Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

SOME GRAPHS DETERMINED BY THEIR SIGNLESS LAPLACIAN (DISTANCE) SPECTRA*

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Abstract. In literature, there are some results known about spectral determination of graphs with many edges. In [M. Cámara and W.H. Haemers. Spectral characterizations of almost complete graphs. *Discrete Appl. Math.*, 176:19–23, 2014.], Cámara and Haemers studied complete graph with some edges deleted for spectral determination. In fact, they found that if the deleted edges form a matching, a complete graph K_m provided $m \leq n-2$, or a complete bipartite graph, then it is determined by its adjacency spectrum. In this paper, the graph $K_n \backslash K_{l,m}$ $(n > l+m)$ which is obtained from the complete graph K_n by removing all the edges of a complete bipartite subgraph $K_{l,m}$ is studied. It is shown that the graph $K_n \backslash K_{1,m}$ with $m \geq 4$ is determined by its signless Laplacian spectrum, and it is proved that the graph $K_n \backslash K_{l,m}$ is determined by its distance spectrum. The signless Laplacian spectral determination of the multicone graph $K_{n-2\alpha} \vee \alpha K_2$ was studied by Bu and Zhou in [C. Bu and J. Zhou. Signless Laplacian spectral characterization of the cones over some regular graphs. *Linear Algebra Appl.*, 436:3634–3641, 2012.] and Xu and He in [L. Xu and C. He. On the signless Laplacian spectral determination of the join of regular graphs. *Discrete Math. Algorithm. Appl.*, 6:1450050, 2014.] only for $n-2\alpha = 1$ or 2. Here, this problem is completely solved for all positive integer $n-2\alpha$. The proposed approach is entirely different from those given by Bu and Zhou, and Xu and He.

Key words. Signless Laplacian spectrum, Distance spectrum.

AMS subject classifications. 05C50.

1. Introduction. Graphs considered in this paper are all simple. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *degree* of a vertex v_i is the number of neighbors of v_i in G and is denoted by $d_i(G)$. Throughout the paper, we assume that the sequence $\{d_i(G)\}_{i=1}^{n}$ is non-increasing, i.e., $d_i(G) \geq d_{i+1}(G)$, $i = 1, 2, \ldots, n-1$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ real symmetric matrix whose (i,j)-entry is 1 if $v_iv_j \in E(G)$ and 0 otherwise. The *adjacency spectrum* or *spectrum* of G is the multiset of eigenvalues of $A(G)$. The matrix $L(G) = D_g(G) - A(G)$ (resp., $Q(G) = D_g(G) + A(G)$), where $D_g(G) = diag(d_1(G), d_2(G), \ldots, d_n(G))$ is the *Laplacian matrix* (resp., *signless Laplacian*) of G and the *L-spectrum* (resp., *Q-spectrum*) of G is the spectrum of $L(G)$ (resp., $Q(G)$). Two graphs are *cospectral* (resp., L-cospectral, *Q-cospectral*) if they have same spectrum (resp., L-spectrum, Q-spectrum). We say that a graph G *is determined by its spectrum* (resp., L-spectrum, Q-spectrum) or simply G is *DS* (resp., *DLS*, *DQS*) if there is no non-isomorphic graph cospectral to G.

One of the interesting problems in spectral graph theory is to characterize graphs which are determined by their spectra. This question was raised by Günthard and Primas [12] with motivations from Hückel theory. In [25, 26], Dam and Haemers gave a survey of (partial) answers to the posed question. In literature,

*Received by the editors on April 6, 2018. Accepted for publication on May 28, 2020. Handling Editor: Bryan L. Shader. Corresponding Author: Kinkar Chandra Das.

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The second author was supported by the Sungkyun research fund, Sungkyunkwan University, 2017, and National Research Foundation of the Korean government with grant no. 2017R1D1A1B03028642.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

there are several papers addressing the problem of characterizing graphs which are DS, DLS and DQS. For some recent papers on this topic, see [8, 14, 16, 22, 24].

The *distance* between the vertices v_i and v_j of G is the length of a shortest path between them. It is denoted by $d(v_i, v_j)$. The maximum of all distances between any pair of vertices of G is the *diameter* of G. The *distance matrix* $D(G)$ of a connected graph G is the real symmetric matrix of order n with (i, j)-entry equal to $d(v_i, v_j)$. The *distance spectrum* or *D-spectrum* of G is the spectrum of $D(G)$. Two connected graphs are *D-cospectral* if they have same D-spectrum. A connected graph G is DDS if there is no non-isomorphic graph D-cospectral to G. In [18], Lin et al. showed that the complete graph, the complete bipartite graph and the complete split graph are DDS. Further, they conjectured that the complete multipartite graph is DDS and this conjecture was confirmed by Jin and Zhang in [17]. In [28], Xue et al. proved that the path graph and double star graph is determined by their distance spectrum. In [8], Das and Liu proved that the kite graph $Ki_{n,n-1}$ (for definition, see [8]) is DDS.

In [4], Cámara and Haemers proved that the graph $K_n \backslash K_{l,m}$ is DS. In [29], Zhou and Bu showed that if G is a disconnected DLS graph, then the join graph $G \vee K_r$ is DLS. From this, it follows that the graph $K_n \backslash K_{l,m}$ with $m > 1$ and $\frac{l}{m} > \frac{5}{3}$ is DLS, since $K_l \cup K_m$ with $m > 1$ and $\frac{l}{m} > \frac{5}{3}$ is DLS, see [1]. Motivated by these results, in Section 3 of this paper, we show that the graph $K_n \backslash K_{1,m}$ with $m \geq 4$ is determined by its signless Laplacian spectrum and we also prove that the graph $K_n \backslash K_{l,m}$ is determined by its distance spectrum. Recently, the signless Laplacian spectral determination of the join of graphs has been studied, for example, we have the following joins which are DQS.

