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A NOTE ON PARALLEL DISTINGUISHABILITY OF TWO QUANTUM OPERATIONS*

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Abstract. In this work, the authors consider a homogeneous system of linear equations of the form $A_\alpha^{\otimes N}\mathbf{x} = 0$ arising from the distinguishability of two quantum operations by N uses in parallel, where the coefficient matrix A_α depends on a real parameter α. It was conjectured by Duan et al. that the system has a non-trivial nonnegative solution if and only if α lies in a certain interval R_N depending on N. The authors affirm the necessity part of the conjecture and establish the sufficiency of the conjecture for $N \leq 10$ by presenting explicit non-trivial nonnegative solutions for the linear system.

Key words. Quantum channels, Parallel distinguishability.

AMS subject classifications. 46N60, 15A69.

1. Introduction. Let $M_{m,n}$ (respectively, M_n) be the set of $m \times n$ (respectively, $n \times n$) complex matrices. Denote by H_n the set of $n \times n$ Hermitian matrices and by D_n the set of $n \times n$ density matrices, which are positive semidefinite matrices with trace one.

In the mathematical framework of quantum mechanics, density matrices are used to describe the state of a quantum system. Quantum operations [5, 6] are trace-preserving, completely-positive linear maps from M_n to M_m. It is known [2, 4] that for a quantum operation $\mathcal{E} : M_n \longrightarrow M_m$, there exists a set of matrices $\{E_1, \ldots, E_{n_0}\} \subset M_{m,n}$, called a set of Choi-Kraus operators of $\mathcal{E}$, such that

$$\sum_{j=1}^{n_0} E_j^* E_j = I_n \quad \text{and} \quad \mathcal{E}(X) = \sum_{j=1}^{n_0} E_j X E_j^* \qquad \text{for any } X \in M_n.$$

For example, the identity map on M_ℓ, denoted by $\mathcal{I}_\ell$, has $\{I_\ell\}$ as Choi-Kraus operator.

Two quantum operations $\mathcal{E} : M_n \longrightarrow M_m$ and $\mathcal{F} : M_n \longrightarrow M_m$, with Choi-Kraus operators given by $\{E_j\}_{j=1}^{n_0}$ and $\{F_k\}_{k=1}^{n_1}$ are *distinguishable by N uses in parallel* if for some integers ℓ, r, there exists a nonzero vector $\mathbf{x} \in \mathbb{C}^{\ell^r \cdot n^N}$ such that

$$Y_1 = (\mathcal{I}_\ell^{\otimes r} \otimes \mathcal{E}^{\otimes N})(\mathbf{x}\mathbf{x}^*) = \sum_{j_1,\ldots,j_N \in \{1,\ldots,n_0\}} (I_\ell^{\otimes r} \otimes E_{j_1} \otimes \cdots \otimes E_{j_N})\mathbf{x}\mathbf{x}^*(I_\ell^{\otimes r} \otimes E_{j_1}^* \otimes \cdots \otimes E_{j_N}^*)$$

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and

$$Y_2 = (\mathcal{I}_\ell^{\otimes r} \otimes \mathcal{F}^{\otimes N})(\mathbf{x}\mathbf{x}^*) = \sum_{k_1,\ldots,k_N \in \{1,\ldots,n_1\}} (I_\ell^{\otimes r} \otimes F_{k_1} \otimes \cdots \otimes F_{k_N})\mathbf{x}\mathbf{x}^*(I_\ell^{\otimes r} \otimes F_{k_1}^* \otimes \cdots \otimes F_{k_N}^*)$$

are orthogonal, that is, $\mathrm{Tr}(Y_1^* Y_2) = 0$. One may see [3] and its references for the background of the concept. In particular, the following results were obtained in [3, Theorems 1 and 2].

Proposition 1.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be two quantum operations with Choi-Kraus operators $\{E_j\}_{j=1}^{n_0}$ and $\{F_k\}_{k=1}^{n_1}$, respectively. Then $\mathcal{E}$ and $\mathcal{F}$ can be perfectly distinguished by N uses in parallel if and only if there exists a density matrix $\rho \in (S_{\mathcal{E},\mathcal{F}}^{\otimes N})^\perp$, where*

$$S_{\mathcal{E},\mathcal{F}} = \mathrm{Span}\,\{E_j^* F_k \mid 1 \le j \le n_0, 1 \le k \le n_1\} \quad \text{and} \quad S_{\mathcal{E},\mathcal{F}}^{\otimes N} = \mathrm{Span}\,\{R^{\otimes N} : R \in S_{\mathcal{E},\mathcal{F}}\}.$$

Proposition 1.2. *Any non-empty subset $T \subseteq M_n$ can be realized as a spanning set of $S_{\mathcal{E},\mathcal{F}}$ of some pair of quantum operations $\mathcal{E}, \mathcal{F}$.*

Here we give a short proof of Proposition 1.2:

Proof. Suppose $\mathrm{Span}\,T$ has a basis $\{A_1,\ldots,A_m\} \subseteq M_n$. Consider the block diagonal matrix $\mathbf{A} = A_1 \oplus \cdots \oplus A_m$. If $\mathbf{A}$ has rank $\tilde{k}$, then $\mathbf{A} = [B_1 \cdots B_m]^*[C_1 \cdots C_m]$, where $B_1,\ldots,B_m,C_1,\ldots,C_m$ are $k \times n$ matrices with $k = \max\{\tilde{k}, n\}$. Let $M > 0$ be such that $I_n - \frac{1}{M}\sum_{j=1}^m B_j^* B_j = B_{m+1}^* B_{m+1}$ and $I_n - \frac{1}{M}\sum_{j=1}^m C_j^* C_j = C_{m+1}^* C_{m+1}$ for some $k \times n$ matrices B_{m+1}, C_{m+1}. Let $E_1,\ldots,E_{m+1},F_1,\ldots,F_{m+1} \in M_{3k,n}$ be such that

$$E_j^* = \frac{1}{\sqrt{M}}[B_j^* | 0_{n,2k}], \quad F_j^* = \frac{1}{\sqrt{M}}[C_j^* | 0_{n,2k}], \qquad j = 1,\ldots,m,$$

$$E_{m+1}^* = [0_{n,k} | B_{m+1}^* | 0_{n,k}], \quad F_{m+1}^* = [0_{n,2k} | C_{m+1}^*].$$

Then $\sum_{j=1}^{m+1} E_j^* E_j = \sum_{j=1}^{m+1} F_j^* F_j = I_n$, and

$$[E_1 \cdots E_{m+1}]^*[F_1 \cdots F_{m+1}] = \frac{1}{M}(A_1 \oplus \cdots \oplus A_m \oplus 0_n).$$

If the quantum channels from M_n to M_{3k} have the sets of Choi-Kraus operators $\{E_1,\ldots,E_{m+1}\}$ and $\{F_1,\ldots,F_{m+1}\}$, then $\mathrm{Span}\,\{E_i^* F_j : 1 \le i,j \le m+1\} = \mathrm{Span}\,\{A_1,\ldots,A_m\}$. ∎

In [3], the authors considered the quantum channels $\mathcal{E}$ and $\mathcal{F}$ with $S_{\mathcal{E},\mathcal{F}}$ equal to the span of the set

$$T_\alpha = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix} \right\}, \quad \alpha \in [0, 2\pi).$$

It is easy to see that the following conditions for a density operator $\rho = (\rho_{ij}) \in M_{3^N}$ are equivalent.

