Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 503-510, July 2020.

ZERO-NONZERO PATTERNS THAT ALLOW OR REQUIRE AN INERTIA SET RELATED TO DYNAMICAL SYSTEMS*

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Abstract. The inertia of an $n \times n$ real matrix B, denoted by $\mathrm{i}(B)$, is the ordered triple $\mathrm{i}(B) = (i_+(B),\, i_-(B), i_0(B))$, in which $i_+(B)$, $i_-(B)$ and $i_0(B)$ are the numbers of its eigenvalues (counting multiplicities) with positive, negative and zero real parts, respectively. The inertia of an $n \times n$ zero-nonzero pattern $\mathcal{A}$ is the set $\mathrm{i}(\mathcal{A}) = \{\mathrm{i}(B) \mid B \in Q(\mathcal{A})\}$. For $n \geq 2$, let $\mathbb{S}_n^* = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0), (n, 0, 0), (n-1, 0, 1), (n-1, 1, 0)\}$. An $n \times n$ zero-nonzero pattern $\mathcal{A}$ allows $\mathbb{S}_n^*$ if $\mathbb{S}_n^* \subseteq \mathrm{i}(\mathcal{A})$ and requires $\mathbb{S}_n^*$ if $\mathbb{S}_n^* = \mathrm{i}(\mathcal{A})$. In this paper, it is shown that there are no zero-nonzero patterns for order $n \geq 2$ that require $\mathbb{S}_n^*$. Also, a complete characterization of zero-nonzero star patterns of order $n \geq 3$ that allow $\mathbb{S}_n^*$ is given.

Key words. Zero-nonzero pattern, Eigenvalues, Inertia, Digraph.

AMS subject classifications. 15B35, 05C20, 15A18, 05C50.

1. Introduction.

An $n \times n$ *zero-nonzero pattern* is an $n \times n$ matrix $\mathcal{A}$ with entries from $\{*, 0\}$, where $*$ is nonzero. A real matrix $B = (b_{ij})$ is a *realization* of $\mathcal{A}$ if $b_{ij} \neq 0$ if and only if the (i, j)-entry of $\mathcal{A}$ is $*$. The *zero-nonzero pattern class* $Q(\mathcal{A})$ is the set of all realizations of $\mathcal{A}$. Two zero-nonzero patterns are *equivalent* if one can be obtained from the other by any combination of transposition and permutation similarity. If we assign each nonzero entry of $\mathcal{A}$ a sign ($+$ or $-$), then we can get a sign pattern $\mathcal{B}$, whose entries come from the set $\{+, -, 0\}$. The sign pattern $\mathcal{B}$ is called a *signing* of $\mathcal{A}$.

The *inertia* of an $n \times n$ real matrix B, denoted by $\mathrm{i}(B)$, is the ordered triple $\mathrm{i}(B) = (i_+(B),\, i_-(B), i_0(B))$, in which $i_+(B)$, $i_-(B)$ and $i_0(B)$ are the numbers of its eigenvalues (counting multiplicities) with positive, negative and zero real parts, respectively. The *inertia* of an $n \times n$ zero-nonzero pattern $\mathcal{A}$ is the set $\mathrm{i}(\mathcal{A}) = \{\mathrm{i}(B) \mid B \in Q(\mathcal{A})\}$ ([11]).

In [2], motivated by the possible onset of instability in dynamical systems associated with a zero eigenvalue, the inertia set S_n with $n \geq 2$ is defined as

$$\mathbb{S}_n = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0)\}.$$

For a zero-nonzero pattern $\mathcal{A}$, $(i_+, i_-, i_0) \in \mathrm{i}(\mathcal{A})$ if and only if its *reversal* $(i_-, i_+, i_0) \in \mathrm{i}(\mathcal{A})$. Thus, in [2], the inertia set $S_n^* = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0), (n, 0, 0), (n-1, 0, 1), (n-1, 1, 0)\}$ was introduced.

