ON THE EXPONENT OF $R$-REGULAR PRIMITIVE MATRICES

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Abstract. Let $P_{nr}$ be the set of $n$-by-$n$ $r$-regular primitive $(0,1)$-matrices. In this paper, an explicit formula is found in terms of $n$ and $r$ for the minimum exponent achieved by matrices in $P_{nr}$. Moreover, matrices achieving that exponent are given in this paper. Gregory and Shen conjectured that $b_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1$ is an upper bound for the exponent of matrices in $P_{nr}$. Matrices achieving the exponent $b_{nr}$ are presented for the case when $n$ is not a multiple of $r$. In particular, it is shown that $b_{2r+1,r}$ is the maximum exponent attained by matrices in $P_{2r+1,r}$. When $n$ is a multiple of $r$, it is conjectured that the maximum exponent achieved by matrices in $P_{nr}$ is strictly smaller than $b_{nr}$. Matrices attaining the conjectured maximum exponent in that set are presented. It is shown that the conjecture is true when $n = 2r$.

Key words. $r$-Regular matrices, Primitive matrices, Exponent of primitive matrices.

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1. Introduction. A nonnegative square matrix $A$ is called primitive if there exists a positive integer $k$ such that $A^k$ is positive. The smallest such $k$ is called the exponent of $A$. We denote the exponent of a primitive matrix $A$ by $\text{exp}(A)$.

A $(0,1)$-matrix $A$ is said to be $r$-regular if every column sum and every row sum is constantly $r$.

Consider the set $P_{nr}$ of all primitive $r$-regular $(0,1)$-matrices of order $n$, where $2 \leq r \leq n$. Notice that, for $n > 1$, $n$-by-$n$ 1-regular matrices are permutation matrices, which are not primitive. An interesting problem is to find the following two positive integers:

$$l_{nr} = \min\{\text{exp}(A) : A \in P_{nr}\}, \quad u_{nr} = \max\{\text{exp}(A) : A \in P_{nr}\},$$

as well as finding matrices attaining those exponents. In this paper, we call the integers $l_{nr}$ and $u_{nr}$ the optimal lower bound and the optimal upper bound for the exponent of matrices in $P_{nr}$, respectively.
In the literature, numerous papers can be found about good upper bounds for the exponent of general primitive matrices $A$ of order $n$. In [8] Wielandt stated, without proof, that

$$\exp(A) \leq (n - 1)^2 + 1.$$  

Recently, the proof was found in Wielandt’s unpublished diaries and published in [5]. There are many improvements of Wielandt’s bound for special classes of primitive matrices. The problem of finding an upper bound for the exponent of matrices in $P_{nr}$ has been considered by several authors in Discrete Mathematics, in particular, by some researchers in Graph Theory [2, 4, 6, 7]. In the literature, several such bounds can be found. In [4], it is shown that $\exp(A) \leq \frac{2n(3n-2)}{(r+1)} - \frac{n+2}{r+1}$. In [7], it is shown that, if $A \in P_{nr}$, then $\exp(A) \leq 3n^2/r^2$. Also, it is conjectured there that, if $A \in P_{nr}$, then $\exp(A) \leq \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$, where $\lfloor \cdot \rfloor$ denotes the floor function, that rounds a number to the next smaller integer. J. Shen proved that this conjecture is true when $r = 2$ [6], however it remains open for $r > 2$.

In this paper, we give an explicit expression for $l_{nr}$ in terms of $n$ and $r$, and construct matrices attaining that exponent. We also construct matrices whose exponent is $\left\lfloor \frac{n}{r} \right\rfloor^2 + 1$ when $n = gr + c$, with $0 < c < r$, which proves that $u_{nr} \geq \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$ in those cases. Moreover, we prove that $u_{nr} = \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$ when $g = 2$ and $c = 1$. When $n = gr$, with $g = 2$, we determine $u_{nr}$; when $g \geq 3$, we give a conjecture for the value of $u_{nr}$ and present matrices achieving the conjectured optimal upper bound exponent. According to this conjecture, $u_{nr}$ would be smaller than $\left\lfloor \frac{n}{r} \right\rfloor^2 + 1$.