(a) The density operator $\rho \in (\mathrm{Span}\,(T_\alpha)^{\otimes N})^\perp$.
(b) The diagonal density operator $\hat{\rho} = \mathrm{diag}\,(\rho_{11},\ldots,\rho_{3^N,3^N}) \in (\mathrm{Span}\,(T_\alpha)^{\otimes N})^\perp$.
(c) The vector $\mathbf{x} = (\rho_{11},\ldots,\rho_{3^N,3^N})^t \in \mathbb{C}^{3^N}$ satisfies the homogeneous equation

$$A_\alpha^{\otimes N}\mathbf{x} = 0 \quad \text{with} \quad A_\alpha = \begin{bmatrix} 1 & e^{i\alpha} & 0 \\ 0 & 1 & e^{i\alpha} \end{bmatrix}. \tag{1.1}$$

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By the above fact, one can focus on finding a non-trivial nonnegative vector $\mathbf{x} \in \mathbb{C}^{3^N}$ satisfying (1.1). Furthermore, the following remarks and conjecture were made in [3].

REMARK 1.3. If $\alpha \in [0, \frac{\pi}{2})$, the space $\mathrm{Span}(T_\alpha)$ contains a positive operator. In this case $(\mathrm{Span}(T_\alpha)^{\otimes N})^\perp$ does not contain a density matrix for any positive integer N. This makes the corresponding pair of quantum operations $\mathcal{E}, \mathcal{F}$, satisfying $S_{\mathcal{E},\mathcal{F}} = \mathrm{Span}(T)$, indistinguishable. By taking the complex conjugate of equation (1.1), we see that there is a non-trivial nonnegative solution to $A_\alpha^N \mathbf{x} = 0$ if and only if there is a non-trivial nonnegative solution to $A_{-\alpha}^N \mathbf{x} = A_{2\pi-\alpha}^N \mathbf{x} = 0$. Hence, we only need to focus on the case when $\alpha \in \left[\frac{\pi}{2}, \pi\right]$.

CONJECTURE 1.4. *Let $\alpha \in \left[\frac{\pi}{2}, \pi\right]$. The equation (1.1) has a non-trivial nonnegative solution if and only if $\alpha \in [\frac{\pi}{2} + \frac{\pi}{2N}, \pi]$.*

In [3], the authors gave explicit solutions of the equation (1.1) for $N \leq 4$. Furthermore, in Section IV of the paper, it was shown that one may reduce the complexity of the equation (1.1) by finding solution with some symmetries imposed on its entries, and reduce the equation to another equation $C_{\alpha,N}\mathbf{y} = 0$, where $C_{\alpha,N}$ is an $(N+1) \times \dfrac{(N+1)(N+2)}{2}$ matrix with full column rank. In Section 2, we will set up the system $C_{\alpha,N}\mathbf{y} = 0$ and obtain another symmetry for the solution. In Section 3, we prove the necessity part of Conjecture 1.4, that is, if $\alpha \in [\frac{\pi}{2}, \pi]$ and equation (1.1) has a non-trivial nonnegative solution, then $\alpha \in [\frac{\pi}{2} + \frac{\pi}{2N}, \pi]$. In Section 4, we present explicit non-trivial nonnegative solutions (1.1) for $\alpha \in [\frac{\pi}{2} + \frac{\pi}{2N}, \pi]$ and $N \leq 10$. In Section 5, we provide some additional remarks that may help in studying the sufficiency part of the conjecture.

2. A reduction of the linear system. First, we label the entries of a vector $\mathbf{x} \in \mathbb{C}^{3^N}$ using ternary numbers. That is, we use the ternary number $(j_0, \ldots, j_{N-1}) \in \{0,1,2\}^N$, for the j-th entry of $\mathbf{x}$ when

$$j = 1 + \sum_{p=0}^{N-1} j_p 3^{N-1-p}.$$

For example, we will label the entries of $\mathbf{x} \in \mathbb{C}^{3^2}$ with $00, 01, 02, 10, 11, 12, 20, 21, 22$. In the same manner, we label the columns of $A_\alpha^{\otimes N}$ using ternary numbers. Meanwhile, we label the rows of $A_\alpha^{\otimes N}$ using binary numbers.

In [3, Section IV], it was shown that one may reduce the complexity of the equation (1.1) by finding solution $\mathbf{x} = [x_J]_{J\in\{0,1,2\}^N} \in \mathbb{C}^{3^N}$ with entries labeled by $J \in \{0,1,2\}^N$ such that $x_J = x_{\hat{J}}$ whenever $J = P\hat{J}$ for a permutation matrix $P \in M_N$, i.e., the ternary sequences J and $\hat{J}$ have the same numbers of $0, 1, 2$ terms. We summarize the result in the following.

