

A NOTE ON THE LARGEST EIGENVALUE OF NON-REGULAR GRAPHS*

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Abstract. The spectral radius of connected non-regular graphs is considered. Let λ_1 be the largest eigenvalue of the adjacency matrix of a graph G on n vertices with maximum degree Δ . By studying the λ_1 -extremal graphs, it is proved that if G is non-regular and connected, then $\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}$. This improves the recent results by B.L. Liu et al.

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1. Introduction. Let $G = (V, E)$ be a simple graph on vertex set V and edge set E , where $|V| = n$. The eigenvalues of the adjacency matrix of G are called the eigenvalues of G . The largest eigenvalue of G , denoted by $\lambda_1(G)$, is called the spectral radius of G . Let D denote the diameter of G . We suppose throughout the paper that G is a simple graph. For any vertex u , let $\Gamma(u)$ be the set of all neighbors of u and $d(u) = |\Gamma(u)|$ be the degree of u . A nonincreasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of non-negative integers is called (*connected*) *graphic* if there exists a (connected) simple graph on n vertices, for which d_1, d_2, \dots, d_n are the degrees of its vertices. Let Δ and δ be the maximum and minimum degree of vertices of G , respectively. A graph is called regular if $d(u) = \Delta$ for any $u \in V$. It is easy to see that the spectral radius of a regular graph is Δ with $(1, 1, \dots, 1)^T$ as a corresponding eigenvector. We will use $G - e$ ($G + e$) to denote the graph obtained from G by deleting (adding) the edge e . For other notations in graph theory, we follow from [2].

Stevanović [8] first found a lower bound of $\Delta - \lambda_1$ for the connected non-regular graphs. Then the results from [8] were improved in [9, 4, 7, 3]. In [4, 7], the authors showed that

$$\Delta - \lambda_1 \geq \frac{1}{n(D + 1)} \quad ([4, 7]) \quad (1.1)$$

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and

$$D \leq \frac{3n + \Delta - 5}{\Delta + 1} \quad ([7]). \quad (1.2)$$

B.L. Liu et al. obtained

$$\Delta - \lambda_1 \geq \frac{\Delta + 1}{n(3n + 2\Delta - 4)} \quad ([7]). \quad (1.3)$$

Recently, S.M. Cioabă [3] improved (1.1) as follows:

$$\Delta - \lambda_1 > \frac{1}{nD} \quad ([3]). \quad (1.4)$$

Thus combining (1.2) and (1.4), the inequality (1.3) can be improved as follows:

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 5)}. \quad (1.5)$$

In this note we improve the inequality (1.2) on λ_1 -extremal graphs. Furthermore, we obtain the following inequality which improves (1.5).

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}.$$

2. Preparation. Firstly, we state a well-known result which is just Frobenius's theorem applied to graphs.

LEMMA 2.1. *Let G be a connected graph and $\lambda_1(G)$ be its spectral radius. Then $\lambda_1(G + uv) > \lambda_1(G)$ for any $uv \notin E$.*

DEFINITION 2.2. [7] Let G be a connected non-regular graph. Then the graph G is called λ_1 -extremal if $\lambda_1(G) \geq \lambda_1(G')$ for any other connected non-regular graph G' with the same number of vertices and maximum degree as G .

THEOREM 2.3. *Let G be a λ_1 -extremal graph on n vertices with maximum degree Δ . Define*

$$V_{<\Delta} = \{u : u \in V \text{ and } d(u) < \Delta\}.$$

Then G must have one of the following properties:

- (1) $|V_{<\Delta}| \geq 2$ and $V_{<\Delta}$ induces a complete graph.
- (2) $|V_{<\Delta}| = 1$.
- (3) $V_{<\Delta} = \{u, v\}$, $uv \notin E(G)$ and $d(u) = d(v) = \Delta - 1$.

Proof. By contradiction, suppose that G is a λ_1 -extremal graph without properties (1), (2) and (3). It follows that $|V_{<\Delta}| \geq 2$. Then there are two cases.

Case 1: $V_{<\Delta} = \{u, v\}$, $uv \notin E(G)$, $d(u) < \Delta - 1$ and $d(v) \leq \Delta - 1$. Then the graph $G + uv$ has the same maximum degree as G . By Lemma 2.1, we obtain $\lambda_1(G + uv) > \lambda_1(G)$, contradicting the choice of G .

Case 2: $|V_{<\Delta}| > 2$ and $V_{<\Delta}$ does not induce a complete graph. Then there exist two vertices $u, v \in V_{<\Delta}$ and $uv \notin E(G)$. Similarly arguing to case 1, we obtain $\lambda_1(G + uv) > \lambda_1(G)$, contradicting the choice of G .