2. Notation and Auxiliary Results. In the sequel we will use the following notation: If $A$ is an $n$-by-$m$ matrix, we denote by $A(i, j)$ the entry of $A$ in the position $(i, j)$. By $A(i_1 : i_2, j_1 : j_2)$, with $i_2 \geq i_1$ and $j_2 \geq j_1$, we denote the submatrix of $A$ lying in rows $i_1, i_1 + 1, \ldots, i_2$ and columns $j_1, j_1 + 1 \ldots, j_2$. We abbreviate $A(i_1 : i_1, j_1 : j_2)$ to $A(i_1, j_1 : j_2)$ and $A(1 : n, j_1 : j_2)$ to $A(:, j_1 : j_2)$. Similar abbreviations are used for the columns of $A$. The $m$-by-$n$ matrix whose entries are all equal to one is denoted by $J_{mn}$. Unspecified entries in matrices are represented by a $\ast$.

Some of the proofs in this paper involve the concept of digraph associated with a $(0,1)$- matrix.

**Definition 2.1.** Let $A$ be a $(0,1)$-matrix of size $n$-by-$n$. The digraph $G(A)$ associated with $A$ is the directed graph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E$ where $(i, j) \in E$ if and only if $A(i, j) = 1$.

Notice from the previous definition that $A$ is the adjacency matrix of $G(A)$.

A digraph $G$ is said to be $r$-regular if and only if its adjacency matrix is an...
$r$-regular matrix. Note that the outdegree and the indegree of each vertex of an $r$-regular digraph are exactly $r$. A digraph is said to be primitive if and only if its adjacency matrix is primitive. Clearly, for $A \in P_{nr}$, $\exp(A) = k$ if and only if any two vertices in $G(A)$ are connected by a walk of length $k$ and, if $k > 1$, there are at least two vertices that are not connected by a walk of length $k - 1$.

It is important to notice that if $A$ is an $r$-regular primitive matrix and $B = P^T A P$ for some permutation matrix $P$, then, for any positive integer $k$, $B^k = P^T A^k P$. Thus, $\exp(A) = \exp(B)$. Also $G(A)$ and $G(B)$ are isomorphic digraphs. Therefore, throughout the paper, we will work on the set of equivalence classes under permutation similarity. Notice also that $A \in P_{nr}$ if and only $A^t \in P_{nr}$.

Next we include some simple observations about $r$-regular primitive matrices that will be useful to prove some of the main results in the paper.

**Lemma 2.2.** Let $A \in P_{n,r}$ and let $k$ be any positive integer. Then, every row of $A^k$ contains at most $r^k$ nonzero entries.

**Proof.** We prove the result by induction on $k$. Let $A \in P_{n,r}$. Then, every row of $A$ contains $r$ nonzero entries since $A$ is $r$-regular. Therefore, the result is true for $k = 1$.

Assume that every row of $A^{k-1}$ contains at most $r^{k-1}$ nonzero entries. Then, any $r \times n$ submatrix of $A^{k-1}$ has at most $r^k$ nonzero columns. Because $A^k = AA^{k-1}$, the result follows.

**Lemma 2.3.** Let $A \in P_{nr}$ and let $k > 1$ be a positive integer. If $A^k(i,j) = 0$, then there are at least $r$ zero entries in the $i$-th row of $A^{k-1}$; also there are at least $r$ zero entries in the $j$-th column of $A^{k-1}$.

**Proof.** Notice that $A^k(i,j) = A^{k-1}(i,:)A(:,j) = 0$. Since $A$ is $r$-regular, $r$ entries of $A(:,j)$ are ones. Taking into account that $A^{k-1}(i,:) \geq 0$, the first result follows. The second claim can be proven in a similar way taking into account that $A^k(i,j) = A(i,:)A^{k-1}(:,j) = 0$.