PROPOSITION 2.1. *If there is a non-trivial nonnegative solution $\mathbf{x}$ satisfying equation (1.1), then there is a non-trivial nonnegative solution $\hat{\mathbf{x}} = [\hat{x}_J]_{J\in\{0,1,2\}^N}$ such that*

$$\hat{x}_{j_0,\ldots,j_{N-1}} = \hat{x}_{k_0,\ldots,k_{N-1}}$$

whenever there exists a permutation $\sigma \in S_N$ such that

$$(j_{\sigma(0)}, \ldots, j_{\sigma(N-1)}) = (k_0, \ldots, k_{N-1}).$$

For a triple (N_0, N_1, N_2) of nonnegative integers with $N_0 + N_1 + N_2 = N$, define the set

$$[N_0, N_1, N_2] = \{(j_0, j_1, \ldots, j_{N-1}) \in \{0,1,2\}^N : N_\ell = \#\{p : j_p = \ell\} \quad \text{for all } \ell \in \{0,1,2\}\}, \tag{2.2}$$

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of all ternary labels of length N that contains N_0 digits equal to 0, N_1 digits equal to 1 and N_2 digits equal to 2. For example, when $N=2$,

$$[1,1,0]=\{01,10\},\quad [0,1,1]=\{12,21\},\quad [2,0,0]=\{00\}.$$

Proposition 2.1 states that if there is a non-trivial nonnegative solution to $A_\alpha^N \mathbf{x}=0$, then there is a non-trivial nonnegative solution $\hat{\mathbf{x}}=[a_J]_{J\in\{0,1,2\}^N}$ such that $a_J=a_K$ whenever $J,K\in[N_0,N_1,N_2]$. Using this symmetry, $\hat{\mathbf{x}}$ has at most p_N distinct entries, where p_N is the number of nonnegative integer triples (N_0,N_1,N_2) satisfying $N_0+N_1+N_2=N$. The total number of such triples equals the sum of solutions of $N_0+N_1=k$ for $k=0,\ldots,N$, and hence,

$$p_N = 1+\cdots+(N+1)=\frac{(N+1)(N+2)}{2}. \tag{2.3}$$

For example, when $N=2$, we see from equation (2.3), that $\hat{x}$ has at most $p_2=6$ distinct entries. In fact, we may assume that the solution has the form:

$$\hat{\mathbf{x}}^T=\begin{bmatrix} x_{00} & x_{01} & x_{02} & x_{10} & x_{11} & x_{12} & x_{20} & x_{21} & x_{22}\end{bmatrix}=\begin{bmatrix} a & b & c & b & d & e & c & e & f\end{bmatrix}.$$

In the following, it is convenient to replace A_α by the matrix

$$A_\alpha=\begin{bmatrix}1 & 0 & -e^{2i\alpha}\\ 0 & 1 & e^{i\alpha}\end{bmatrix}=\begin{bmatrix}1 & -e^{i\alpha}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & e^{i\alpha} & 0\\ 0 & 1 & e^{i\alpha}\end{bmatrix}=\begin{bmatrix}1 & 0 & -z^2\\ 0 & 1 & z\end{bmatrix}\quad \text{with } z=e^{i\alpha}. \tag{2.4}$$

Now, let us define a $3^N\times p_N$ matrix Q_N by labeling its rows by ternary numbers in the usual order and labeling its first $N+1$ columns by $[N,0,0],[N-1,0,1],\ldots,[1,0,N-1],[0,0,N]$, then its next N columns by $[N-1,1,0],[N-2,1,1],\ldots,[0,1,N-1]$ and so on; then setting the (i,j)-th entry of Q_N equal to 1 precisely when the ternary label of the i-th row is an element of the j-th column label as defined in equation (2.2). We can then define the following $2^N\times p_N$ matrix

$$B_{\alpha,N}=A_\alpha^{\otimes N}Q_N. \tag{2.5}$$

Notice that for a non-trivial nonnegative solution $\hat{\mathbf{x}}$ satisfying the symmetry described in Proposition 2.1, we have

$$A_\alpha^{\otimes N}\hat{\mathbf{x}}=A_\alpha^{\otimes N}Q_N\mathbf{y}=B_{\alpha,N}\mathbf{y}$$

for some nonzero nonnegative vector $\mathbf{y}\in\mathbb{C}^{p_N}$. Observe $B_{\alpha,N}$ for $N=2,3$ given below,

$$B_{\alpha,2}=\begin{array}{cc} & \begin{array}{cccccc} {[200]} & {[101]} & {[002]} & {[110]} & {[011]} & {[020]}\end{array}\\ \begin{array}{c} {[00]}\\ {[01]}\\ {[10]}\\ {[11]}\end{array} & \begin{bmatrix} 1 & -2z^2 & z^4 & 0 & 0 & 0\\ 0 & z & -z^3 & 1 & -z^2 & 0\\ 0 & z & -z^3 & 1 & -z^2 & 0\\ 0 & 0 & z^2 & 0 & 2z & 1\end{bmatrix}\end{array},$$

$$B_{\alpha,3}=\begin{array}{cc} & \begin{array}{cccccccccc} {[300]} & {[201]} & {[102]} & {[003]} & {[210]} & {[111]} & {[012]} & {[120]} & {[021]} & {[030]}\end{array}\\ \begin{array}{c} {[000]}\\ {[001]}\\ {[010]}\\ {[011]}\\ {[100]}\\ {[101]}\\ {[110]}\\ {[111]}\end{array} & \begin{bmatrix} 1 & -3z^2 & 3z^4 & -z^6 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & z & -2z^3 & z^5 & 1 & -2z^2 & z^4 & 0 & 0 & 0\\ 0 & z & -2z^3 & z^5 & 1 & -2z^2 & z^4 & 0 & 0 & 0\\ 0 & 0 & z^2 & -z^4 & 0 & 2z & -2z^3 & 1 & -z^2 & 0\\ 0 & z & -2z^3 & z^5 & 1 & -2z^2 & z^4 & 0 & 0 & 0\\ 0 & 0 & z^2 & -z^4 & 0 & 2z & -2z^3 & 1 & -z^2 & 0\\ 0 & 0 & z^2 & -z^4 & 0 & 2z & -2z^3 & 1 & -z^2 & 0\\ 0 & 0 & 0 & z^3 & 0 & 0 & 3z^2 & 0 & 3z & 1\end{bmatrix}\end{array}.$$

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Proposition 2.2. *Let $B_{\alpha,N}$ be defined as in equation (2.5). Then:*

(a) $\operatorname{rank}(B_{\alpha,N}) = N+1$.
(b) *The first $N+1$ columns of $B_{\alpha,N}$ are linearly independent.*
(c) *If the digits of the binary labels of rows J and K have the same number of zeros (equivalently, the same number of ones), then the J-th and K-th rows of $B_{\alpha,N}$ are identical.*
(d) *Let $J = 00\cdots\underbrace{11\ldots1}_{j}$. Then*

$$(B_{\alpha,N})_{J;[N_0,N_1,N_2]} = \binom{N-j}{N_0}\cdot(-z^2)^{N-j-N_0}\cdot\binom{j}{N_1}z^{j-N_1}, \tag{2.6}$$

where we agree that $\binom{n}{k}=0$ whenever $k>n$.

Proof. Let A_α be defined as in equation (2.4). Denote its entries by $a_{j,\ell}$ where $j\in\{0,1\}$ and $\ell\in\{0,1,2\}$. One can check that if $J=(j_1,\ldots,j_N)\in\{0,1\}^N$ and $L=(\ell_1,\ldots,\ell_N)\in\{0,1,2\}^N$, then the (J,L)-th entry of $A_\alpha^{\otimes N}$ is $\prod_{s=1}^N a_{j_s,\ell_s}$.

