns$. It is easy to see that

$$f(s\alpha) = -s(n-s)(1-\alpha)^2 \neq 0 \quad \text{and} \quad f(n\alpha-1) = s(1-\alpha)(s-n+1) \neq 0.$$

If $s\alpha \neq n\alpha - 1$, then G has four distinct A_α-eigenvalues, a contradiction. This implies that $s\alpha = n\alpha - 1$, i.e., $\alpha = \frac{1}{n-s}$. Conversely, if $\alpha = \frac{1}{n-s}$, then Table 1 shows that $M_\alpha(G) = n-2$. Thus, we complete the proof. ∎

Lemma 3.11. *Let $G \cong K_{s,n-s}$ with $1 \leq s \leq n-1$. Then $M_\alpha(G) = n-2$ if and only if $s = 1$ or $s = \frac{n}{2}$.*

Proof. Suppose that $M_\alpha(G) = n-2$. Let $t = n-s \geq 1$. From (3.10), the A_α-characteristic polynomial of G is as follows:

$$P_\alpha^G(x) = (x-s\alpha)^{s-1}(x-t\alpha)^{t-1}\big(x^2 - \alpha(s+t)x + 2\alpha st - st\big).$$

Let $f(x) = x^2 + \alpha(s+t)x + 2\alpha st - st$. Suppose $s > t$. If $t \geq 2$, then $s\alpha$ is a multiple A_α-eigenvalue. By Lemma 3.4, we obtain that $m(s\alpha) = n-2$, and so $f(x) = 0$. But, $f(s\alpha) = -st(1-\alpha)^2 \neq 0$, a contradiction. If $t = 1$ or $s = t$, then Table 1 shows that $M_\alpha(G) = n-2$. ∎

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Graphs	A_α-spectra
$K_{1,n-1}$	$\frac{n\alpha+\sqrt{n^2\alpha^2+4(n-1)(1-\alpha)}}{2}, \alpha, \ldots, \alpha, \frac{n\alpha-\sqrt{n^2\alpha^2+4(n-1)(1-\alpha)}}{2}$
$K_{\frac{n}{2},\frac{n}{2}}$	$\frac{n}{2}, \frac{n\alpha}{2}, \ldots, \frac{n\alpha}{2}, \frac{n(2\alpha-1)}{2}$
$K_s \vee (n-s)K_1(\alpha=\frac{1}{n-s})$	$\frac{ns+s-s^2+(n-s-1)\sqrt{s(4n-3s)}}{2(n-s)}, \frac{s}{n-s}, \ldots, \frac{s}{n-s}, \frac{ns+s-s^2-(n-s-1)\sqrt{s(4n-3s)}}{2(n-s)}$
$K_1 \vee K_{\frac{n-1}{2},\frac{n-1}{2}}(\alpha=\frac{4}{n+1})$	$\frac{n^2+4n-5}{2(n+1)}, 2, \ldots, 2, \frac{-n^2+8n+1}{2(n+1)}$
$sK_1 \vee (K_1 \cup K_{2s-3})(\alpha=\frac{3}{n-1})$	$\frac{s^2-2s+2+(s-2)\sqrt{(3s-1)(s-1)}}{s-1}, 2, \ldots, 2, \frac{s^2-2s+2-(s-2)\sqrt{(3s-1)(s-1)}}{s-1}$
$K_1 \vee 2K_{\frac{n-1}{2}}(\alpha=\frac{2}{n+1})$	$\frac{(n-1)^2}{2(n+1)}, \frac{n^2+2n-3}{2(n+1)}, 0, 0, \ldots, 0$

TABLE 1. *The A_α-spectra of graphs in Theorem 3.12.*

According to Lemma 3.3 and Lemmas 3.5-3.11, we obtain the following theorem immediately.