**Lemma 2.4.** Let $A \in P_{nr}$ and $i \in \{1, \ldots, n\}$. Then, the number of nonzero entries in the $i$-th row (column) of $A^k$, $k \geq 1$, is a nondecreasing sequence in $k$.

**Proof.** Suppose that in the $i$-th row of $A^k$ there are exactly $s$ nonzero entries. We want to show that in the $i$-th row of $A^{k+1}$ there are at least $s$ nonzero entries. Denote by $S$ the set $\{j \in \{1, \ldots, n\} : A^k(i,j) \neq 0\}$. Since the outdegree of each node of $G(A)$ is exactly $r$, there are $rs$ arcs with origin in the vertices in $S$. Since the indegree of each node of $G$ is exactly $r$, then the $rs$ arcs with origin in $S$ have their terminus in at least $rs/r = s$ vertices. Thus, with origin in the $i$-th node of $G(A)$, there are walks of length $k + 1$ to at least $s$ distinct vertices. The result for columns follows taking
into account that $A^t \in P_{nr}$. 

Note that the last lemma implies that each row (column) of $A^k$ has at least $r$ nonzero entries.

If $i \in \{1, \ldots, n\}$ is such that $A(i, i) = 1$, then Lemma 2.4 may be refined. We consider this situation in the next lemma, as it will allow us to get an interesting corollary. We assume that $n \geq 2r$ since, by Lemma 2.3, if $n < 2r$, $A^2(i, :) = 0$ is positive.

**Lemma 2.5.** Let $A \in P_{nr}$, with $n \geq 2r$, and $i \in \{1, \ldots, n\}$. Suppose that $A(i, i) = 1$. Let $s_k$ be the number of nonzero entries in $A^k(i, :)$, $k \geq 1$. If $s_k < n$, then the number of nonzero entries in the $i$-th row of $A^{k+1}$ is at least $s_k + 1$. In particular, the $i$-th row of $A^{n-2r+3}$ is positive.

**Proof.** By a possible permutation similarity of $A$, we assume that $i = 1$ and $A(1, :) = [J_{1r} \ 0]$. Let $k \in \{2, \ldots, n\}$. Clearly, the first $r$ entries of $A^k(1, :) = a_{s_k} \ 0$ are nonzero. If $k = 2$, since $A$ is not reducible, $A^2(1, :)$ has more than $r$ nonzero entries. Now suppose that $k > 2$ and $s_k < n$. With a possible additional permutation similarity, we assume, without loss of generality, that $A^k(1, :) = [a_1 \cdots \ a_{s_k} \ 0]$, where $a_i > 0$, $i = 1, \ldots, k$. We show that $s_{k+1} \geq s_k + 1$. Suppose that $A^{k-1}(1, :) = [b_1 \cdots \ b_n]$, where $b_1, b_2, \ldots, b_r, b_{i_1}, \ldots, b_{i_{s_k-1}-r}$ are positive integers, with $r \leq i_1 < \cdots < i_{s_k-1-r} \leq n$. Because $A^{k} = AA^{k-1}$, then $i_{s_k-1-r} \leq s_k$; also, as $A^k = A^{k-1}A$ then

$$A = \begin{bmatrix} J_{1r} & 0 & 0 \\ \ast & R_{11} & 0 \\ \ast & R_{21} & R_{22} \\ \ast & R_{31} & R_{32} \end{bmatrix},$$

for some blocks $R_{ij}$, where $R_{11}$ and $R_{22}$ are $(r-1)$-by-$(s_k-r)$ and $(s_k-r)$-by-$(n-s_k)$ matrices, respectively. Since all the entries of

$$[b_2 \cdots \ b_n] \begin{bmatrix} R_{11}^t & R_{21}^t & R_{31}^t \end{bmatrix}^t$$