1. $K_1 \vee (cK_2 \cup (n - 2c - 1)K_1)$ with $n \geq 2c + 1$ and $c \geq 0$ [20].
2. $K_1 \vee C_n$, where C_n is the cycle with n vertices [19].
3. $G \vee K_1$, where G is an r-regular graph on n vertices and $r = 1$ or $n - 2$ or 2 with $n \geq 11$, and $G \vee K_1$, where G is an $(n - 3)$-regular with $\overline{G}$ having no triangles [3].
4. $G \vee K_2$, where G is an r-regular graph on n vertices and $r = 1$ or $n - 2$, and $G \vee K_2$, where G is an $(n - 3)$-regular with $\overline{G}$ having no triangles [27].
5. $G \vee K_m$, where G is an $(n - 2)$-regular graph and $\overline{K}_n \vee K_2$ for $n \neq 3$ [21].
6. The complete split graph $\overline{K}_n \vee K_m$ for $n \neq 3$ [7].

Motivated by these results, in Section 4, we prove that the join graph $G \vee K_n$ is determined by its signless Laplacian spectrum, where G is a 1-regular graph. This result extends the following known theorem.

Theorem 1.1. ([3, 27]) *Let G be a 1-regular graph. Then for $r = 1, 2$, $G \vee K_r$ is determined by its signless Laplacian spectrum.*

2. Some preliminary results.

In this section, we give some necessary theorems and lemmas required to prove our main results. We denote the eigenvalues of a Hermitian matrix M of order m by $\theta_1(M) \geq \theta_2(M) \geq \cdots \geq \theta_m(M)$ and also, let $\gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_n(G)$ and $\eta_1(G) \geq \eta_2(G) \geq \cdots \geq \eta_n(G)$ be the signless Laplacian eigenvalues and distance eigenvalues of G, respectively.

Theorem 2.1. ([15]) *Let M be a Hermitian matrix of order n.*

(i) If M_k is a principal submatrix of M of order k with $1 \leq k \leq n$, then for $1 \leq i \leq k$, $\theta_{n-k+i}(M) \leq \theta_i(M_k) \leq \theta_i(M)$.
(ii) If $M = N + P$, where N and P are Hermitian matrices of order n. Then for $1 \leq i, j \leq n$, we have:
  (a) $\theta_i(N) + \theta_j(P) \leq \theta_{i+j-n}(M)$ $(i + j > n)$;
  (b) $\theta_{i+j-1}(M) \leq \theta_i(N) + \theta_j(P)$ $(i + j - 1 \leq n)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

A connected bipartite graph G with vertex partition sets U and V is called as *balanced* if the cardinalities of U and V are same.

THEOREM 2.2. ([5, 6, 8, 11]) *Let G be a connected graph of order n.*

(i) We have $\gamma_2(G) \leq n-2$, for $n \geq 2$. Moreover, $\gamma_{k+1}(G) = n-2$ $(1 \leq k \leq n-1)$ if and only if $\overline{G}$ has either k balanced bipartite components or $k+1$ bipartite components.
(ii) $d_{n-1}(G) \geq \gamma_{n-1}(G) - 1$. Furthermore, if the equality holds, then $d_{n-1}(G) = d_n(G)$.
(iii) If G is a bipartite graph, then the Q-spectrum of G is equal to its L-spectrum.
(iv) The largest Laplacian eigenvalue of G is at most n and $\gamma_1(G) \leq 2d_1(G)$.
(v) The multiplicity of 0 as an eigenvalue pertaining to $Q(G)$ is the number of connected bipartite components of G.

LEMMA 2.3. *Let $l+m \leq n-1$. Then the signless Laplacian spectrum of $K_n \backslash K_{l,m}$ consists of $n-2$ of multiplicity $n-l-m$, $n-m-2$ of multiplicity $l-1$, $n-l-2$ of multiplicity $m-1$, and the two roots of the quadratic polynomial $x^2 + (l+m-3n+4)x + 2n^2 - (2l+2m+6)n + (4m+2)l + 2m + 4$.*

Proof. The signless Laplacian matrix Q of $K_n \backslash K_{l,m}$ can be written as follows:

$$\begin{bmatrix} J_{n-m-l} + (n-2)I_{n-m-l} & J & J \\ J & J_l + (n-m-2)I_l & 0 \\ J & 0 & J_m + (n-l-2)I_m \end{bmatrix},$$

where J is a matrix whose all entries are 1 and I_m is the identity matrix of order m. From the above matrix, we see that the matrices $Q-(n-2)I_n$, $Q-(n-m-2)I_n$ and $Q-(n-l-2)I_n$ have rank at most $l+m+1$, $n-l+1$ and $n-m+1$, respectively. This implies that the matrices $Q-(n-2)I_n$, $Q-(n-m-2)I_n$ and $Q-(n-l-2)I_n$ have nullity at least $n-l-m-1$, $l-1$ and $m-1$, respectively. Thus, the spectrum of Q consists of $n-2$ with multiplicity $n-l-m-1$, $n-m-2$ with multiplicity $l-1$ and $n-l-2$ with multiplicity $m-1$. Now observe that the given partition of Q is equitable (see [2]) with the quotient matrix

$$Q_1 = \begin{bmatrix} 2n-m-l-2 & l & m \\ n-m-l & n-m+l-2 & 0 \\ n-m-l & 0 & n-l+m-2 \end{bmatrix}.$$

The spectrum of Q_1 consists of $n-2$ and the two roots of the polynomial $x^2 + (l+m-3n+4)x + 2n^2 - (2l+2m+6)n + (4m+2)l + 2m + 4$. As the spectrum of Q_1 is contained in the spectrum of Q, see [2], we are done. ∎

The following lemma gives the D-spectrum of $K_n \backslash K_{l,m}$. As the proof of the lemma is in similar lines of the above lemma, we omit the details.