An $n \times n$ zero-nonzero pattern (resp. sign pattern) $\mathcal{A}$ *allows* $\mathbb{S}_n^*$ (resp. $\mathbb{S}_n$) if $\mathbb{S}_n^* \subseteq \mathrm{i}(\mathcal{A})$ (resp. $\mathbb{S}_n \subseteq \mathrm{i}(\mathcal{A})$), and *requires* $\mathbb{S}_n^*$ (resp. $\mathbb{S}_n$) if $\mathbb{S}_n^* = \mathrm{i}(\mathcal{A})$ (resp. $\mathbb{S}_n = \mathrm{i}(\mathcal{A})$).

A related set of refined inertias $\mathbb{H}_n$ was introduced for sign patterns in [3]. The *refined inertia* of a square real matrix B, denoted by $\mathrm{ri}(B)$, is the ordered 4-tuple $(n_+(B), n_-(B), n_z(B), 2n_p(B))$, where $n_+(B)$ (resp., $n_-(B)$) is the number of eigenvalues of B with positive (resp., negative) real part, $n_z(B)$ is the number of

*Received by the editors on August 4, 2018. Accepted for publication on May 28, 2020. Handling Editor: Bryan L. Shader.
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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 503-510, July 2020.

zero eigenvalues of B, and $2n_p(B)$ is the number of pure imaginary (nonzero) eigenvalues of B. For $n \geq 2$, $\mathbb{H}_n = \{(0,n,0,0),(0,n-2,0,2),(2,n-2,0,0)\}$. The set $\mathbb{H}_n$ can signal the onset of periodic solutions by a pair of nonzero pure imaginary eigenvalues in dynamical systems. For further results on $\mathbb{H}_n$, see [4, 5, 6, 8, 9]. In [1], the concept $\mathbb{H}_n$ for sign patterns is expanded to $\mathbb{H}_n^*$ for zero-nonzero patterns. For further results on $\mathbb{H}_n^*$, see [7].

A zero-nonzero pattern $\mathcal{A}$ is *reducible* if it is permutation similar to a pattern of the form

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$, $\mathcal{A}_{22}$ are square and non-vacuous. A zero-nonzero pattern is *irreducible* if it is not reducible.

There is a natural association between digraphs and zero-nonzero patterns. For an $n \times n$ zero-nonzero pattern $\mathcal{A}$, the *associated digraph* $D(\mathcal{A})$ has n vertices $v_1, v_2, \ldots, v_n$, an arc from vertex v_i to vertex v_j if and only if the (i,j)-entry of $\mathcal{A}$ is nonzero, and a loop at vertex v_i if and only if the (i,i)-entry of $\mathcal{A}$ is nonzero. Two digraphs are equivalent if and only if their associated zero-nonzero patterns are equivalent.

A *simple cycle* of length l or an l-cycle in a digraph D is a sequence of arcs of the form $C = v_{i_1}v_{i_2}, v_{i_2}v_{i_3}, \ldots, v_{i_l}v_{i_1}$, where $v_{i_1}, \ldots, v_{i_l}$ are distinct vertices. A *composite cycle* C in a digraph D is a vertex disjoint union of simple cycles, say $C = C_1 \cup C_2 \cup \cdots \cup C_k$. If the length of C_i is l_i, then the length of C is $\sum_{i=1}^{k} l_i$.

In [2], it is shown that a zero-nonzero pattern of order 2 allows $\mathbb{S}_2^*$ if and only if every entry in the zero-nonzero pattern is $*$. The authors also describe all irreducible nonequivalent zero-nonzero patterns of order 3 and 4 that allow $\mathbb{S}_n^*$. For zero-nonzero patterns requiring $\mathbb{S}_n^*$, it is proved that for $2 \leq n \leq 4$, there are no irreducible zero-nonzero patterns of order n that require $\mathbb{S}_n^*$. For $n \geq 5$, this question is open.