are nonzero, then also all the entries of

$$[a_2 \cdots \ a_{s_k} \ 0] \begin{bmatrix} R_{11}^t & R_{21}^t & R_{31}^t \end{bmatrix}^t$$

are nonzero, which implies that $A^{k+1}(1, i) \neq 0$ for $i = 1, \ldots, s_k$. Since $A$ is not reducible, it also follows that $R_{22}$ is nonzero. Therefore, $A^{k+1}(1, :)$ has at least $s_k + 1$ nonzero entries. Clearly, $A^{n-2r+2}(1, :)$ has at most $r - 1$ zero entries, which implies, by Lemma 2.3, that $A^{n-2r+3}(1, :)$ is positive. 

The next result is a simple consequence of Lemma 2.5. It gives an upper bound for the exponent of matrices in $P_{nr}$ with nonzero trace. Another such upper bound
can be found in [4]: if $A \in P_{nr}$ has $p$ nonzero diagonal entries, then $\exp(A) \leq \max\{2(n - r + 1) - p, n - r + 1\}$. It is easy to check that there are values of $n$ and $r$ for which the upper bound given in Corollary 2.6 for the exponent of matrices with nonzero trace is smaller than those in [4] and [7]. Check with $n = 30$ and $r = 15$, for instance.

**Corollary 2.6.** Let $A \in P_{nr}$, with $n \geq 2r$, and suppose that $\text{trace}(A) \neq 0$. Then, $\exp(A) \leq 2n - 4r + 6$.

**Proof.** Let $i \in \{1, \ldots, n\}$ be such that $A(i, i) \neq 0$. According to Lemma 2.5, the $i$-th row and the $i$-th column of $A^{n-2r+3}$ have no zero entries. Therefore, from any vertex in $G(A)$ there is a walk of length $n - 2r + 3$ to vertex $i$; also, there is a walk of length $n - 2r + 3$ from vertex $i$ to any vertex. Thus, any two vertices are connected by a walk of length $2n - 4r + 6$. Ο

Finally, we include the following technical lemma.

**Lemma 2.7.** Let $D_{rk}$, $k < r$, denote an $r$-by-$k$ matrix with exactly $r - 1$ nonzero entries in each column. Then, at least one row of $D_{rk}$ has no zero entries. Moreover, if $k < r - 1$, then at least two rows of $D_{rk}$ have no zero entries.

**Proof.** Notice that the number $t$ of nonzero entries in $D_{rk}$ is $r(k - 1)$ since every column contains $r - 1$ nonzero entries. Assume that all rows of $D_{rk}$ have at least one zero entry. Then, the number $m$ of zero entries in $D_{rk}$ would be at least $r$. This implies that

$$t = rk - m \leq rk - r < k(r - 1),$$

which is a contradiction. The second claim can be proven in a similar way. Ο

### 3. Optimal lower bound

In this section, we determine the optimal lower bound $l_{nr}$ for the exponent of matrices in $P_{nr}$ in terms of $n$ and $r$. We also present matrices achieving this exponent.

**Lemma 3.1.** Let $A \in P_{nr}$. Then,

$$\exp(A) \geq \lceil \log_r(n) \rceil.$$

**Proof.** Taking into account Lemma 2.2, each row of $A$ has at most $r^k$ nonzero entries. Since $r^k \geq n$ if and only if $k \geq \log_r(n)$, the result follows. Ο

Next we prove that there exist matrices in $P_{nr}$ whose exponent is $\lceil \log_r(n) \rceil$.

**Definition 3.2.** Let $B = [b_{ij}]$ be an $m$-by-$n$ real (complex) matrix. We call the
indicator matrix of $B$, which we denote by $M(B)$, the $m$-by-$n$ $(0,1)$-matrix $[\mu_{ij}]$, with $\mu_{ij} = 1$ if $b_{ij} \neq 0$ and $\mu_{ij} = 0$ if $b_{ij} = 0$.