We first prove (c). Since J and K have the same number of zeros, there exists $\sigma\in S_N$ such that $J=\sigma(K)$.

Let N_0,N_1,N_2 be nonnegative integers with $N_0+N_1+N_2=N$, $\tau\in S_N$, write

$$\tau([N_0,N_1,N_2]) := \{\tau(L)|L\in[N_0,N_1,N_2]\}.$$

It is easy to verify that $\tau([N_0,N_1,N_2])=[N_0,N_1,N_2]$ for any τ. Then

$$\begin{aligned}(B_{\alpha,N})_{J;[N_0,N_1,N_2]} &= \sum_{L=\ell_1\ell_2\cdots\ell_N\in[N_0,N_1,N_2]}(A_\alpha^{\otimes N})_{J,L}\\ &= \sum_{L=\ell_1\ell_2\cdots\ell_N\in[N_0,N_1,N_2]}\prod_{s=1}^N(A_\alpha)_{j_s,\ell_s}\\ &= \sum_{\sigma(L)\in[N_0,N_1,N_2]}\prod_{s=1}^N(A_\alpha)_{j_{\sigma(s)},\ell_{\sigma(s)}}\\ &= \sum_{L\in[N_0,N_1,N_2]}\prod_{s=1}^N(A_\alpha)_{k_s,\ell_s}\\ &= (B_{\alpha,N})_{K;[N_0,N_1,N_2]},\end{aligned}$$

where $J=j_1j_2\cdots j_N$ and $K=k_1k_2\cdots k_N$. Thus, the J-th and K-th rows of $B_{\alpha,N}$ are identical.

Note that

$$(B_{\alpha,N})_{J;[N_0,N_1,N_2]} = \sum_{L=\ell_1\ell_2\cdots\ell_N\in[N_0,N_1,N_2]}\prod_{s=1}^{N-j}a_{0,\ell_s}\prod_{t=N-j+1}^{N}a_{1,\ell_t}. \tag{2.7}$$

Let $L\in[N_0,N_1,N_2]$ corresponding to a nonzero term in the formula (2.7). Since $a_{10}=0$, then $\{s|\ell_s=0\}\subseteq[N-j]$. Since $\#\{s|\ell_s=0\}=N_0$, there are $\binom{N-j}{N_0}$ different choices for the positions of 0s in L. Now suppose that the positions of 0s have been chosen, then for $s\in[N-j]\backslash\{s|\ell_s=0\}$, ℓ_s can't be 1 since $a_{01}=0$. Thus, there are $N-j-N_0$ terms of $-z^2$ in the first product.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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For the second product, $\ell_{N-j+1}\cdots\ell_N$ must contain all the N_1 1s in L since $a_{01}=0$, and there are $\binom{j}{N_1}$ different choices. After the positions of 1s been chosen, all the ℓ_t left must be 2, and thus, the corresponding terms in the product are z, yielding that the formula (2.6) holds.

By formula (2.6), the submatrix of $B_{\alpha,N}$ obtained by taking only the rows labeled by

$$00\cdots00,\ 00\cdots01,\ \dots,\ 11\cdots11,$$

and columns labeled by

$$[N,0,0],\ [N-1,0,1],\ \dots,\ [1,0,N-1],\ [0,0,N],$$

is an upper triangular matrix with nonzero diagonals. Thus, (a) and (b) hold. $\square$

Now, define the $(N+1)\times p_N$ matrix $C_{\alpha,N}$ as the submatrix of $B_{\alpha,N}$ obtained by taking only the rows labeled by $00\cdots00$, $00\cdots01,\dots,11\cdots11$. This makes $C_{\alpha,N}$ a full row rank matrix such that $B_{\alpha,N}\mathbf{y}=0$ if and only if $C_{\alpha,N}\mathbf{y}=0$. We illustrate $C_{\alpha,2}$ and $C_{\alpha,3}$ below,

$$C_{\alpha,2}=\begin{array}{c} \\ [00] \\ [01] \\ [11] \end{array}\begin{array}{c} \begin{array}{cccccc} [200] & [101] & [002] & [110] & [011] & [020] \end{array} \\ \begin{bmatrix} 1 & -2z^2 & z^4 & 0 & 0 & 0 \\ 0 & z & -z^3 & 1 & -z^2 & 0 \\ 0 & 0 & z^2 & 0 & 2z & 1 \end{bmatrix} \end{array},$$

$$C_{\alpha,3}=\begin{array}{c} \\ [000] \\ [001] \\ [011] \\ [111] \end{array}\begin{array}{c} \begin{array}{cccccccccc} [300] & [201] & [102] & [003] & [210] & [111] & [012] & [120] & [021] & [030] \end{array} \\ \begin{bmatrix} 1 & -3z^2 & 3z^4 & -z^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & -2z^3 & z^5 & 1 & -2z^2 & z^4 & 0 & 0 & 0 \\ 0 & 0 & z^2 & -z^4 & 0 & 2z & -2z^3 & 1 & -z^2 & 0 \\ 0 & 0 & 0 & z^3 & 0 & 0 & 3z^2 & 0 & 3z & 1 \end{bmatrix} \end{array}.$$

Notice that we can write $C_{\alpha,N}$ as

$$C_{\alpha,N}=\left[\Gamma_N\ \middle|\ \begin{matrix}0_{1,N}\\ D_{N,1}\Gamma_{N-1}\end{matrix}\ \middle|\ \begin{matrix}0_{2,N-1}\\ D_{N,2}\Gamma_{N-2}\end{matrix}\ \middle|\ \cdots\ \middle|\ \begin{matrix}0_{N,1}\\ D_{N,N}\Gamma_{0}\end{matrix}\right],$$

where for $n=N,\dots,0$, and $k\le n$, $D_{n,k}$ is the diagonal matrix $\operatorname{diag}\Big(\binom{k}{k},\binom{k+1}{k},\dots,\binom{n}{k}\Big)\in M_{n-k+1}$, and

$$\Gamma_n=\begin{bmatrix}1&&&&\\&z&&&\\&&\ddots&&\\&&&\ddots&\\&&&&z^n\end{bmatrix}\begin{bmatrix}1&\binom{n}{1}(-z^2)&\binom{n}{2}(-z^2)^2&\cdots&\binom{n}{n}(-z^2)^n\\0&1&\binom{n-1}{1}(-z^2)&\cdots&\binom{n-1}{n-1}(-z^2)^{n-1}\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\end{bmatrix}.$$

So, Γ_n is an $(n+1)\times(n+1)$ upper triangular matrix whose (j,k)-th entry is given by

$$\begin{cases}(-1)^{k-j}\binom{n+1-j}{k-j}z^{j-1+2(k-j)} & \text{if } k\ge j,\\ 0 & \text{if } j>k.\end{cases}$$

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3. Necessity of Conjecture 1.4. To prove the necessity of Conjecture 1.4, we demonstrate another symmetry one may impose on the solution of $\mathbf{x}$ of the equation (1.1).