LEMMA 2.4. *The distance spectrum of $K_n \backslash K_{l,m}$ consists of -1 with multiplicity $n-3$ and the three roots of the polynomial $x^3 - (n-3)x^2 - (3lm + 2n - 3)x - ml^2 - m(m-n+3)l - n + 1$.*

LEMMA 2.5. ([10]) *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices. Then the Q-spectrum of $G_1 \vee G_2$ consists of $\gamma_j(G_1) + n_2$ $(j = 2, 3, \ldots, n_1)$, $\gamma_j(G_2) + n_1$, $(j = 2, 3, \ldots, n_2)$ and the two roots of the quadratic polynomial $x^2 - \Big(2(r_1+r_2) + (n_1+n_2)\Big)x + 2(2r_1r_2 + r_1n_1 + r_2n_2)$.*

LEMMA 2.6. *Let $Q_1 = J_{n-m+1} + (n-2)I_{n-m+1}$ be a square matrix of order $n-m+1$ with $n > m+1$. Then $n-2$ is an eigenvalue of multiplicity $n-m$ and $2n-m-1$ is the remaining eigenvalue of Q_1.*

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue θ of Q_1. Then

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

$Q_1\mathbf{x} = \theta\,\mathbf{x}$. One can easily see that the eigenvalue $n-2$ with corresponding eigenvectors

$$(\underbrace{1,\,-1}_{2},0,\ldots,\,0)^T,\ \ (\underbrace{1,\,0,\,-1}_{3},0,\ldots,\,0)^T,\ \ \ldots,\ \ \text{and}\ \ (\underbrace{1,\,0,\ldots,\,0,\,-1}_{n-m+1})^T$$

satisfy the above relation. Since these $n-m$ eigenvectors are linearly independent, $n-2$ is an eigenvalue with multiplicity at least $n-m$. Since $\sum\limits_{i=1}^{n-m+1} \theta_i(Q_1) = (n-1)\,(n-m+1)$, $2n-m-1$ is the remaining eigenvalue. ∎

Corollary 2.7. *Let* $Q_1 = J_{n-m+1} + (n-2)I_{n-m+1}$ *be a square matrix of order* $n-m+1$ *with* $n > m+1$. *Then* $\theta_{n-m+1}(Q_1) = n-2$.

Proof. Since $n > m+1$, by Lemma 2.6, we get the required result. ∎

Lemma 2.8. *Let* G *be a graph of order* 2α, *where the spectrum of the signless Laplacian of* G *is*

$$Q_S(G) = \left(4\alpha-4, \underbrace{2\alpha-2,\ldots,2\alpha-2}_{\alpha}, \underbrace{2\alpha-4,\ldots,2\alpha-4}_{\alpha-1}\right).$$

Then G *is regular of degree* $(2\alpha-2)$.

Proof. From the signless Laplacian spectrum of G, one can easily see that

$$\sum_{i=1}^{2\alpha} d_i = \sum_{i=1}^{2\alpha} \gamma_i = 4\alpha(\alpha-1) \quad \text{and} \quad \sum_{i=1}^{2\alpha} d_i^2 = \sum_{i=1}^{2\alpha} \gamma_i^2 - \sum_{i=1}^{2\alpha} \gamma_i = 8\alpha(\alpha-1)^2.$$

Thus, we have

$$\sum_{i=1}^{2\alpha} \Big(d_i - 2(\alpha-1)\Big)^2 = 0, \quad \text{that is,}\ \ d_i = 2\alpha-2, \quad i = 1,\,2,\ldots,\,2\alpha.$$

Hence, G is regular of degree $(2\alpha-2)$. ∎

Let $\lambda_1(G)$ denote the spectral radius of the graph G and also let L_G be the line graph of G. The following result was obtained in [9, 13]:

Lemma 2.9. ([9, 13]) *Let* G *be a connected graph of order* n. *Then* $\gamma_1(G) = 2 + \lambda_1(L_G)$.

Corollary 2.10. *Let* G *be a connected graph of order* n *with* m *edges. Then* $\gamma_1(G) \le m+1$ *with equality holding if and only if* $L_G \cong K_m$.

Proof. It is well known that the adjacency spectral radius $\lambda_1(G) \le n-1$ with equality holding if and only if $G \cong K_n$. Then by Lemma 2.9, we have $\gamma_1(G) \le m+1$ as m is the number of edges in G. Moreover, the equality holds if and only if $L_G \cong K_m$. ∎

3. Signless Laplacian (distance) spectral characterization of $K_n\backslash K_{l,m}$. Let $\lambda_1(G) \ge \lambda_2(G) \ge \cdots \ge \lambda_n(G)$ be the spectrum of G. The following theorem gives a general form of Theorem 2.2 (ii).

Theorem 3.1. *Let* G *be a graph with* n *vertices and let* H_i *be the graph induced by the vertices* $v_{n-i+1}, v_{n-i+2}, \ldots, v_n$ *of* G, *where* $2 \le i \le n$. *Then*

$$\gamma_{n-i+1}(G) - \lambda_1(H_i) \le d_{n-i+1}(G).$$

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Proof. Let $A(H_i)$ be the adjacency matrix of H_i with its rows and columns indexed by $v_{n-i+1}, v_{n-i+2}, \dots, v_n$. Let $Q_i = D_i + A(H_i)$, where $D_i = diag(d_{n-i+1}(G), d_{n-i+2}(G), \dots, d_n(G))$. Then clearly, Q_i is a principal submatrix of $Q(G)$ and so by Theorem 2.1 (i), we have $\theta_1(Q_i) \geq \gamma_{n-i+1}(G)$. Hence, $\gamma_{n-i+1}(G) \leq \theta_1(Q_i) \leq d_{n-i+1}(G) + \lambda_1(H_i)$ by Theorem 2.1 (ii). ∎

In literature, complete graph K_n with some edges deleted is studied for spectral determination. In the following theorem, it is shown that if the deleted edges form a star graph with at least 4 edges, then it is determined by its signless Laplacian spectrum.