**Definition 3.3.** Let $v = (v_1, v_2, \ldots, v_n)$ be a row vector in $\mathbb{R}^n$. Let $s$ be an integer such that $0 < s \leq n$. Define the $s$-shift operator $f_s : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_s(v_1, v_2, \ldots, v_n) = (v_{n-s+1}, v_{n-s+2}, \ldots, v_n, v_1, v_2, \ldots, v_{n-s}).$$

The $s$-generalized circulant matrix associated with $v$ is the $n$-by-$n$ matrix whose $k$-th row is given by $f_s^{k-1}(v)$, for $k = 1, \ldots, n$, where $f_s^{k-1}$ denotes the composition of $f_s$ with itself $k-1$ times.

Note that $f_s^n(v_1, \ldots, v_n) = (v_1, \ldots, v_n)$, as the position of $v_1$ after $n$ $s$-shifts is $ns+1$ modulo $n$, that is, 1.

Let $0 < s \leq r$ be an integer. We denote by $T_{nr}$ the $s$-generalized circulant matrix associated with $u_r = \sum_{i=1}^r e_i$, where $e_i$ denotes the $i$-th column of the $n$-by-$n$ identity matrix. For instance,

$$T_{152}^5 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

**Lemma 3.4.** For $r \geq 2$, the matrix $T_{nr}$ is $r$-regular and primitive. Moreover, $\exp(T_{nr}) = [\log_r(n)]$.

**Proof.** First we prove that $T_{nr}$ is an $r$-regular matrix. By construction, it is easy to see that the row sum is constantly $r$. In order to determine the column sum note that there are exactly $nr$ entries equal to one in $T_{nr}$. We denote by $s_i$, $i \geq 1$, the remainder of the division of $i$ by $n$, if $i$ is not a multiple of $n$, and $s_i = n$ otherwise. By construction again, the ones in the $i$-th row occur in positions $s_{(i-1)r+1}, \ldots, s_{ir}$. The sequence of columns in which the ones occur, starting in the first row, then the second row and so on, is just the sequence $s_1, s_2, s_3, \ldots, s_{nr}$, that is, $1, \ldots, n, 1, \ldots, n, \ldots, 1, \ldots, n$. Clearly, each $j \in \{1, 2, \ldots, n\}$ appears exactly $r$ times in that sequence.

Now we prove that $T_{nr}$ is primitive by computing its exponent. We first show, by induction on $k$, that the first $\min\{n, r^k\}$ entries of the first row of $(T_{nr})^k$ are nonzero and, if $r^k < n$, the last $n - r^k$ entries of the first row of $(T_{nr})^k$ are zero. If $k = 1$, this claim is trivially true. Now suppose that the claim is valid for $k = p$. Note that, for each integer $1 \leq k \leq n$, all the columns of the submatrix of $T_{nr}$ indexed by the first $r^k$ rows and the first $\min\{n, r^{k+1}\}$ columns are nonzero. Also, if $r^{k+1} < n$, the
submatrix of $T_{nr}^{p}$ indexed by the first $r^{k}$ rows and the last $n - r^{k+1}$ columns is 0. Taking into account this observation, it follows that the first $\min\{n, r^{p+1}\}$ entries of $(T_{nr}^{p+1})^{r}(1:) = (T_{n}^{p})^{r}(1:)T_{nr}^{r}$ are nonzero while the last $n - \min\{n, r^{p+1}\}$ are zero.

Using similar arguments, we can show that, in general, the $i$-th row of $M((T_{nr}^{r})^{k})$ is $f_{r}^{(i-1)r^{k-1}}(u_{k})$, where $u_{k} = \sum_{j=1}^{\min\{r^{k}, n\}} e_{j}$.