Combining the above two cases, the proof follows. \square

Using the properties mentioned in Theorem 2.3, we give the following definition.

DEFINITION 2.4. Let G be a connected non-regular graph on n vertices with maximum degree Δ . Then

- the graph G is called *type-I* if it has property (1),
- the graph G is called *type-II* if it has property (2),
- the graph G is called *type-III* if it has property (3).

LEMMA 2.5. [6] Let G be a simple connected graph with n vertices, m edges and spectral radius $\lambda_1(G)$. Then

$$\lambda_1(G) \leq \frac{\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - \delta n)}}{2}$$

and equality holds if and only if G is either a regular graph or a graph in which each vertex has degree either δ or $n - 1$.

We first consider the λ_1 -extremal graphs with $\Delta = 2$ or $\Delta = n - 1$. When $\Delta = 2$, the λ_1 -extremal graph is the path with $\lambda_1(P_n) = 2\cos(\frac{\pi}{n+1})$. When $\Delta = n - 1$, similarly arguing to Theorem 2.3, we know that the λ_1 -extremal graph is $K_n - e$. By Lemma 2.5, we obtain

$$\lambda_1(K_n - e) = \frac{n - 3 + \sqrt{(n + 1)^2 - 8}}{2}. \quad (2.1)$$

Theorem 2.3 shows that the λ_1 -extremal graphs must be type-I, type-II or type-III, but in what follows, we will prove that when $2 < \Delta < n - 1$, any type-III graph is not λ_1 -extremal.

LEMMA 2.6. [5] Let $\pi = (d_1, d_2, \dots, d_n)$ be a nonincreasing sequence of non-negative integers. Then π is graphic if and only if

$$\sum_{i=1}^n d_i \text{ is even and } \sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\}, \text{ for all } k = 1, 2, \dots, n-1. \quad (2.2)$$

LEMMA 2.7. Let $\pi = (d_1, d_2, \dots, d_n)$ be a nonincreasing sequence of positive

integers and $d_{n-1} \geq 2$, $d_n \geq 1$. Then π is graphic if and only if it is connected graphic.

Proof. If π is connected graphic, then it is obviously graphic. Conversely, suppose that π is graphic and G is a disconnected graph with the degree sequence π . Without loss of generality, suppose that G has two components G_1 and G_2 . Noticing $d_{n-1} \geq 2$ and $d_n \geq 1$, we suppose that any vertex in G_1 has degree at least two and any edge $u_2v_2 \in E(G_2)$. Then it follows that there exists one edge u_1v_1 in G_1 which is not the cut edge, i.e. $G_1 - u_1v_1$ is still connected. Otherwise, G_1 is a tree, a contradiction. Consider $G' = G - u_1v_1 - u_2v_2 + u_1u_2 + v_1v_2$. It is easy to see that G' is a connected graph with the degree sequence π . \square

LEMMA 2.8. Let $\pi = (d_1, d_2, \dots, d_n) = (\Delta, \Delta, \dots, \Delta, \Delta - 1, \Delta - 1)$ and $\pi' = (d'_1, d'_2, \dots, d'_n) = (\Delta, \Delta, \dots, \Delta, \Delta - 2)$ with $2 < \Delta < n - 1$. If π is connected graphic, then π' is connected graphic.

Proof. Since $2 < \Delta < n - 1$, we obtain $d'_{n-1} \geq 3$ and $d'_n \geq 1$. Then by Lemma 2.7, we need only to prove that π' is graphic. Let G be a connected graph with the degree sequence π . Since π is graphic, by Lemma 2.6, we obtain

$$\sum_{i=1}^n d_i \text{ is even and } \sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\}, \text{ for all } k = 1, 2, \dots, n-1.$$

For π' we will prove that (2.2) is still true. Obviously, $\sum_{i=1}^n d'_i = \sum_{i=1}^n d_i$ is even. Then we need only to prove that the inequality is true. We split our proof into four cases.

Case 1: $1 \leq k \leq \Delta - 2$, then $k \leq n - 4$ and

$$\begin{aligned} \sum_{i=1}^k d'_i &= \sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\} \\ &= k(k-1) + k(n-k) \\ &= k(k-1) + \sum_{i=k+1}^n \min\{k, d'_i\}. \end{aligned}$$

Case 2: $1 < k = \Delta - 1$, then $k \leq n - 3$ and

$$k(k-1) + k(n-k) - \sum_{i=1}^k d_i = k(k-1) + k(n-k) - k\Delta = k(n-k-2) \geq k > 1.$$