PROPOSITION 3.1. *Suppose* $\mathbf{x} \in \mathbb{R}^{3^N}$ *satisfies equation* (1.1)*, and* $\hat{\mathbf{x}}$ *is obtained from* $\mathbf{x}$ *by exchanging the entries* $\mathbf{x}_j = \mathbf{x}_{3^N-j}$ *whenever* $1 \le j \le (3^N-1)/2$. *Then* $A^{\otimes N}\hat{\mathbf{x}} = 0$.

Note that if the entries of $\mathbf{x}$ and $\hat{\mathbf{x}}$ are labeled by $x_{j_1\cdots j_N}$ and $\hat{x}_{j_1\cdots j_N}$ using ternary sequences $j_1\cdots j_N \in \{0,1,2\}^N$, then $x_{j_1\cdots j_N} = \hat{x}_{(2-j_1)\cdots(2-j_N)}$.

Proof. Let $\tilde{A}_\alpha = \begin{bmatrix} 1 & e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then for A_α defined in (1.1),

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} A_\alpha = \tilde{A}_\alpha = J\overline{\tilde{A}_\alpha}K.$$

Thus, $\mathbf{x} \in \mathrm{Null}(A_\alpha^{\otimes N})$ if and only if $\mathbf{x} \in \mathrm{Null}(\tilde{A}_\alpha^{\otimes N})$. Additionally, if $\mathbf{x}$ is real,

$$\mathbf{x} \in \mathrm{Null}(\tilde{A}_\alpha^{\otimes N}) \iff \tilde{A}_\alpha^{\otimes N}\mathbf{x} = \mathbf{0} \iff \overline{\tilde{A}_\alpha^{\otimes N}}\mathbf{x} = \mathbf{0} \iff K^{\otimes N}\mathbf{x} \in \mathrm{Null}(\tilde{A}_\alpha^{\otimes N}).$$

So, $\hat{x}_{j_1\cdots j_N} = \hat{x}_{2-j_1\cdots 2-j_N}$. Thus, we can assume that $x_{i_1 i_2\cdots i_n} = x_{(2-i_1)(2-i_2)\cdots(2-i_N)}$. ∎

By the above proposition and the discussion in Section 2, we see that the system $A_\alpha^{\otimes N}\mathbf{x} = 0$ has a non-trivial nonnegative solution if and only if the system $C_{\alpha,N}\mathbf{y} = 0$ has a non-trivial nonnegative solution $\mathbf{y}$. We have the following.

THEOREM 3.2. *Let* $\alpha \in [\frac{\pi}{2}, \pi]$. *If the equation* $C_{\alpha,N}\mathbf{y} = 0$ *has a non-trivial nonnegative solution* $\mathbf{y}$*, then* $\alpha \in \left[\frac{\pi}{2} + \frac{\pi}{2N}, \pi\right]$.

Proof. We consider the reduced equation $C_{\alpha,N}\mathbf{y} = 0$ with $z = e^{i\alpha}$ as shown in Section 2. Let

$$\begin{aligned}\mathbf{y} &= (y_{[N00]}, y_{[(N-1)01]}, \ldots, y_{[00N]}, y_{[(N-1)10]}, \ldots, y_{[01(N-1)]}, \ldots, \ldots, y_{[0N0]})^t \\ &= (y_{0,0}, y_{0,1}, \ldots, y_{0,N}, y_{1,0}, \ldots, y_{1,N-1}, \ldots, y_{N,0})^t\end{aligned}$$

be a nonnegative solution of $C_{\alpha,N}\mathbf{y} = 0$. We will show that if $\alpha \in [\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2N})$, then $\mathbf{y}$ is a zero vector, which is a contradiction.

Case 1. N is even. Since $y_{j,k} = y_{j,N-j-k}$, we may rewrite the first equation of the linear system as

$$\sum_{k=0}^{N/2-1} (-1)^k \binom{N}{k} (z^{2k} + z^{2N-2k}) y_{0,k} + (-1)^{N/2} \binom{N}{N/2} z^N y_{0,N/2} = 0. \tag{3.8}$$

Divided by z^N, (3.8) reduces to

$$\sum_{k=0}^{N/2-1} (-1)^k \binom{N}{k} 2\cos((N-2k)\alpha) y_{0,k} + (-1)^{N/2} \binom{N}{N/2} y_{0,N/2} = 0. \tag{3.9}$$

Let $\theta = \alpha - \frac{\pi}{2}$. Since we assume that $\alpha \in [\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2N})$, then $\theta \in [0, \frac{\pi}{2N})$ so that $\cos(m\theta)$ are all positive for $1 \le m \le N$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 36, pp. 198-209, April 2020.

Now replace α in (3.9) with θ, we have

$$(-1)^{N/2}\left(\sum_{k=0}^{N/2-1}\binom{N}{k}2\cos((N-2k)\theta)y_{0,k}+\binom{N}{N/2}y_{0,N/2}\right)=0.$$

Since all the coefficients of $y_{0,k}$ are nonnegative, $y_{0,k}=0$ for all $k=0,1,\ldots,N$.

Case 2. N is odd. Since $y_{j,k}=y_{j,N-j-k}$, we may rewrite the first equation of the linear system as

$$\sum_{k=0}^{(N-1)/2}(-1)^k\binom{N}{k}(z^{2k}-z^{2N-2k})y_{0,k}=0.$$

Dividing the equation by iz^N, and replacing α with $\theta=\alpha-\frac{\pi}{2}$, we get

$$(-1)^{\frac{N+1}{2}}\left(\sum_{k=0}^{(N-1)/2}\binom{N}{k}2\cos((N-2k)\theta)y_{0,k}\right)=0,$$

by the same reason as the even case, $y_{0,k}$ needs to be 0 for all $k=0,1,\ldots,N$.