Therefore, any row of $(T_{nr}^{r})^{k}$ has exactly $\min\{r^{k}, n\}$ nonzero entries. Thus, $(T_{nr}^{r})^{k}$ is positive if and only if $r^{k} \geq n$, which implies the result. \[ \square \]

**Theorem 3.5.** Suppose that $2 \leq r \leq n$. Then, $l_{rn} = \lceil \log_{r}(n) \rceil$.

**Proof.** Follows from Lemma 3.1 and Lemma 3.4. \[ \square \]

### 4. Optimal upper bound.

Although stated in terms of graphs, the following conjecture is given in [7]: If $A \in P_{nr}$, then $\exp(A) \leq \left\lfloor \frac{n}{r} \right\rfloor^{2} + 1$. In [6] this conjecture was proven for $r = 2$. Notice that this conjecture is trivially true for $r \geq \frac{n}{2} + 1$. Hence, in the sequel we assume that $n \geq 2r$.

Given any $q \geq 2$, an $r$-regular primitive digraph with $n = qr + 1$ vertices whose exponent is $\left\lfloor \frac{n}{r} \right\rfloor^{2} + 1$ can be found in [7]. A matrix with such a graph is the following:

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & J_{rr} \\
J_{rr} & 0 & \cdots & 0 & 0 & 0 \\
0 & J_{rr} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & J_{rr} & 0 & 0 \\
0 & 0 & \cdots & T_{1}^{r-1} & J_{r1} & 0
\end{bmatrix}
\tag{4.1}
\]

In the next two subsections we generalize the structure of the matrix $A$ by defining the matrices $E_{nr}$ for all possible combinations of $n$ and $r$.

#### 4.1. The case in which $n$ is not a multiple of $r$.

Generalizing the structure of the matrix in (4.1), in this section we define the $n$-by-$n$ matrices $E_{nr}$, when $n = gr + c$ for some positive integers $g \geq 2$ and $0 < c < r$, as follows:
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$E_{nr} = \begin{bmatrix}
  0 & 0 & J_{rr} \\
  J_{cr} & 0 & 0 \\
  T_{1}^{r} & J_{rc} & 0
\end{bmatrix}$, if $n = 2r + c$, \hfill (4.2)

$E_{nr} = \begin{bmatrix}
  0 & 0 & \cdots & 0 & 0 & J_{rr} \\
  J_{rr} & 0 & \cdots & 0 & 0 & 0 \\
  0 & J_{rr} & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & 0 & J_{rr} & 0 \\
  0 & 0 & \cdots & 0 & T_{1}^{r} & J_{rc} & 0
\end{bmatrix}$, \hfill (4.3)

if $n = gr + c$, with $g \geq 3$. \hfill (4.4)

Note that we can replace $T_{1}^{r} - c$ by any matrix in $P_{r,c}$ without changing the exponent of $E_{nr}$.

Next we show that $\exp(E_{nr}) = \lfloor \frac{n}{r} \rfloor^2 + 1$, which implies that $u_{nr} \geq \lfloor \frac{n}{r} \rfloor^2 + 1$.

**Lemma 4.1.** If $n = 2r + c$, where $0 < c < r$, then $\exp(E_{nr}) = \lfloor \frac{n}{r} \rfloor^2 + 1 = 5$.

**Proof.** It is easy to check that

$M(E_{nr}^2) = \begin{bmatrix}
  J_{rr} & J_{rc} & 0 \\
  0 & 0 & J_{cr} \\
  J_{rr} & 0 & J_{rr}
\end{bmatrix}$, $M(E_{nr}^3) = \begin{bmatrix}
  J_{rr} & 0 & J_{rr} \\
  J_{cr} & J_{cc} & 0 \\
  J_{rr} & J_{rc} & J_{rr}
\end{bmatrix}$,

$M(E_{nr}^4) = \begin{bmatrix}
  J_{rr} & J_{rc} & J_{rr} \\
  J_{cr} & 0 & J_{cr} \\
  J_{rr} & J_{rc} & J_{rr}
\end{bmatrix}$.