Thus

$$\sum_{i=1}^k d'_i = \sum_{i=1}^k d_i < k(k-1) + k(n-k) - 1 = k(k-1) + \sum_{i=k+1}^n \min\{k, d'_i\}.$$

Case 3: $\Delta \leq k \leq n - 2$, then

$$\begin{aligned} \sum_{i=1}^k d'_i &= \sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\} \\ &= k(k-1) + (n-k-2)\Delta + 2\Delta - 2 \\ &= k(k-1) + \sum_{i=k+1}^n \min\{k, d'_i\}. \end{aligned}$$

Case 4: $k = n - 1$, then

$$\begin{aligned} k(k-1) + \Delta - \sum_{i=1}^k d_i &= (n-1)(n-2) + \Delta - [(n-1)\Delta - 1] \\ &= (n-2)(n-1-\Delta) + 1 \geq 4, \end{aligned}$$

where the last inequality holds since $2 < \Delta < n - 1$. Hence

$$\begin{aligned} \sum_{i=1}^k d'_i &= \sum_{i=1}^k d_i + 1 < k(k+1) + \Delta - 2 \\ &= k(k-1) + \sum_{i=k+1}^n \min\{k, d'_i\}. \end{aligned}$$

Combining the above four cases, the inequality is true. Then by Lemma 2.6, the π' is graphic. This completes the proof. \square

As we know, *majorization* on degree sequences is defined as follows: for two sequences $\pi = (d_1, d_2, \dots, d_n)$, $\pi' = (d'_1, d'_2, \dots, d'_n)$ we write $\pi \preceq \pi'$ if and only if $\sum_{i=1}^n d_i = \sum_{i=1}^n d'_i$ and $\sum_{i=1}^j d_i \leq \sum_{i=1}^j d'_i$ for all $j = 1, 2, \dots, n$. We claim that G has the *greatest maximum* eigenvalue if $\lambda_1(G) \geq \lambda_1(G')$ for any other graph G' in the class \mathcal{C}_π , where $\mathcal{C}_\pi = \{G : G \text{ is a connected graph with the degree sequence } \pi\}$.

LEMMA 2.9. [1] *Let π and π' be two distinct degree sequences with $\pi \preceq \pi'$. Let G and G' be graphs with the greatest maximum eigenvalues in classes \mathcal{C}_π and $\mathcal{C}_{\pi'}$, respectively. Then $\lambda_1(G) < \lambda_1(G')$.*

THEOREM 2.10. *Let G be a connected graph with degree sequence*

$$\pi = (\Delta, \Delta, \dots, \Delta, \Delta - 1, \Delta - 1)$$

and $2 < \Delta < n - 1$. Then there exists a connected graph G' with degree sequence $\pi' = (\Delta, \Delta, \dots, \Delta, \Delta - 2)$ such that $\lambda_1(G) < \lambda_1(G')$.

Proof. By Lemma 2.8, there exists a connected graph G' with degree sequence $\pi' = (\Delta, \Delta, \dots, \Delta, \Delta - 2)$. We suppose that G (G') is the graph with the greatest

maximum eigenvalue in \mathcal{C}_π ($\mathcal{C}_{\pi'}$). It is obvious that $\pi \preceq \pi'$. By Lemma 2.9, we obtain $\lambda_1(G) < \lambda_1(G')$. \square

THEOREM 2.11. *Let G be a λ_1 -extremal graph on n vertices with the maximum degree Δ and $2 < \Delta < n - 1$. Then G must be either type-I or type-II.*

Proof. Suppose that G is a type-III graph with the greatest maximum eigenvalue in class \mathcal{C}_π , where $\pi = (\Delta, \Delta, \dots, \Delta, \Delta - 1, \Delta - 1)$. By Theorem 2.10, there exists a graph G' with degree sequence $\pi' = (\Delta, \Delta, \dots, \Delta, \Delta - 2)$ and greatest maximum eigenvalue in class \mathcal{C}'_π such that $\lambda_1(G') > \lambda_1(G)$. It follows that G is not λ_1 -extremal. \square

REMARK. Although Theorem 2.11 shows that the λ_1 -extremal graph with $2 < \Delta < n - 1$ must be type-I or type-II, there exist some graphs with property (1) or (2) which are not λ_1 -extremal. Let G_1, G_2 be connected graphs with degree sequences $(3, 3, 3, 3, 2, 2)$, $(5, 5, 5, 5, 5, 5, 2)$, respectively. Clearly G_1 (G_2) is a type-I (type-II) graph. However, by checking the Table 1 of [7], we know they are not the λ_1 -extremal graphs. After some computer experiments, we give a conjecture about the λ_1 -extremal graphs as follows:

CONJECTURE 2.12. *Let G be a connected non-regular graph on n vertices and $2 < \Delta < n - 1$. Then G is λ_1 -extremal if and only if G is a graph with the greatest maximum eigenvalue in classes \mathcal{C}_π and $\pi = (\Delta, \Delta, \dots, \Delta, \delta)$, where*

$$\delta = \begin{cases} (\Delta - 1), & \text{when } n\Delta \text{ is odd,} \\ (\Delta - 2), & \text{when } n\Delta \text{ is even.} \end{cases}$$

3. Main Results.

THEOREM 3.1. *Let G be a type-I or type-II graph on n vertices with diameter D . Then*

$$D \leq \frac{3n + \Delta - 8}{\Delta + 1}. \quad (3.1)$$

Proof. Since G is a type-I or type-II graph, we have $\Delta \geq 3$. Let u, v be two vertices at distance D and $P : u = u_0 \leftrightarrow u_1 \leftrightarrow \dots \leftrightarrow u_D = v$ be the shortest path connecting u and v . We first claim that $|V_{<\Delta} \cap V(P)| \leq 2$. Otherwise, G must be a type-I graph and suppose $\{u_p, u_q, u_r\} \subseteq V(P) \cap V_{<\Delta}$ with $p < q < r$. Then by definition of type-I graph, we obtain that $u_p u_q, u_q u_r$ and $u_p u_r \in E(G)$. Therefore, P is not the shortest path connecting u and v , a contradiction.

Then there are two cases.

Case 1: $V_{<\Delta} \cap V(P) = \emptyset$. Define $T = \{i : i \equiv 0 \pmod{3} \text{ and } i \leq (D - 3)\} \cup \{D\}$. Thus $|T| = \lceil \frac{D+1}{3} \rceil$. Let $d(u_i, u_j)$ denote the distance between u_i and u_j . Since P is

the shortest path connecting u and v , we have $d(u_i, u_j) \geq 3$ and $\Gamma(u_i) \cap \Gamma(u_j) = \emptyset$ for any distinct $i, j \in T$. Notice that $u_i \in V(P)$ for any $i \in T$. We obtain

$$|\Gamma(u_i) - V(P)| = \begin{cases} \Delta - 1, & \text{if } i \in \{0, D\}, \\ \Delta - 2, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} n &\geq |V(P)| + \sum_{i \in T} |\Gamma(u_i) - V(P)| \\ &\geq D + 1 + (|T| - 2)(\Delta - 2) + 2(\Delta - 1) \\ &\geq D + 1 + \left(\frac{D+1}{3} - 2\right)(\Delta - 2) + 2(\Delta - 1). \end{aligned}$$

Thus

$$D \leq \frac{3n - \Delta - 7}{\Delta + 1}.$$

Case 2: Either $V_{<\Delta} \cap V(P) = \{u_p, u_q\}$ with $q = p + 1$ or $V_{<\Delta} \cap V(P) = \{u_p\}$. The proof is similar to the proof of [7]. We obtain the same result

$$D \leq \frac{3n + \Delta - 8}{\Delta + 1}.$$

Combining the above two cases, the proof follows. \square

LEMMA 3.2. [3] *Let G be a connected non-regular graph on n vertices with maximum degree Δ and diameter D . Then*

$$\Delta - \lambda_1 > \frac{1}{nD}.$$

THEOREM 3.3. *Let G be a connected non-regular graph on n vertices with maximum degree Δ . Then*

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}.$$

Proof. Without loss of generality, we suppose that G is a λ_1 -extremal graph. Since G is connected and non-regular, then $n \geq 3$ and $\Delta \geq 2$. When $\Delta = n - 1$ and $n \geq 5$, the λ_1 -extremal graph is $K_n - e$ with $D = 2$. Then by Lemma 3.2, we obtain

$$\lambda_1(K_n - e) < \Delta - \frac{1}{2n} < \Delta - \frac{\Delta + 1}{n(3n + \Delta - 8)}.$$

When $\Delta = n - 1$ and $n = 3$, the λ_1 -extremal graph is P_3 with $\lambda_1(P_3)=1.4142$. When $\Delta = n - 1$ and $n = 4$, the λ_1 -extremal graph is $K_4 - e$ with $\lambda_1(K_4 - e)=2.5616$. By direct calculation, we know that the inequality is true. When $2 < \Delta < n - 1$, applying Theorem 3.1 and Lemma 3.2, we obtain the result. When $\Delta = 2$ and $n > 3$, the λ_1 -extremal graph is P_n . By adding some edges to P_n , we can attain $K_n - e$. Then following the Lemma 2.1, we obtain $\lambda_1(P_n) < \lambda_1(K_n - e)$. This completes the proof. \square

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