THEOREM 3.2. *If $n > m+1$ and $m \geq 4$, then $K_n \backslash K_{1,m}$ is DQS.*

Proof. Let G be a graph Q-cospectral with $K_n \backslash K_{1,m}$. Then by Lemma 2.3, the Q-spectrum of G is

$$\left.\begin{array}{l} \left(3n - m - 5 \pm \sqrt{m^2 + (2n-14)m + (n+1)^2}\right)/2, \\ n-2 \text{ with multiplicity } n-m-1, \\ n-3 \text{ with multiplicity } m-1. \end{array}\right\} . \tag{3.1}$$

This implies

$$\left.\begin{array}{r} 2|E(G)| = \sum_{i=1}^{n} d_i(G) = \sum_{i=1}^{n} \gamma_i(G) = n^2 - 2m - n, \\ \sum_{i=1}^{n} d_i(G)(d_i(G)+1) = \sum_{i=1}^{n} \gamma_i^2(G) = n^3 - n^2 + m^2 - (4n-3)m. \end{array}\right\} . \tag{3.2}$$

Since $n > m+1$, from (3.1), one can easily see that

$$\gamma_1(G) = \left(3n - m - 5 + \sqrt{m^2 + (2n-14)m + (n+1)^2}\right)/2 > 2(n-2),$$

and $\gamma_1(G) \leq 2d_1(G)$ by Theorem 2.2 (iv). It follows that $d_1(G) = n-1$, and hence, G is connected. Since

$$\left(3n - m - 5 - \sqrt{m^2 + (2n-14)m + (n+1)^2}\right)/2 < n-3,$$

by Theorem 2.2 (ii) and (3.1), we have $d_{n-1}(G) \geq n-4$. Furthermore, if $d_{n-1}(G) = n-4$, then $d_{n-1}(G) = d_n(G)$. Let v_n be a vertex of degree $d_n(G)$ and let n_i be the number of vertices in $V(G)\backslash\{v_n\}$ of degree $n-i$, $i = 1, 2, 3, 4$. Suppose $d_{n-m+1}(G) = n-1$. Then $Q_1 = J_{n-m+1} + (n-2)I_{n-m+1}$ is a principal submatrix of $Q(G)$ and so by Theorem 2.1 (i) with Corollary 2.7, we have $\gamma_{n-m+1}(G) \geq \theta_{n-m+1}(Q_1) = n-2$. Since $\gamma_{n-m+1}(G) = n-3$, this is a contradiction. Thus, $1 \leq n_1 \leq n-m$.

From (3.2), we have

$$n_4 + n_3 + n_2 + n_1 = n - 1, \tag{3.3}$$

$$(n-4)n_4 + (n-3)n_3 + (n-2)n_2 + (n-1)n_1 = n^2 - 2m - n - d_n(G), \tag{3.4}$$

$$(n-4)^2 n_4 + (n-3)^2 n_3 + (n-2)^2 n_2 + (n-1)^2 n_1 = n^3 - 2n^2 - (4m-1)n + m^2 + 5m - d_n^2(G). \tag{3.5}$$

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Suppose $d_{n-1}(G) = n-4$. Then $d_n(G) = d_{n-1}(G) = n-4$ and by equations (3.3)–(3.5), we have

$$n_3 + 2n_2 + 3n_1 = 3n - 2m, \tag{3.6}$$

$$(n-3)n_3 + 2(n-2)n_2 + 3(n-1)n_1 = 3n^2 - 3n + m^2 - (2n+3)m. \tag{3.7}$$

Eliminating n_3 from (3.6) and (3.7), we have

$$n_2 = (1/2)m^2 - (9/2)m + 3n - 3n_1. \tag{3.8}$$

Substituting (3.8) in (3.6), we have

$$n_3 = -m^2 + 7m - 3n + 3n_1. \tag{3.9}$$

Since $m \geq 4$ and $n_1 \leq n-m$, from (3.9), we have $n_3 = 0$ and $n_1 = n-4$. These results with (3.3) and (3.4), we get $n_2 + n_4 = 3$ and $n_4 = m-3$. Therefore, by (3.8), it follows that $n_2 = 2$ and $n_4 = 1$. This is impossible because if $n_1 = n-4$, $n_4 = 1$ and $d_n(G) = n-4$, then we must have $n_2 = 0$. Hence, $d_{n-1}(G) \geq n-3$ and $n_4 = 0$ holds in equations (3.3)–(3.5). Now from (3.3) and (3.4), we have

$$n_2 + 2n_1 = 3n - 2m - d_n(G) - 3. \tag{3.10}$$

Eliminating n_3 from (3.3) and (3.5), we have

$$(2n-5)n_2 + 4(n-2)n_1 = 5n^2 + m^2 - 4mn + 5m - 14n + 9 - d_n^2(G). \tag{3.11}$$

Solving equations (3.10) and (3.11) for n_1, we have

$$n_1 = (1/2)m^2 - (1/2)n^2 + (n-5/2)d_n(G) - (1/2)d_n^2(G) - (5/2)m + (7/2)n - 3. \tag{3.12}$$

If $d_n(G) \geq n-2$, then by (3.12), we have $n_1 \geq n-3$, a contradiction, since $n_1 \leq n-m$ and $m \geq 4$. Thus, $1 \leq d_n(G) \leq n-3$.