Finally, we get that $M(E_{nr}^5) = J_{nn}$, which implies the result. \hfill \square

**Lemma 4.2.** If $n = gr + c$, with $g \geq 3$ and $0 < c < r$, then $\exp(E_{nr}) = \lfloor \frac{n}{r} \rfloor^2 + 1 = g^2 + 1$.

**Proof.** Consider the digraph $G$ associated with $E_{nr}$. Let us group the vertices of $G$ in the following way: We call $B_1$ the set of vertices from $(g - 1)r + c + 1$ to $gr + c$; we call $B_2$ the set of vertices from $(g - 1)r + 1$ to $(g - 1)r + c$; we call $B_i$, $i = 3, \ldots, g + 1$, the set of vertices from $(g - i + 1)r + 1$ to $(g - i + 2)r$.

Suppose that $u$ and $v$ are two vertices in the same block $B_i$. Then there is a path from $u$ to $v$ of length $g$ and another one of length $g + 1$, except if $u, v \in B_2$,
in which case there is just a path of length $g + 1$. Therefore, a walk from $u$ to $v$ has length $t$ if and only if $t = \alpha g + \beta (g + 1)$, for some nonnegative integers $\alpha, \beta$, with $\beta > 0$ if $u, v \in B_2$. In particular, no vertex in $B_2$ lies on a closed walk of length $g^2$ since $\alpha g + \beta (g + 1) = g^2$ implies $\beta = 0$. Thus, $\exp(E_{nr}) > g^2$.

Because

$$g^2 + 1 = (g - 1)g + (g + 1),$$

it follows that there is a walk of length $g^2 + 1$ from any vertex to any other in the same block $B_i$, $i = 1, \ldots, g + 1$.

Now consider a vertex $u$ in $B_i$ and a vertex $v$ in $B_j$, where $i, j \in \{1, \ldots, g + 1\}$ and $i \neq j$. Let $s$ be the distance from $u$ to $v$. Note that $s \leq g$. We will show that there is a walk of length $g^2 + 1$ from $u$ to $v$. Suppose that $s > 1$. In this case we have

$$g^2 - s + 1 = (s - 2)g + (g - s + 1)(g + 1).$$

Thus, $u$ lies on a closed walk of length $g^2 - s + 1$, which implies that there is a walk of length $g^2 + 1$ from $u$ to $v$.

Now suppose that $s = 1$. If $u \notin B_2$, $u$ lies on a closed walk of length $g^2$, which implies that there is a walk of length $g^2 + 1$ from $u$ to $v$. If $u \in B_2$, then $v \in B_3$ and $v$ lies on a closed walk of length $g^2$, which implies that there is a walk of length $g^2 + 1$ from $u$ to $v$.

We have shown that the vertices in $B_2$ do not lie on any closed walk of length $g^2$. On the other hand, between any two vertices there is a walk of length $g^2 + 1$. Thus $E_{nr}^{g^2}$ is not positive, while $E_{nr}^{g^2+1}$ is positive. Therefore, $\exp(E_{nr}^{g^2+1}) = g^2 + 1$. \qed

The following theorem follows in a straightforward way from Lemmas 4.1 and 4.2.

**Theorem 4.3.** If $n = gr + c$, with $0 < c < r$, then $u_{nr} \geq \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$.

We now show that, when $n = 2r + 1$, $u_{nr} = \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$.

**Theorem 4.4.** Let $n = 2r + 1$. Then, $u_{nr} = \left\lfloor \frac{n}{r} \right\rfloor^2 + 1 = 5$.

**Proof.** Clearly, by Theorem 4.3, $u_{nr} \geq 5$. We now show that if $A \in P_{nr}$ and $\exp(A) > 4$, then $\exp(A) = 5$, which means that there are no matrices in $P_{nr}$ with exponent greater than 5, and, therefore, $u_{nr} = 5$. The strategy we follow allows us to characterize, up to a permutation similarity, all the matrices in $P_{nr}$ that achieve exponent 5.