In this paper, we study zero-nonzero patterns that allow or require $\mathbb{S}_n^*$. In Section 3, we first prove that there are no irreducible zero-nonzero patterns of order $n \geq 5$ that require $\mathbb{S}_n^*$. So together with the result in [2], there are no irreducible zero-nonzero patterns of order $n \geq 2$ that require $\mathbb{S}_n^*$. Moreover, we can prove that there are no reducible zero-nonzero patterns of order $n \geq 2$ that require $\mathbb{S}_n^*$. In Section 4, we give a complete characterization of zero-nonzero star patterns of order $n \geq 3$ that allow $\mathbb{S}_n^*$.

2. Preliminaries.

Remark 2.1. Let $\mathcal{A}$ be a zero-nonzero pattern of order n. If $\mathcal{A}$ requires $\mathbb{S}_n^*$, then by the definition of $\mathbb{S}_n^*$, it is clear that for any $B \in Q(\mathcal{A})$, either $i_+(B) \leq 1$ or $i_-(B) \leq 1$.

Lemma 2.2. *Let $n \geq 4$ and $n \times n$ matrix*

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ (n=4), \quad B = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix} \ (n \geq 5).$$

Then $i_+(B) \geq 2$ and $i_-(B) \geq 2$.

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Proof. When $n = 4$, the characteristic polynomial of B is

$$\det(\lambda I - B) = \begin{vmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = \lambda^4 + 1 = (\lambda^2 + \sqrt{2}\lambda + 1)(\lambda^2 - \sqrt{2}\lambda + 1).$$

So, $\mathrm{i}(B) = (2, 2, 0, 0)$ and the lemma holds.

When $n \geq 5$, the characteristic polynomial of B is

$$\det(\lambda I - B) = \begin{vmatrix} \lambda & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ -1 & & & & \lambda \end{vmatrix} = \lambda^n - 1.$$

It is easy to see that $\sigma(B) = \left\{\cos(\frac{2k\pi}{n}) + \mathrm{i}\sin(\frac{2k\pi}{n}) \mid k = 1, 2, \ldots, n\right\}$, and so $i_+(B) \geq 2$ and $i_-(B) \geq 2$. The lemma holds. ∎

A zero-nonzero pattern $\mathcal{B}$ is a *subpattern* of a zero-nonzero pattern $\mathcal{A}$ if $\mathcal{B}$ is obtained from $\mathcal{A}$ by replacing some (or possibly none) of the nonzero entries of $\mathcal{A}$ with zeros. $\mathcal{A}$ is a *superpattern* of $\mathcal{B}$ if $\mathcal{B}$ is a subpattern of $\mathcal{A}$.

LEMMA 2.3. *Let $\mathcal{A}$ be a zero-nonzero pattern of order n and $\mathcal{A}_1$ be a subpattern of $\mathcal{A}$. If there exists $B_0 \in Q(\mathcal{A}_1)$ such that $i_+(B_0) \geq 2$ and $i_-(B_0) \geq 2$, then $\mathcal{A}$ does not require $\mathbb{S}_n^*$.*

Proof. Note that the eigenvalues of a matrix can be arranged so that they are continuous functions of the entries ([10]). If the condition holds, then there exists a matrix $B \in Q(\mathcal{A})$ such that $i_+(B) \geq 2$ and $i_-(B) \geq 2$. So $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Remark 2.1. ∎

LEMMA 2.4. *Let $\mathcal{A}$ be a zero-nonzero pattern of order n and assume that $\mathcal{A}$ has k nonzero diagonal entries. If $\mathcal{A}$ requires $\mathbb{S}_n^*$, then $1 \leq k \leq 3$.*

Proof. First suppose $k = 0$. Since the trace of $\mathcal{A}$ is zero, it is impossible to find $B \in Q(\mathcal{A})$ with $\mathrm{i}(B) = (0, n-1, 1)$. This contradicts to $\mathcal{A}$ requiring $\mathbb{S}_n^*$.