THEOREM 3.12. *Let G be a connected graph of order $n \geq 3$. Then $M_\alpha(G) = n-2$ if and only if*
(1) $G \cong K_{1,n-1}$, or
(2) $G \cong K_{\frac{n}{2},\frac{n}{2}}$ with $n \geq 4$, or
(3) $G \cong K_s \vee (n-s)K_1$ with $2 \leq s \leq n-2$ and $\alpha = \frac{1}{n-s}$, or
(4) $G \cong K_1 \vee K_{\frac{n-1}{2},\frac{n-1}{2}}$ with $n \geq 5$ and $\alpha = \frac{4}{n+1}$, or
(5) $G \cong sK_1 \vee (K_1 \cup K_{n-s-1})$ with $s \geq 3, n = 3s-2$ and $\alpha = \frac{3}{n-1}$, or
(6) $G \cong K_1 \vee 2K_{\frac{n-1}{2}}$ with $n \geq 5$ and $\alpha = \frac{2}{n+1}$.

Now, let us consider the disconnected graphs that have an A_α-eigenvalue with multiplicity $n-2$.

COROLLARY 3.13. *Let G be a disconnected graph of order $n \geq 3$. Then $M_\alpha(G) = n-2$ if and only if*
(1) $G \cong 2K_2$, or
(2) $G \cong 2K_2 \cup (n-4)K_1$ with $n \geq 5$ and $\alpha = 1/2$, or
(3) $G \cong K_2 \cup (n-2)K_1$ with $\alpha \neq 1/2$, or
(4) $G \cong K_{1,s-1} \cup (n-s)K_1$ with $3 \leq s \leq n-1$ and $\alpha = 0$, or
(5) $G \cong K_{\frac{s}{2},\frac{s}{2}} \cup (n-s)K_1$ with $4 \leq s \leq n-1$ and $\alpha = 0$, or
(6) $G \cong K_1 \vee 2K_{\frac{s-1}{2}} \cup (n-s)K_1$ with $3 \leq s \leq n-1$ and $\alpha = \frac{2}{s+1}$.

Proof. Let λ be the A_α-eigenvalue G with multiplicity $n-2$. Let G_1 be a component of G with order $s \geq 3$. Thus, all A_α-eigenvalues of $G - G_1$ are equal to λ, this implies that $G - G_1$ is the union of some isolated vertices, and so $\lambda = 0$. Hence, we see that $\lambda = 0$ is an A_α-eigenvalue of G_1 with multiplicity $s-2$. According to Theorem 3.12 and Table 1, we obtain that $G_1 \cong K_{1,s-1}$ with $\alpha = 0$, $G_1 \cong K_{\frac{s}{2},\frac{s}{2}}$ with $\alpha = 0$ or $G_1 \cong K_1 \vee 2K_{\frac{s-1}{2}}$ with $\alpha = \frac{2}{s+1}$. So in the following we may assume that $G \cong sK_2 \cup (n-2s)K_1$. Clearly, its A_α-eigenvalues are

$$\underbrace{1, 1, \ldots, 1}_{s}, \underbrace{2\alpha-1, 2\alpha-1, \ldots, 2\alpha-1}_{s}, \underbrace{0, 0, \ldots, 0}_{n-2s}.$$

Therefore, we obtain that $G \cong 2K_2$, $G \cong 2K_2 \cup (n-4)K_1$ with $\alpha = 1/2$ or $G \cong K_2 \cup (n-2)K_1$ with $\alpha \neq 1/2$. Thus, we complete the proof. ∎

4. Conclusions. For graphs on n vertices, P_n is the only graph with diameter $n-1$. Note also that P_n has n distinct A_α-eigenvalues. Thus, $M_\alpha(P_n) = n - d(P_n) = 1$. For K_n, clearly $M_\alpha(K_n) = n - d(K_n) = n-1$. Hence, we see that P_n and K_n achieve the upper bound (1.1). In Theorem 3.12, we determine all connected

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graphs with $M_\alpha(G) = n - 2$. The diameter of any graph in Theorem 3.12 is two. Therefore, the graphs, with diameter two, satisfying the equality in (1.1) are also characterized by Theorem 3.12. Motivated by these results, we propose the following problem:

PROBLEM 4.1. *Characterize all graphs G with $M_\alpha(G) = n - d(G)$ for $d(G) = 3, 4, \ldots, n-2$.*

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