Now, define

$$\phi(x) = (1/2)m^2 - (1/2)n^2 + nx - (1/2)x^2 - (5/2)m + (7/2)n - (5/2)x - 3.$$

Then

$$\phi'(x) = n - x - 5/2 > 0, 1 \leq x \leq n-3.$$

Hence, $\phi(x)$ is an increasing function on $1 \leq x \leq n-3$. If $n-m \leq d_n(G) \leq n-3$, then $\phi(n-m) = n-3 \leq n_1 \leq n-m$, a contradiction, since $m \geq 4$. Thus, $d_n(G) \leq n-m-1$. Since $|E(G)| = \dfrac{1}{2}(n-2)(n-1)+n-m-1$, we have

$$\begin{aligned}(n-2)(n-1) + 2(n-m-1) &= 2|E(G)| \\ &\leq d_n(G) + d_n(G)(n-1) + (n-d_n(G)-1)(n-2) \\ &= (n-1)(n-2) + 2d_n(G).\end{aligned} \tag{3.13}$$

If $d_n(G) \leq n-m-2$, then from (3.13), we get a contradiction. Thus, we have $d_n(G) = n-m-1$. Again since $|E(G)| = \dfrac{1}{2}(n-2)(n-1) + n-m-1$, we have $G \backslash \{v_n\} \cong K_{n-1}$. Hence, $G \cong K_n \backslash K_{1,m}$. ∎

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THEOREM 3.3. *Let G be a connected graph on n vertices with diameter 2 and having a distance eigenvalue -1 with multiplicity $n-3$. Then $G \cong K_n \backslash K_{l,m}$, where $l, m \geq 1$ and $l+m \leq n-1$.*

Proof. Since -1 is a distance eigenvalue of G with multiplicity $n-3$, it follows that the symmetric matrix $D(G)+I$ is of rank 3. Thus, we can assume that

$$D(G)+I = \begin{bmatrix} D_1 & X \\ X^T & D_2 \end{bmatrix},$$

where D_1 is a nonsingular matrix of order 3. Since the nullity of $D(G)+I$ is $n-3$ and D_1 is a nonsingular matrix of order 3, we have $D_2 = X^T D_1^{-1} X$. Thus, $x^T D_1^{-1} x = 1$ for each column x of X, since D_2 is a matrix with 1 as its diagonal entries. Now as $D_1 - I$ is a principal submatrix of $D(G)$, the distance matrix of G with diameter 2 and $rank(D_1) = 3$, we have

$$D_1 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{or} \quad D_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

In the first case,

$$D_1^{-1} = (1/5) \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

and it is easy to see that the only possible columns x of X satisfying $1 = x^T D_1^{-1} x$ are $[1,2,2]^T$, $[2,1,2]^T$ and $[2,2,1]^T$. In the second case,

$$D_1^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

and so $x^T D_1^{-1} x = 1$ implies that x is one of the three vectors $[1,1,1]^T$, $[1,1,2]^T$ and $[1,2,1]^T$. Thus, these two cases leads to the following two possibilities for $D(G)+I$:

$$D(G)+I = \begin{bmatrix} J_k & 2J & 2J \\ 2J & J_l & 2J \\ 2J & 2J & J_m \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} J_k & J & J \\ J & J_l & 2J \\ J & 2J & J_m \end{bmatrix}. \qquad \square$$

The first possibility is impossible because G is a connected graph. Thus, we must have

$$D(G)+I = \begin{bmatrix} J_k & J & J \\ J & J_l & 2J \\ J & 2J & J_m \end{bmatrix}, \quad \text{i.e.,} \quad G \cong K_n \backslash K_{l,m}.$$

THEOREM 3.4. *The graph $K_n \backslash K_{l,m}$ is DDS.*

Proof. Let G be a graph D-cospectral with $K_n \backslash K_{l,m}$. If the diameter of G is at least 3, then G has P_4, the path of length 3 as its induced subgraph and $D(P_4)$ as a principal submatrix of $D(G)$. Therefore, by Theorem 2.1 (i), $\eta_2(G) \geq -0.586$ and $\eta_3(P_4) \cong -1.162 \geq \eta_{n-1}(G)$, which is a contradiction to Lemma 2.4. Thus, the diameter of G is at most 2. Clearly $G \ncong K_n$ and so G is of diameter 2. Hence, by Theorem 3.3, $G \cong K_n \backslash K_{l_1,m_1}$. Since G and $K_n \backslash K_{l,m}$ have same Q-spectrum, from Lemma 2.4, we have $lm = l_1 m_1$ and $l+m = l_1 + m_1$. Solving these equations, we get $l = l_1$ and $m = m_1$. Thus, $G \cong K_n \backslash K_{l,m}$. $\square$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

4. Signless Laplacian spectral characterization of the join graph $\alpha K_2 \vee K_{n-2\alpha}$.

In this section, we prove that the multicone graph $\alpha K_2 \vee K_{n-2\alpha}$ is determined by its signless Laplacian spectrum which complement the works of Bu and Zhou in [3], and Xu and He in [27].

THEOREM 4.1. *Let n and α be positive integers with $n-2\alpha \geq 1$. Then the graph $\alpha K_2 \vee K_{n-2\alpha}$ is DQS.*

Proof. Let $G \cong \alpha K_2 \vee K_{n-2\alpha}$ $(n-2\alpha \geq 1)$. When $\alpha = 1$, G is a complete graph and so G is QDS. For $n-2\alpha = 1$, by Theorem 1.1, $G \cong \alpha K_2 \vee K_1$ is determined by its Q-spectrum. Otherwise, $\alpha \geq 2$ and $n - 2\alpha \geq 2$. Let H be a graph Q-cospectral with G. Then by Lemma 2.5, the Q-spectrum of H is

$$\left.\begin{aligned}
&\gamma_1(H), \gamma_n(H) = \left(3n - 4\alpha \pm \sqrt{n^2 + 8n(\alpha-1) - 16\alpha^2 + 16}\right)/2,\\
&\gamma_2(H) = \gamma_3(H) = \cdots = \gamma_{n-2\alpha}(H) = n-2,\\
&\gamma_{n-2\alpha+1}(H) = \gamma_{n-2\alpha+2}(H) = \cdots = \gamma_{n-\alpha-1}(H) = n-2\alpha+2,\\
&\gamma_{n-\alpha}(H) = \gamma_{n-\alpha+1}(H) = \cdots = \gamma_{n-1}(H) = n-2\alpha.
\end{aligned}\right\} \tag{4.14}$$