Now suppose $k > 4$. Without loss of generality, assume that the nonzero diagonal entries of $\mathcal{A}$ are $a_{11}, a_{22}, \ldots, a_{kk}$. By emphasizing $a_{11}, a_{22}, a_{33}, a_{44}$ and taking $a_{11} > 0, a_{22} > 0, a_{33} < 0, a_{44} < 0$, we can get a matrix $B \in Q(\mathcal{A})$ such that $i_+(B) \geq 2$ and $i_-(B) \geq 2$. By Remark 2.1, $\mathcal{A}$ does not require $\mathbb{S}_n^*$. Thus, the result follows. ∎

LEMMA 2.5. *Let $\mathcal{A}$ be a zero-nonzero pattern of order n. If $D(\mathcal{A})$ contains a composite cycle that consists of one 2-cycle and two loops, then $\mathcal{A}$ does not require $\mathbb{S}_n^*$.*

Proof. Without loss of generality, assume that the 2-cycle is $v_1v_2v_1$ and two loops are on v_3 and v_4. Take an $n \times n$ matrix $B_0 = (b_{ij})$ with $b_{12} = b_{21} = b_{33} = 1$, $b_{44} = -1$, and other entries equal to zero. It is clear that $i_+(B_0) = 2$ and $i_-(B_0) = 2$. Thus, $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 2.3. ∎

A zero-nonzero pattern $\mathcal{A}$ is *combinatorially singular* if B is singular for all $B \in Q(\mathcal{A})$, and $\mathcal{A}$ is *combinatorially nonsingular* if B is nonsingular for all $B \in Q(\mathcal{A})$.

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Lemma 2.6. ([2]) *If an $n \times n$ zero-nonzero pattern $\mathcal{A}$ requires or allows $\mathbb{S}_n^*$, then $\mathcal{A}$ is not combinatorially singular, and not combinatorially nonsingular.*

Lemma 2.7. *If an $n \times n$ zero-nonzero pattern $\mathcal{A}$ allows $\mathbb{S}_n$, then $\mathcal{A}$ allows $\mathbb{S}_n^*$. And if an $n \times n$ sign pattern $\mathcal{A}$ allows or requires $\mathbb{S}_n$, then the associated zero-nonzero pattern allows $\mathbb{S}_n^*$.*

Proof. Note that for zero-nonzero pattern $\mathcal{A}$, the inertia $(i_+, i_-, i_0) \in i(\mathcal{A})$ if and only if its reversal $(i_-, i_+, i_0) \in i(\mathcal{A})$. ∎

3. Zero-nonzero patterns of order n do not require $\mathbb{S}_n^*$.

Lemma 3.1. *Let $\mathcal{A}$ be a zero-nonzero pattern of order $n \geq 5$. If $D(\mathcal{A})$ satisfies one of the following two conditions, then $\mathcal{A}$ does not require $\mathbb{S}_n^*$.*

(1) $D(\mathcal{A})$ contains a s-cycle with $s \geq 4$;

(2) $D(\mathcal{A})$ contains a k-cycle and a t-cycle that are vertex disjoint with $1 \leq k, t \leq 3$ and $k + t \geq 4$.

Proof. Denote $V(D(\mathcal{A})) = \{v_1, v_2, \ldots, v_n\}$. Consider the following two cases.

Case 1. $D(\mathcal{A})$ contains a s-cycle C_s with $s \geq 4$.

Without loss of generality, assume that $C_s = v_1 v_2 \cdots v_s v_1$. If $s = 4$, then take an $n \times n$ matrix $B_0 = (b_{ij})$ with $b_{12} = -1$, $b_{23} = \cdots = b_{s-1,s} = b_{s,1} = 1$, and the other entries of B_0 equal to zero. If $s > 4$, then take an $n \times n$ matrix $B_0 = (b_{ij})$ with $b_{12} = b_{23} = \cdots = b_{s-1,s} = b_{s,1} = 1$, and the other entries of B_0 equal to zero. By Lemma 2.2, $i_+(B_0) \geq 2$ and $i_-(B_0) \geq 2$. Thus, $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 2.3.