For $y_{1,0},y_{1,1},\ldots,y_{1,N-1}$, since it is already proved that when $\alpha\in[\frac{\pi}{2},\frac{\pi}{2}+\frac{\pi}{2N})$, $y_{0,0}=y_{0,1}=\cdots=y_{0,N}=0$, the second equation of $C_{\alpha,N}\mathbf{y}=0$ becomes the same as the first equation of $C_{\alpha,N-1}\mathbf{y}=0$. By induction on N, since $[\frac{\pi}{2},\frac{\pi}{2}+\frac{\pi}{2N})\subseteq[\frac{\pi}{2},\frac{\pi}{2}+\frac{\pi}{2(N-1)})$, we have $y_{1,0}=y_{1,1}=\cdots=y_{1,N-1}=0$. Furthermore, by induction on j, we have $y_{j,0}=\cdots=y_{j,N-j}=0$ for $j=0,1,\ldots,N$, which means $\mathbf{y}=0$, completing the proof. ∎

COROLLARY 3.3. *Let $\alpha\in[\frac{\pi}{2},\pi]$. If the equation $A_\alpha^{\otimes N}\mathbf{x}=0$ has a non-trivial nonnegative solution $\mathbf{x}$, then $\alpha\in\left[\frac{\pi}{2}+\frac{\pi}{2N},\pi\right]$.*

4. Explicit solution of the system $C_{\alpha,N}\mathbf{y}=0$ when $N\leq 10$.

Note that if $\alpha\in[\frac{\pi}{2}+\frac{\pi}{2N},\pi]\subseteq[\frac{\pi}{2}+\frac{\pi}{2(N+1)},\pi]$ and $\mathbf{x}\in\mathbb{C}^{3^N}$ satisfy $A_\alpha^{\otimes N}\mathbf{x}=0$, then for any nonnegative vector $\mathbf{y}\in\mathbb{C}^3$, we have $A_\alpha^{\otimes(N+1)}(\mathbf{x}\otimes\mathbf{y})=0$. Thus, for $N\leq 10$, it is enough to find a non-trivial nonnegative solution to $C_{\alpha,N}\mathbf{y}=0$ when $\alpha\in\left[\frac{\pi}{2}+\frac{\pi}{2N},\frac{\pi}{2}+\frac{\pi}{2(N-1)}\right)$. In the next lemma, we determine the exact location of $e^{ik\alpha}$ in the Argand plane. Let Q_1,Q_2,Q_3,Q_4 denote the four quadrants of the complex plane.

LEMMA 4.1. *For $N\geq 2$, let*

$$\theta\in\left[\frac{\pi}{2N},\ \frac{\pi}{2(N-1)}\right)\quad\textit{and}\quad\alpha=\theta+\frac{\pi}{2}.$$

Then $N\alpha\in Q_{N+2 \pmod 4}$, and for $k<N$ we have $k\alpha\in Q_{k+1 \pmod 4}$.

Proof. Note that $[a,b]\subseteq Q_r$ if and only if there exists ℓ such that $r\equiv\ell+1 \pmod 4$ and $\frac{\ell}{2}\pi\leq a\leq b\leq\frac{\ell+1}{2}\pi$. Since $\alpha\in\left[\pi\left(\frac12+\frac{1}{2N}\right),\pi\left(\frac12+\frac{1}{2(N-1)}\right)\right)$, then $N\alpha\in\left[\pi\left(\frac{N+1}{2}\right),\pi\left(\frac{N+1}{2}+\frac{1}{2(N-1)}\right)\right)$. Note that $\frac{1}{2(N-1)}\leq\frac12$, and hence, if $\ell=N+1$, then $\alpha\in Q_r$ where $r\equiv\ell+1\equiv N+2 \pmod 4$. On the other hand, if $0\leq k<N$, then $\frac{k}{2(N-1)}\leq\frac12$. Note that $k\alpha\in\left[\pi\left(\frac{k}{2}+\frac{k}{2N}\right),\pi\left(\frac{k}{2}+\frac{k}{2(N-1)}\right)\right)$. Thus, if $\ell=k$ then $k\alpha\in Q_r$ where $r\equiv\ell+1\equiv k+1 \pmod 4$. ∎

We now present some explicit non-trivial nonnegative solutions to $C_{\alpha,N}\mathbf{y}=0$. One can use the preceding lemma to verify that the given $\mathbf{y}$ is nonzero and nonnegative.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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1. For $N=1$, we have $\alpha=\pi$ and a non-trivial nonnegative solution given by

$$\begin{array}{rc} & \begin{array}{ccc} [100] & [001] & [010] \end{array} \\ \mathbf{y}^T = & \left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right]. \end{array}$$

2. For $N=2$, we have $\alpha\in\left[\frac{3\pi}{4},\pi\right)$ and a non-trivial nonnegative solution given by

$$\begin{array}{rc} & \begin{array}{cccccc} [200] & [101] & [002] & [110] & [011] & [020] \end{array} \\ \mathbf{y}^T = & \left[\begin{array}{cccccc} 1 & \cos 2\alpha & 1 & -\cos\alpha & -\cos\alpha & 1 \end{array}\right]. \end{array}$$

3. For $N=3$, we have $\alpha\in\left[\frac{2\pi}{3},\frac{3\pi}{4}\right)$ and a non-trivial nonnegative solution given by

$$\begin{array}{rc} & \begin{array}{cccccccccc} [300] & [201] & [102] & [003] & [210] & [111] & [012] & [120] & [021] & [030] \end{array} \\ \mathbf{y}^T = & \left[\begin{array}{cccccccccc} 3\sin\alpha & \sin 3\alpha & \sin 3\alpha & 3\sin\alpha & -\sin 2\alpha & -\sin 2\alpha & -\sin 2\alpha & \sin\alpha & \sin\alpha & 0 \end{array}\right]. \end{array}$$

4. For $N=4$, we have $\alpha\in\left[\frac{5\pi}{8},\frac{2\pi}{3}\right)$ and a non-trivial nonnegative solution given by

$$\mathbf{y}^T=\left[\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4\right], \text{ where}$$

$$\begin{array}{rcl} & & \begin{array}{ccccc} [400] & [301] & [202] & [103] & [004] \end{array} \\ \mathbf{a}_0 & = & \left[\begin{array}{ccccc} 6 & 0 & -2\cos 4\alpha & 0 & 6 \end{array}\right], \\ & & \begin{array}{cccc} [310] & [211] & [112] & [013] \end{array} \\ \mathbf{a}_1 & = & \left[\begin{array}{cccc} -3\cos\alpha & \cos 3\alpha & \cos 3\alpha & -3\cos\alpha \end{array}\right], \\ & & \begin{array}{cccccc} [220] & [121] & [022] & [130] & [031] & [040] \end{array} \\ \left[\mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4\right] & = & \left[\begin{array}{cccccc} 2 & 0 & 2 & -3\cos\alpha & -3\cos\alpha & 6 \end{array}\right]. \end{array}$$