Let $\overline{H}$ be the complement graph of H. From Theorem 2.1 (ii) and (4.14), we have

$$n-2 \leq \gamma_{n-2\alpha}(H) + \gamma_{2\alpha+2}(\overline{H}) \leq \theta_2(Q(H) + Q(\overline{H})) = n-2,$$

$$n-2 = \theta_n(Q(H) + Q(\overline{H})) \leq \gamma_{n-2\alpha+1}(H) + \gamma_{2\alpha}(\overline{H}) = n - 2\alpha + 2 + \gamma_{2\alpha}(\overline{H}),$$

$$n - 2\alpha + 2 + \gamma_{\alpha+3}(\overline{H}) = \gamma_{n-\alpha-1}(H) + \gamma_{\alpha+3}(\overline{H}) \leq \theta_2(Q(H) + Q(\overline{H})) = n-2,$$

$$n-2 = \theta_n(Q(H) + Q(\overline{H})) \leq \gamma_{n-\alpha}(H) + \gamma_{\alpha+1}(\overline{H}) = n - 2\alpha + \gamma_{\alpha+1}(\overline{H})$$

and

$$n - 2\alpha + \gamma_3(\overline{H}) = \gamma_{n-1}(H) + \gamma_3(\overline{H}) \leq \theta_2(Q(H) + Q(\overline{H})) = n-2.$$

Thus,

$$\left.\begin{aligned}
&\gamma_3(\overline{H}) = \gamma_4(\overline{H}) = \cdots = \gamma_{\alpha+1}(\overline{H}) = 2\alpha - 2,\\
&\gamma_{\alpha+3}(\overline{H}) = \gamma_{\alpha+4}(\overline{H}) = \cdots = \gamma_{2\alpha}(\overline{H}) = 2\alpha - 4,\\
&\gamma_{2\alpha+2}(\overline{H}) = \gamma_{2\alpha+3}(\overline{H}) = \cdots = \gamma_n(\overline{H}) = 0.
\end{aligned}\right\} \tag{4.15}$$

Since G and H are Q-cospectral, it is easy to see that

$$\sum_{i=1}^{n} \gamma_i(\overline{H}) = 2|E(\overline{H})| = \sum_{i=1}^{n} d_i(\overline{H}) = \sum_{i=1}^{n} d_i(\overline{G}) = 4\alpha(\alpha-1), \tag{4.16}$$

$$\sum_{i=1}^{n} \gamma_i^2(\overline{H}) = \sum_{i=1}^{n} d_i(\overline{H})(d_i(\overline{H})+1) = \sum_{i=1}^{n} d_i(\overline{G})(d_i(\overline{G})+1) = 4\alpha(\alpha-1)(2\alpha-1). \tag{4.17}$$

If $\alpha = 2$, then from (4.16), $|E(\overline{H})| = 4$. Thus, we have the following possibilities for $\overline{H}$.

$$\overline{H} \cong \begin{cases}
H_1 \cup (n-5)K_1, \quad H_1 \text{ is a tree on 5 vertices},\\
H_1 \cup (n-4)K_1, \quad H_1 \text{ is a connected graph with 4 vertices and 4 edges},\\
K_3 \cup K_2 \cup (n-5)K_1 \text{ or } P_4 \cup K_2 \cup (n-6)K_1 \text{ or } S_{1,3} \cup K_2 \cup (n-6)K_1, \text{ or}\\
P_3 \cup P_3 \cup (n-6)K_1 \text{ or } P_3 \cup K_2 \cup K_2 \cup (n-7)K_1 \text{ or } 4K_2 \cup (n-8)K_1.
\end{cases}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

By Sage [23], using the Q-spectrum of connected graphs with at most 5 vertices, one can easily check that all these cases for $\overline{H}$, except $\overline{H} \cong K_{2,2} \cup (n-4)K_1$ contradicts either (4.15) or (4.17). Thus, $\overline{H} \cong K_{2,2} \cup (n-4)K_1$, and hence, the theorem is true for $1 \leq \alpha \leq 2$. Now let $\alpha \geq 3$. The following claim is important for our proof.

Claim: $\overline{H}$ has at least one connected component H_1 with $\gamma_2(H_1) \geq 2\alpha - 2$.

Proof of Claim: Let $\overline{H} = H_1 \cup H_2 \cup \cdots \cup H_k \cup r\,K_1$, where H_i $(1 \leq i \leq k)$ is i-th connected component of order n_i in $\overline{H}$ $(r = n - n_1 - n_2 - \cdots - n_k)$. Then $\gamma(\overline{H}) = \gamma(H_1) \cup \gamma(H_2) \cup \cdots \cup \gamma(H_k) \cup \{\underbrace{0, \ldots, 0}_{r}\}$, where $\gamma(H_i)$ denotes the signless Laplacian spectrum of H_i. Suppose to the contrary that $\gamma_2(H_i) < 2\alpha - 2$ for all i, $1 \leq i \leq k$. Since $\gamma_1(\overline{H}) \geq \gamma_2(\overline{H}) \geq \gamma_3(\overline{H}) = \gamma_4(\overline{H}) = \cdots = \gamma_{\alpha+1}(\overline{H}) = 2\alpha - 2$, there exists $\alpha + 1$ connected components $H_{i_1}, H_{i_2}, \ldots, H_{i_{\alpha+1}}$ such that

$$\gamma_1(\overline{H}) = \gamma_1(H_{i_1}), \quad \gamma_2(\overline{H}) = \gamma_1(H_{i_2}), \quad \ldots, \quad \gamma_{\alpha+1}(\overline{H}) = \gamma_1(H_{i_{\alpha+1}}).$$

Since $\gamma_2(H_{i_j}) < 2\alpha - 2$ $(1 \leq j \leq \alpha + 1)$ and $\overline{H}$ has at most α non-zero signless Laplacian eigenvalues which are strictly less than $2\alpha - 2$, there exist at least one connected component say H_{i_j} such that $\gamma_2(H_{i_j}) = 0$. Therefore, H_{i_j} is of order 2, since $\gamma_2(H_{i_j}) = 0$ and 0 can be a signless Laplacian eigenvalue of a connected graph with multiplicity at most 1. So $H_{i_j} \cong K_2$. This implies that $2 = \gamma_1(H_{i_j}) \geq 2\alpha - 2 > 2$ as $\alpha \geq 3$. This is a contradiction, and the claim is proven.