Case 2. $D(\mathcal{A})$ contains a k-cycle C_k and a t-cycle C_t that are vertex disjoint with $1 \leq k, t \leq 3$ and $k + t \geq 4$.

Without loss of generality, assume that $C_k = v_1 v_2 \cdots v_k v_1$ and $C_t = v_{k+1} v_{k+2} \cdots v_{k+t} v_{k+1}$. Up to equivalence, there are the following four cases.

- $k = 3$ and $t = 1$;
- $k = 2$ and $t = 2$;
- $k = 2$ and $t = 3$;
- $k = 3$ and $t = 3$.

If $k = 3$ and $t = 1$, take an $n \times n$ matrix $B_0 = (b_{ij})$ with $b_{12} = b_{23} = b_{31} = 1$, $b_{44} = 1$ and the other entries equal to zero. Then $i_+(B_0) = 2$ and $i_-(B_0) = 2$, and $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 2.3.

If $k = 2$ and $t = 2$, take an $n \times n$ matrix $B_0 = (b_{ij})$ with $b_{12} = b_{21} = b_{34} = b_{43} = 1$, and the other entries equal to zero. Then $i_+(B_0) = 2$ and $i_-(B_0) = 2$, and $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 2.3.

If $k = 2$ and $t = 3$, take an $n \times n$ matrix $B_0 = (b_{ij})$ with $b_{12} = b_{21} = b_{34} = b_{45} = b_{53} = 1$, and the other entries equal to zero. Then $i_+(B_0) = 2$ and $i_-(B_0) = 3$, and $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 2.3.

If $k = 3$ and $t = 3$, take an $n \times n$ matrix $B_0 = (b_{ij})$ with $b_{12} = b_{23} = b_{31} = 1$, $b_{45} = b_{56} = b_{64} = -1$, and the other entries equal to zero. Then $i_+(B_0) = 3$ and $i_-(B_0) = 3$, and $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 2.3. ∎

Theorem 3.2. *There are no irreducible zero-nonzero patterns of order $n \geq 2$ that require $\mathbb{S}_n^*$.*

Proof. Based on the result in [2], we only need to prove the result when $n \geq 5$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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Let $D(\mathcal{A})$ be the associated digraph of $\mathcal{A}$ with vertex set $\{v_1, v_2, \ldots, v_n\}$. By Lemma 2.6, we may assume that $\mathcal{A}$ is not combinatorially singular. Thus, there exists a composite cycle $C = C_1 \cup C_2 \cup \cdots \cup C_k$ of length n in $D(\mathcal{A})$. Denote the lengths of $C_1, C_2, \ldots, C_k$ by $l_1, \ldots, l_k$, respectively. Without loss of generality, assume that $l_1 \leq \cdots \leq l_k$.

Consider the following four cases.

Case 1. $l_k \geq 4$. Then $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 3.1.

Case 2. $l_k = 3$. Note that $k \geq 2$ since $n \geq 5$. Then $\mathcal{A}$ does not require $\mathbb{S}_n^*$ by Lemma 3.1.

Case 3. $l_k = 2$. If $l_{k-1} = 2$, then $D(\mathcal{A})$ contains two vertex disjoint 2-cycles that do not have common vertices. By Lemma 3.1, $\mathcal{A}$ does not require $\mathbb{S}_n^*$. Otherwise, since $n \geq 5$, it is clear that $D(\mathcal{A})$ contains a composite cycle that consists of one 2-cycle and two loops. By Lemma 2.5, $\mathcal{A}$ does not require $\mathbb{S}_n^*$.