5. For $N=5$, we have $\alpha\in\left[\frac{3\pi}{5},\frac{5\pi}{8}\right)$ and a non-trivial nonnegative solution given by

$$\mathbf{y}^T=\left[\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5\right], \text{ where}$$

$$\begin{array}{rcl} & & \begin{array}{cccccc} [500] & [401] & [302] & [203] & [104] & [005] \end{array} \\ \mathbf{a}_0 & = & \left[\begin{array}{cccccc} 20\sin\alpha & 0 & -2\sin 5\alpha & -2\sin 5\alpha & 0 & 20\sin\alpha \end{array}\right], \\ & & \begin{array}{ccccc} [410] & [311] & [212] & [113] & [014] \end{array} \\ \mathbf{a}_1 & = & \left[\begin{array}{ccccc} -4\sin 2\alpha & \sin 4\alpha & 2\sin 4\alpha & \sin 4\alpha & -4\sin 2\alpha \end{array}\right], \\ & & \begin{array}{cccc} [320] & [221] & [122] & [023] \end{array} \\ \mathbf{a}_2 & = & \left[\begin{array}{cccc} 3\sin\alpha & -\sin 3\alpha & -\sin 3\alpha & 3\sin\alpha \end{array}\right], \\ & & \begin{array}{cccccc} [230] & [131] & [032] & [140] & [041] & [\ 050] \end{array} \\ \left[\mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5\right] & = & \left[\begin{array}{cccccc} -\sin 2\alpha & 0 & -\sin 2\alpha & 2\sin\alpha & 2\sin\alpha & 0 \end{array}\right]. \end{array}$$

6. For $N=6$, we have $\alpha\in\left[\frac{7\pi}{12},\frac{3\pi}{5}\right)$ and a non-trivial nonnegative solution given by

$$\mathbf{y}^T=\left[\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5 \quad \mathbf{a}_6\right], \text{ where}$$

$$\begin{array}{rcl} \mathbf{a}_0 & = & \left[\begin{array}{ccccccc} 20 & 0 & 0 & 2\cos 6\alpha & 0 & 0 & 20 \end{array}\right], \\ \mathbf{a}_1 & = & \left[\begin{array}{cccccc} -10\cos\alpha & 0 & -\cos 5\alpha & -\cos 5\alpha & 0 & -10\cos\alpha \end{array}\right], \\ \left[\mathbf{a}_2 \quad \mathbf{a}_3\right] & = & \left[\begin{array}{ccccccccc} 3 & 0 & \cos 4\alpha & 0 & 3 & \cos 3\alpha & 0 & 0 & \cos 3\alpha \end{array}\right], \\ \left[\mathbf{a}_4 \quad \mathbf{a}_5 \quad \mathbf{a}_6\right] & = & \left[\begin{array}{cccccc} -2\cos 2\alpha & 0 & -2\cos 2\alpha & 0 & 0 & 5 \end{array}\right]. \end{array}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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7. For $N = 7$, we have $\alpha \in \left[\frac{4\pi}{7}, \frac{7\pi}{12}\right)$ and a non-trivial nonnegative solution given by

$$\mathbf{y}^T = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7\end{bmatrix}, \text{ where}$$

$$\begin{aligned}
\mathbf{a}_0 &= \begin{bmatrix} 140\sin\alpha & 0 & 0 & 4\sin 7\alpha & 4\sin 7\alpha & 0 & 0 & 140\sin\alpha \end{bmatrix},\\
\mathbf{a}_1 &= \begin{bmatrix} -30\sin 2\alpha & 0 & -2\sin 6\alpha & -4\sin 6\alpha & -2\sin 6\alpha & 0 & -30\sin 2\alpha \end{bmatrix},\\
\mathbf{a}_2 &= \begin{bmatrix} 20\sin\alpha & 0 & 2\sin 5\alpha & 2\sin 5\alpha & 0 & 20\sin\alpha \end{bmatrix},\\
\mathbf{a}_3 &= \begin{bmatrix} 2\sin 4\alpha - 4\sin 2\alpha & \sin 4\alpha & 0 & \sin 4\alpha & 2\sin 4\alpha - 4\sin 2\alpha \end{bmatrix},\\
\mathbf{a}_4 &= \begin{bmatrix} -4\sin 3\alpha + 6\sin\alpha & -2\sin 3\alpha & -2\sin 3\alpha & -4\sin 3\alpha + 6\sin\alpha \end{bmatrix},\\
\begin{bmatrix}\mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7\end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 10\sin\alpha & 10\sin\alpha & 0 \end{bmatrix}.
\end{aligned}$$

8. For $N = 8$, we have $\alpha \in \left[\frac{9\pi}{16}, \frac{4\pi}{7}\right)$ and a non-trivial nonnegative solution given by

$$\mathbf{y}^T = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8\end{bmatrix}, \text{ where}$$

$$\begin{aligned}
\mathbf{a}_0 &= \begin{bmatrix} 140 & 0 & 0 & 0 & -4\cos 8\alpha & 0 & 0 & 0 & 140 \end{bmatrix},\\
\mathbf{a}_1 &= \begin{bmatrix} -70\cos\alpha & 0 & 0 & 2\cos 7\alpha & 2\cos 7\alpha & 0 & 0 & -70\cos\alpha \end{bmatrix},\\
\mathbf{a}_2 &= \begin{bmatrix} 20 & 0 & 0 & -2\cos 6\alpha & 0 & 0 & 20 \end{bmatrix},\\
\mathbf{a}_3 &= \begin{bmatrix} 5\cos 3\alpha & -\cos 5\alpha & 0 & 0 & -\cos 5\alpha & 5\cos 3\alpha \end{bmatrix},\\
\mathbf{a}_4 &= \begin{bmatrix} -8\cos 2\alpha & 2\cos 4\alpha & 0 & 2\cos 4\alpha & -8\cos 2\alpha \end{bmatrix},\\
\begin{bmatrix}\mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8\end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 10 & 0 & 10 & -35\cos\alpha & -35\cos\alpha & 140 \end{bmatrix}.
\end{aligned}$$