Let H_1 be a connected component of $\overline{H}$ with $\gamma_2(H_1) \geq 2\alpha - 2$, then from Theorem 2.2 (i), $2\alpha - 2 \leq \gamma_2(H_1) \leq |V(H_1)| - 2$, thus $|V(H_1)| \geq 2\alpha$. From (4.15) and Theorem 2.2 (v), it follows that the number of bipartite components of $\overline{H}$ is either $n - 2\alpha$ or $n - 2\alpha - 1$. First we assume that the number of bipartite components of $\overline{H}$ is exactly $n - 2\alpha$. Then we have $\gamma_{2\alpha+1}(\overline{H}) = 0$. Since $|V(H_1)| \geq 2\alpha$, then we have the following possibilities for $\overline{H}$.

$$\overline{H} \cong \begin{cases} H_1 \cup (n-2\alpha)K_1, H_1 \text{ is a non-bipartite graph of order } 2\alpha, \\ H_1 \cup (n-2\alpha-2)K_1 \cup K_2, H_1 \text{ is a bipartite graph of order } 2\alpha, \text{ or} \\ H_1 \cup (n-2\alpha-1)K_1, H_1 \text{ is a bipartite graph of order } 2\alpha+1. \end{cases}$$

Case I: $\overline{H} \cong H_1 \cup (n - 2\alpha)K_1$, H_1 is a non-bipartite graph of order 2α. Since $|V(H_1)| = 2\alpha$ and $\gamma_2(\overline{H}) \geq 2\alpha - 2$, by Theorem 2.2 (i), we have $\gamma_2(\overline{H}) = 2\alpha - 2$. From (4.15), (4.16) and (4.17), we get

$$\gamma_1(\overline{H}) + \gamma_{\alpha+2}(\overline{H}) = 6\alpha - 8 \quad \text{and} \quad \gamma_1^2(\overline{H}) + \gamma_{\alpha+2}^2(\overline{H}) = 20\alpha^2 - 48\alpha + 32.$$

Solving these equations, we obtain $\gamma_1(\overline{H}) = 4(\alpha - 1)$ and $\gamma_{\alpha+2}(\overline{H}) = 2\alpha - 4$. By Lemma 2.8, H_1 is a $(2\alpha - 2)$-regular graph on 2α vertices. Therefore, $\overline{H}_1 \cong \alpha\,K_2$, and hence, $H \cong G$.

Case II: $\overline{H} \cong H_1 \cup (n - 2\alpha - 2)K_1 \cup K_2$, H_1 is a bipartite graph of order 2α. Then we have $\gamma_{2\alpha}(\overline{H}) = 2\alpha - 4 \leq 2$, that is $\alpha \leq 3$, and hence, $\alpha = 3$. From (4.16), H_1 is a bipartite graph with 6 vertices and 11 edges. This is impossible, since the maximum size of a bipartite graph with 6 vertices is 9.

Case III: $\overline{H} \cong H_1 \cup (n - 2\alpha - 1)K_1$, H_1 is a bipartite graph of order $2\alpha + 1$. From (4.15), (4.16) and Theorem 2.2 (i), (iii) and (iv), we get $8\alpha - 10 = \gamma_1(\overline{H}) + \gamma_2(\overline{H}) + \gamma_{\alpha+2}(\overline{H}) \leq 6\alpha - 2$. Thus, $\alpha = 3$ or 4. Then from (4.16), we see that H_1 is a bipartite graph with 7 vertices and 12 edges, i.e., $H_1 \cong K_{3,\,4}$ $(\alpha = 3)$ or H_1 is a bipartite graph with 9 vertices and 24 edges $(\alpha = 4)$. The second case is clearly impossible, where as the first case contradicts (4.17).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

Next we assume that the number of bipartite components of $\overline{H}$ is exactly $n-2\alpha-1$. Then $\gamma_{2\alpha+1}(\overline{H}) > 0$. Since $|V(H_1)| \geq 2\alpha$, we have the following possibilities for $\overline{H}$.

$$\overline{H} \cong \begin{cases} H_1 \cup (n-2\alpha-2)K_1, & H_1 \text{ is a bipartite graph of order } 2\alpha+2, \\ H_1 \cup (n-2\alpha-3)K_1 \cup P_3, & H_1 \text{ is a bipartite graph of order } 2\alpha, \\ H_1 \cup (n-2\alpha-2)K_1 \cup K_2, & H_1 \text{ is non bipartite graph of order } 2\alpha, \\ H_1 \cup (n-2\alpha-4)K_1 \cup 2K_2, & H_1 \text{ is a bipartite graph of order } 2\alpha, \\ H_1 \cup (n-2\alpha-1)K_1, & H_1 \text{ is a non bipartite graph of order } 2\alpha+1, \text{ or} \\ H_1 \cup (n-2\alpha-3)K_1 \cup K_2, & H_1 \text{ is a bipartite graph of order } 2\alpha+1. \end{cases}$$