Case 4. $l_k = 1$. Since $n \geq 5$, $\mathcal{A}$ has at least five nonzero diagonal vertices. By Lemma 2.4, $\mathcal{A}$ does not require $\mathbb{S}_n^*$. ◻

Theorem 3.3. *There are no reducible zero-nonzero patterns of order $n \geq 2$ that require $\mathbb{S}_n^*$.*

Proof. Let $\mathcal{A}$ be a reducible zero-nonzero pattern of order n. Then there is a permutation zero-nonzero pattern $\mathcal{P}$ such that

$$\mathcal{P}^T \mathcal{A} \mathcal{P} = \begin{bmatrix} \mathcal{A}_m & \# \\ 0 & \mathcal{A}_{n-m} \end{bmatrix},$$

where # is an $m \times (n-m)$ zero-nonzero pattern and $1 \leq m \leq n-1$.

Suppose that $\mathcal{A}$ requires $\mathbb{S}_n^*$. Then $\{(0,n,0),(n,0,0)\} \subseteq \mathrm{i}(\mathcal{A})$, and so $\{(0,m,0),(m,0,0)\} \subseteq \mathrm{i}(\mathcal{A}_m)$ and $\{(0,n-m,0),(n-m,0,0)\} \subseteq \mathrm{i}(\mathcal{A}_{n-m})$. Thus, $\{(m,n-m,0),(n-m,m,0)\} \subseteq \mathrm{i}(\mathcal{A})$. So $m = 1$ or $n-m = 1$.

If $m = 1$ and $n-m = 1$, then $n = 2$, and $\mathcal{A}_m = \mathcal{A}_{n-m} = (*)$. Thus, $\mathrm{i}(\mathcal{A}) = \{(0,2,0),(1,1,0),(2,0,0)\}$, and so $\mathcal{A}$ does not require $\mathbb{S}_n^*$.

If only one of m and $n-m$ is equal to one, then $n \geq 3$. Without loss of generality, assume $m = 1$, that is, $\mathcal{A}_m = (*)$. Then $\mathrm{i}(\mathcal{A}_m) = \{(1,0,0),(0,1,0)\}$. Since $\mathcal{A}$ requires $\mathbb{S}_n^*$, we must have $\{(0,n-2,1),(n-2,0,1)\} \subseteq \mathrm{i}(\mathcal{A}_{n-m})$. Thus, $\{(1,n-2,1),(n-2,1,1)\} \subseteq \mathrm{i}(\mathcal{A})$, a contradiction. ◻

Theorems 3.2 and 3.3 give the main result of this section as follows.

Theorem 3.4. *There are no zero-nonzero patterns of order $n \geq 2$ that require $\mathbb{S}_n^*$.*

4. Zero-nonzero star patterns that allow $\mathbb{S}_n^*$.

Up to equivalence, an $n \times n$ zero-nonzero star pattern can be represented in the following form

$$\mathcal{A} = \begin{bmatrix} a_{11} & * & \cdots & * \\ * & a_{22} & & \\ \vdots & & \ddots & \\ * & & & a_{nn} \end{bmatrix}, \tag{4.1}$$

where $a_{ii} \in \{*, 0\}$ for $i = 1, 2, \ldots, n$.

In [2], it is shown that a zero-nonzero pattern of order 2 allows $\mathbb{S}_2^*$ if and only if every entry in the zero-nonzero pattern is $*$. In this section, we give a complete characterization of zero-nonzero star patterns

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A publication of the International Linear Algebra Society
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of order $n \geq 3$ that allow $\mathbb{S}_n^*$.

THEOREM 4.1. *Let $n \geq 3$ and $\mathcal{A}$ be a zero-nonzero star pattern of order n. Then $\mathcal{A}$ allows $\mathbb{S}_n^*$ if and only if $\mathcal{A}$ is equivalent to one of the following patterns*

$$\mathcal{A}_1 = \begin{bmatrix} * & * & \cdots & \cdots & * \\ * & * & & & \\ \vdots & & * & & \\ \vdots & & & \ddots & \\ * & & & & * \end{bmatrix} \quad \textit{and} \quad \mathcal{A}_2 = \begin{bmatrix} 0 & * & \cdots & \cdots & * \\ * & * & & & \\ \vdots & & * & & \\ \vdots & & & \ddots & \\ * & & & & * \end{bmatrix}.$$

When $\mathcal{A}$ allows $\mathbb{S}_n^*$, $\mathcal{A}$ has a signing that allows $\mathbb{S}_n$.