9. For $N = 9$, we have $\alpha \in \left[\frac{5\pi}{9}, \frac{9\pi}{16}\right)$ and a non-trivial nonnegative solution given by

$$\mathbf{y}^T = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8 & \mathbf{a}_9\end{bmatrix}, \text{ where}$$

$$\begin{aligned}
\mathbf{a}_0 &= \begin{bmatrix} 504\sin\alpha & 0 & 0 & 0 & -4\sin 9\alpha & -4\sin 9\alpha & 0 & 0 & 0 & 504\sin\alpha \end{bmatrix},\\
\mathbf{a}_1 &= \begin{bmatrix} -112\sin 2\alpha & 0 & 0 & 2\sin 8\alpha & 4\sin 8\alpha & 2\sin 8\alpha & 0 & 0 & -112\sin 2\alpha \end{bmatrix},\\
\mathbf{a}_2 &= \begin{bmatrix} 70\sin\alpha & 0 & 0 & -2\sin 7\alpha & -2\sin 7\alpha & 0 & 0 & 70\sin\alpha \end{bmatrix},\\
\mathbf{a}_3 &= \begin{bmatrix} 6\sin 4\alpha - 15\sin 2\alpha & -\sin 6\alpha & -\sin 6\alpha & 0 & -\sin 6\alpha & -\sin 6\alpha & 6\sin 4\alpha - 15\sin 2\alpha \end{bmatrix},\\
\mathbf{a}_4 &= \begin{bmatrix} -10\sin 3\alpha + 20\sin\alpha & 2\sin 5\alpha & 2\sin 5\alpha & 2\sin 5\alpha & 2\sin 5\alpha & -10\sin 3\alpha + 20\sin\alpha \end{bmatrix},\\
\mathbf{a}_5 &= \begin{bmatrix} 4\sin 4\alpha & 0 & 0 & 0 & 4\sin 4\alpha \end{bmatrix},\\
\mathbf{a}_6 &= \begin{bmatrix} 15\sin\alpha - 12\sin 3\alpha & -5\sin 3\alpha & -5\sin 3\alpha & 15\sin\alpha - 12\sin 3\alpha \end{bmatrix},\\
\begin{bmatrix}\mathbf{a}_7 & \mathbf{a}_8 & \mathbf{a}_9\end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 56\sin\alpha & 56\sin\alpha & 0 \end{bmatrix}.
\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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10. For $N = 10$, we have $\alpha \in \left[\frac{11\pi}{20}, \frac{5\pi}{9}\right)$ and a non-trivial nonnegative solution is given by

$$\mathbf{y}^T = \begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 & \mathbf{a}_8 & \mathbf{a}_9 & \mathbf{a}_{10} \end{bmatrix}, \text{ where}$$

$$\begin{aligned}
\mathbf{a}_0 &= \begin{bmatrix} 504 & 0 & 0 & 0 & 0 & 4\cos 10\alpha & 0 & 0 & 0 & 0 & 504 \end{bmatrix}, \\
\mathbf{a}_1 &= \begin{bmatrix} -252\cos\alpha & 0 & 0 & 0 & -2\cos 9\alpha & -2\cos 9\alpha & 0 & 0 & 0 & -252\cos\alpha \end{bmatrix}, \\
\mathbf{a}_2 &= \begin{bmatrix} 70 & 0 & 0 & 0 & 2\cos 8\alpha & 0 & 0 & 0 & 70 \end{bmatrix}, \\
\mathbf{a}_3 &= \begin{bmatrix} 21\cos 3\alpha & 0 & \cos 7\alpha & 0 & 0 & \cos 7\alpha & 0 & 21\cos 3\alpha \end{bmatrix}, \\
\mathbf{a}_4 &= \begin{bmatrix} -30\cos 2\alpha & 0 & -2\cos 6\alpha & 0 & -2\cos 6\alpha & 0 & -30\cos 2\alpha \end{bmatrix}, \\
\mathbf{a}_5 &= \begin{bmatrix} -4\cos 5\alpha & 0 & 0 & 0 & 0 & -4\cos 5\alpha \end{bmatrix}, \\
\mathbf{a}_6 &= \begin{bmatrix} 12\cos 4\alpha + 15 & 0 & 5\cos 4\alpha & 0 & 12\cos 4\alpha + 15 \end{bmatrix}, \\
\begin{bmatrix} \mathbf{a}_7 & \mathbf{a}_8 & \mathbf{a}_9 & \mathbf{a}_{10} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 & -56\cos 2\alpha & 0 - 56\cos 2\alpha & 0 & 0 & 504 \end{bmatrix}.
\end{aligned}$$

5. Final remark. It would be nice to affirm the sufficiency of the Conjecture 1.4 for $N > 10$. Ideally, one can describe a non-trivial nonnegative solution of the linear system for every positive integer N. One may also consider finding an existence proof. In this connection, we have the following proposition. We will continue to use the notation $C_{\alpha,N}$ and consider the reduced system $C_{\alpha,N}\mathbf{y} = 0$.

PROPOSITION 5.1. *Suppose $\alpha \in \left[\frac{\pi}{2} + \frac{\pi}{2N}, \pi\right]$. The following conditions are equivalent.*

(a) *The system $C_{\alpha,N}\mathbf{y} = 0$ has no non-trivial nonnegative solution.*
(b) *There is a complex vector $\mathbf{u} = (\xi_0, \dots, \xi_N)$ with all entries having positive real parts such that all the entries of $\mathbf{u}C_{\alpha,N}$ has positive real parts.*

Proof. We convert the system $C_{\alpha,N}\mathbf{y} = 0$ to a real linear system

$$\tilde{C}_{\alpha,N}\mathbf{y} = 0, \quad \text{where } \tilde{C}_{\alpha,N} = \begin{bmatrix} \Re(C_{\alpha,N}) \\ \Im(C_{\alpha,N}) \end{bmatrix}. \tag{5.10}$$

By Farkas lemma, for example see [1, Section 5.8], the system (5.10) has no non-trivial nonnegative solution if and only if there is a real vector $\mathbf{v} = (a_0, \dots, a_N, b_0, \dots, b_N)$ such that $\mathbf{v}\tilde{C}_{\alpha,N}$ is a positive vector, i.e., all entries are positive. Note that $a_0, a_1, \dots, a_N$ appear in $\mathbf{v}\tilde{C}_{\alpha,N}$ as the jth entries for $j = 1, 1 + (N+1), 1 + (N+1) + N, 1 + (N+1) + N + (N-1), \dots, (N+1)(N+2)/2$. So, $a_0, \dots, a_N > 0$ if the said vector $\mathbf{v}$ exists. Set $\xi_j = a_j - ib_j$ for $j = 0, \dots, N$. Then the system (5.10) has no non-trivial nonnegative solution if and only if condition (b) holds. ☐

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