Case I: $\overline{H} \cong H_1 \cup (n-2\alpha-2)K_1$, H_1 is a bipartite graph of order $2\alpha+2$. From (4.14), we have $\gamma_{n-2\alpha}(H) = n-2$. Therefore, by Theorem 2.2 (i), $\overline{H}$ has either $n-2\alpha-1$ balanced bipartite components or $n-2\alpha$ bipartite components. Since the number of bipartite components of $\overline{H}$ is exactly $n-2\alpha-1$, all $n-2\alpha-1$ bipartite components of $\overline{H}$ are balanced. Since $\overline{H} \cong H_1 \cup (n-2\alpha-2)K_1$ and an isolated vertex K_1 is not balanced, it follows that $n-2\alpha-2=0$ and H_1 is a balanced bipartite component. Therefore, by (4.16), $2|E(\overline{H})| = 4\alpha(\alpha-1) \leq 2(\alpha+1)^2$. This implies that $\alpha = 3$ or 4. If $\alpha = 3$, then H_1 is balanced bipartite graph with 8 vertices and 12 edges, and so H_1 is a graph obtained from $K_{4,4}$ by deleting 4 edges. In this case, by Sage [23], it can be seen that no such graphs satisfies (4.17). If $\alpha = 4$, then H_1 is a balanced bipartite graph with 10 vertices and 24 edges, and so $H_1 \cong K_{5,5}\backslash\{e\}$, where e is an edge in $K_{5,5}$. By Sage [23], this is not possible by (4.17).

Case II: $\overline{H} \cong H_1 \cup (n-2\alpha-3)K_1 \cup P_3$, H_1 is a bipartite graph of order 2α. From (4.14), we have $\gamma_{n-2\alpha}(H) = n-2$. Therefore, by Theorem 2.2 (i), $\overline{H}$ has either $n-2\alpha-1$ balanced bipartite components or $n-2\alpha$ bipartite components. Thus, $\overline{H}$ has $n-2\alpha-1$ balanced bipartite components as the number of bipartite components of $\overline{H}$ is exactly $n-2\alpha-1$. Since P_3 is not balanced, we have a contradiction.

Case III: $\overline{H} \cong H_1 \cup (n-2\alpha-2)K_1 \cup K_2$, H_1 is non bipartite graph of order 2α. Since $|H_1| = 2\alpha$, by Theorem 2.2 (i), we have $\gamma_2(\overline{H}) \leq 2\alpha-2$. From (4.15), we get $\gamma_2(\overline{H}) \geq 2\alpha-2$, and hence, $\gamma_2(\overline{H}) = 2\alpha-2$. Since $\overline{H} = H_1 \cup (n-2\alpha-2)K_1 \cup K_2$, we have that $\overline{H}$ has 2 as its signless Laplacian eigenvalue. Since $\alpha \geq 3$, from (4.15), we have $\gamma_{2\alpha}(\overline{H}) = 2\alpha-4 \geq 2$. From (4.15), we conclude that $\gamma_{2\alpha}(\overline{H}) = 2$ or $\gamma_{2\alpha+1}(\overline{H}) = 2$. First we assume that $\gamma_{2\alpha+1}(\overline{H}) = 2$. Then by (4.15), (4.16) and (4.17), we get

$$\gamma_1(\overline{H}) + \gamma_{\alpha+2}(\overline{H}) = 6\alpha - 10 \quad \text{and} \quad \gamma_1^2(\overline{H}) + \gamma_{\alpha+2}^2(\overline{H}) = 20\alpha^2 - 48\alpha + 28.$$

Solving these equations, we have $\gamma_{\alpha+2}(\overline{H}) = 3\alpha - 5 - \sqrt{\alpha^2+6\alpha-11} < 2\alpha-4$, a contradiction. Next we assume that $\gamma_{2\alpha}(\overline{H}) = 2\alpha-4 = 2$, that is, $\alpha = 3$. Then H_1 is a non bipartite graph with 6 vertices and 11 edges. Using the Q-spectrum of connected graphs with 6 vertices and 11 edges, by Sage [23], one can easily see that all the choices for H_1 contradicts (4.15).

Case IV: $\overline{H} \cong H_1 \cup (n-2\alpha-4)K_1 \cup 2K_2$, H_1 is a bipartite graph of order 2α. Since $\alpha \geq 3$, from (4.15), we have $\gamma_{2\alpha}(\overline{H}) = 2\alpha-4 \geq 2$. Since $\overline{H} \cong H_1 \cup (n-2\alpha-4)K_1 \cup 2K_2$, then $\overline{H}$ has 2 as its signless Laplacian eigenvalue with multiplicity 2 and so $\gamma_{2\alpha}(\overline{H}) = 2$, that is, $2\alpha-4=2$, that is, $\alpha = 3$. From (4.16), H_1 is a bipartite graph with 6 vertices and 10 edges. This is not possible, since there exists no bipartite graph with 6 vertices and 10 edges.

Case V: $\overline{H} \cong H_1 \cup (n-2\alpha-1)K_1$, H_1 is a non bipartite graph of order $2\alpha+1$. From (4.14), we have $\gamma_{n-2\alpha}(H) = n-2$. Therefore, by Theorem 2.2 (i), $\overline{H}$ has either $n-2\alpha-1$ balanced bipartite components

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 461-472, July 2020.

or $n-2\alpha$ bipartite components. Thus, $\overline{H}$ has $n-2\alpha-1$ balanced bipartite components as the number of bipartite components of $\overline{H}$ is exactly $n-2\alpha-1$. Since an isolated vertex K_1 is not balanced, in this case, we must have $n-2\alpha=1$, a contradiction as $n-2\alpha\geq 2$.

Case VI: $\overline{H}\cong H_1\cup(n-2\alpha-3)K_1\cup K_2$, H_1 is a bipartite graph of order $2\alpha+1$. From (4.14), we have $\gamma_{n-2\alpha}(H)=n-2$. Therefore, by Theorem 2.2 (i), $\overline{H}$ has either $n-2\alpha-1$ balanced bipartite components or $n-2\alpha$ bipartite components. Since $\overline{H}=H_1\cup(n-2\alpha-3)K_1\cup K_2$ and an isolated vertex K_1 is not balanced, it follows that $n-2\alpha=3$ and H_1 is a balanced bipartite component of order $2\alpha+1$. This case is not possible because a balanced bipartite graph has even number of vertices. ∎

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