Proof. Let $\mathcal{A}$ be a zero-nonzero star pattern of order n. Up to equivalence, we may assume that $\mathcal{A}$ is in the form (4.1).

Necessity. If at least two of $a_{22}, \ldots, a_{nn}$ are equal to zero, then $\mathcal{A}$ is combinatorially singular, and $\mathcal{A}$ doesn't allow $\mathbb{S}_n^*$ by Lemma 2.6. If exactly one of $a_{22}, \ldots, a_{nn}$ is equal to zero, then $\mathcal{A}$ is combinatorially nonsingular, and $\mathcal{A}$ doesn't allow $\mathbb{S}_n^*$ by Lemma 2.6. Thus, up to equivalence, we may assume $a_{ii} = *$ for $i = 2, 3, \ldots, n$, that is, $\mathcal{A}$ is equivalent to one of patterns $\mathcal{A}_1$ and $\mathcal{A}_2$.

For sufficiency, we consider the following two cases.

Case 1. $\mathcal{A}$ is equivalent to $\mathcal{A}_1$.

Without loss of generality, assume that $\mathcal{A} = \mathcal{A}_1$. Take

$$B = \begin{bmatrix} a & 1 & 1 & \cdots & 1 \\ b & -1 & & & \\ -1 & & -1 & & \\ \vdots & & & \ddots & \\ -1 & & & & -1 \end{bmatrix} \in Q(\mathcal{A}),$$

where $a \neq 0$ and $b \neq 0$. Then the characteristic polynomial of B is

$$f_B(x) = |xI - B| = \begin{vmatrix} x-a & -1 & -1 & \cdots & -1 \\ -b & x+1 & & & \\ 1 & & x+1 & & \\ \vdots & & & \ddots & \\ 1 & & & & x+1 \end{vmatrix}.$$

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By subtracting the second column from the kth column, $k = 3, 4, \ldots, n$, respectively, we have

$$
\begin{aligned}
f_B(x) &= \begin{vmatrix} x-a & -1 & 0 & \cdots & 0 \\ -b & x+1 & -(x+1) & \cdots & -(x+1) \\ 1 & & x+1 & & \\ \vdots & & & \ddots & \\ 1 & & & & x+1 \end{vmatrix} \\
&= (x-a)(x+1)^{n-1} + \begin{vmatrix} -b & -(x+1) & \cdots & -(x+1) \\ 1 & x+1 & & \\ \vdots & & \ddots & \\ 1 & & & x+1 \end{vmatrix}_{n-1} \\
&= (x-a)(x+1)^{n-1} + \begin{vmatrix} n-b-2 & 0 & \cdots & 0 \\ 1 & x+1 & & \\ \vdots & & \ddots & \\ 1 & & & x+1 \end{vmatrix}_{n-1} \\
&= (x-a)(x+1)^{n-1} + (n-b-2)(x+1)^{n-2} \\
&= (x+1)^{n-2}(x^2 + (1-a)x + n - a - b - 2).
\end{aligned}
$$

(1) If $a = \frac{1}{3}$ and $b = n - \frac{5}{2}$, then $f_B(x) = (x+1)^{n-2}(x^2 + \frac{2}{3}x + \frac{1}{6})$, and so $\mathrm{i}(B) = (0, n, 0)$.

(2) If $a = \frac{1}{2}$ and $b = n - \frac{5}{2}$, then $f_B(x) = (x+1)^{n-2}(x^2 + \frac{1}{2}x)$, and so $\mathrm{i}(B) = (0, n-1, 1)$.

(3) If $a = 1$ and $b = n-2$, then $f_B(x) = (x+1)^{n-1}(x-1)$, and so $\mathrm{i}(B) = (1, n-1, 0)$. Thus, $\mathrm{sgn}(B)$ allows $\mathbb{S}_n$. By Lemma 2.7, $\mathcal{A}$ allows $\mathbb{S}_n^*$, and $\mathcal{A}$ has a signing that allows $\mathbb{S}_n$.

Case 2. $\mathcal{A}$ is equivalent to $\mathcal{A}_2$.

Without loss of generality, assume that $\mathcal{A} = \mathcal{A}_2$. Take

$$
C = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ c & -1 & & & \\ -1 & & -1 & & \\ \vdots & & & \ddots & \\ -1 & & & & -1 \end{bmatrix} \in Q(\mathcal{A}),
$$

where $c \neq 0$. By the similar steps as in Case 1, we can get the characteristic polynomial of C

$$
f_C(x) = |xI - C| = (x+1)^{n-2}(x^2 + x + n - c - 2).
$$

(1) If $c = \frac{1}{2}$, then $f_C(x) = (x+1)^{n-2}(x^2 + x + n - \frac{5}{2})$, and so $\mathrm{i}(C) = (0, n, 0)$.

(2) If $c = n-2$, then $f_C(x) = x(x+1)^{n-1}$, and so $\mathrm{i}(C) = (0, n-1, 1)$.

(3) If $c = n$, then $f_C(x) = (x+1)^{n-2}(x+2)(x-1)$, and so $\mathrm{i}(C) = (1, n-1, 0)$. Thus, $\mathrm{sgn}(C)$ allows $\mathbb{S}_n$. By Lemma 2.7, $\mathcal{A}$ allows $\mathbb{S}_n^*$, and $\mathcal{A}$ has a signing that allows $\mathbb{S}_n$. $\square$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 503-510, July 2020.

Acknowledgements. The authors would like to thank the anonymous referee for valuable suggestions which greatly improved the exposition of the paper.

REFERENCES

[1] A.H. Berliner, D.D. Olesky, and P. van den Driessche. Sets of refined inertias of zero-nonzero patterns. *Linear Algebra and its Applications*, 516:243–263, 2017.

[2] A.H. Berliner, D.D. Olesky, and P. van den Driessche. Inertia sets allowed by matrix patterns. *Electronic Journal of Linear Algebra*, 34:343–355, 2018.

[3] E. Bodine, L. Deaett, J.J. McDonald, D.D. Olesky, and P. van den Driessche. Sign patterns that require or allow particular refined inertias. *Linear Algebra and its Applications*, 437(9):2228–2242, 2012.

[4] W. Gao, Z. Li, and L. Zhang. Sign patterns that require $\mathbb{H}_n$ exist for each $n \geq 4$. *Linear Algebra and its Applications*, 489:15–23, 2016.

[5] W. Gao, Z. Li, and L. Zhang. Characterization of star sign patterns that require $\mathbb{H}_n$. *Linear Algebra and its Applications*, 499:43–65, 2016.

[6] W. Gao and Z. Li. Path sign patterns of order $n \geq 5$ do not require $\mathbb{H}_n$. *Linear Algebra and its Applications*, 532:99–126, 2017.

[7] W. Gao. Zero-nonzero patterns of order $n \geq 4$ do not require $\mathbb{H}_n^*$. *Linear Algebra and its Applications*, 554:1–14, 2018.

[8] C. Garnett, D.D. Olesky, and P. van den Driessche. Refined inertias of tree sign patterns. *Electronic Journal of Linear Algebra*, 26:620–635, 2013.

[9] C. Garnett, D.D. Olesky, and P. van den Driessche. A note on sign patterns of order 3 that require particular refined inertias. *Linear Algebra and its Applications*, 450:293–300, 2014.

[10] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[11] I.-J. Kim, J.J. McDonald, D.D. Olesky, and P. van den Driessche. Inertias of zero-nonzero patterns. *Linear and Multilinear Algebra*, 55(3):229–238, 2007.