Suppose that $\exp(A) \geq 5$. Then, there is a zero entry in $A^4$. Without loss of generality, we can assume that $A^4(1,i) = 0$ for some $i \in \{1, \ldots, n\}$. Applying Lemma 2.3 repeatedly, we deduce that there are at least $r$ zero entries in the first row of $A^3$ and $A^2$. 


By a convenient permutation similarity on $A$, we can reduce the proof to the next two cases (and subcases). Throughout the proof, we denote by $D_{rk}$ an $r$-by-$k$ matrix with exactly $r-1$ nonzero entries in each column and by $C_{rr}$ a matrix in $P_{r,r-1}$.

**Case 1.** Let us assume that $A(1,:) = [J_{1r} \ 0]$. Then, $A^2(1,i) \neq 0$ for $i = 1, ..., r$ and we can assume that $A^2(1,r+2:n) = 0$. Therefore,

$$A = \begin{bmatrix} J_{1r} & 0_{1r} & \ast & R_1 & 0_{r-1,r} & \ast & \ast & D_{r+1,r} \end{bmatrix},$$

for some $(r-1)$-by-1 block $R_1$. If $R_1$ is zero, clearly $A$ is reducible, which is a contradiction. If $R_1$ is nonzero, then $M(A^2)(1,:) = [J_{1,r+1} \ 0_{1,r}]$ and $A^3(1,i) = A^2(1,:)A(:,i) \neq 0$ for $i = 1, ..., r+1$. Since $A^3(1,:)$ contains at least $r$ zero entries then $M(A^3)(1,:) = [J_{1,r+1} \ 0_{1,r}]$, which implies that $D_{r+1,r}(1,:) = 0$. Thus,

$$A = \begin{bmatrix} J_{1r} & 0_{1r} & C_{rr} & J_{r1} & 0_{rr} & 0_{rr} & J_{rr} \\ 0_{r1} & 0_{r1} & J_{r1} & 0_{r,r-1} & 0 \end{bmatrix},$$

is reducible, which is again a contradiction.

**Case 2.** Let us assume now that $A(1,:) = [0 \ J_{1r} \ 0_{1r}]$. Notice that there is $i \in \{r+2, ..., n\}$ such that $A^2(1,i) \neq 0$, otherwise $A(1:r+1,r+2:n) = 0$, and $A$ would be reducible. This observation leads to the following subcases:

**Subcase 2.1.** Assume that $A^2(1,i) = 0$ for $i = 1, r+2, ..., n-1$. Then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ 0 & C_{rr} & 0 & J_{r1} \\ J_{r1} & 0 & J_{r,r-1} & 0 \end{bmatrix}.$$

A calculation shows that $\exp(A) = 3$, which is a contradiction.

**Subcase 2.2.** Let us assume that $A^2(1,i) = 0$ for $i = 1, ..., k+1, r+2, ..., 2r-k$, with $0 < k < r-1$. Then,

$$A = \begin{bmatrix} 0 & J_{1k} & J_{1,r-k} & 0_{1,r-k-1} & 0_{1,k+1} \\ 0_{r1} & 0_{rk} & R_1 & 0_{r,r-k-1} & R_2 \\ J_{r_k} & D_{rk} & J_{r,r-k-1} & R_3 \end{bmatrix},$$

for some blocks $R_i$, $i = 1, 2, 3$. Taking into account Lemma 2.7, each column of $R_1$ and $R_2$ is nonzero, which implies that $A^2(1,i) \neq 0$ for $i = k+2, ..., r+1, 2r-k$.
Since $A^2(1,:) = [0_{1,k+1} J_{1,r-k} 0_{r-k-1} J_{1,k+1}]$. Note that the submatrix of $[R_2^T R_3^T]$ indexed by rows $k+1, \ldots, r, 2r-k, \ldots, 2r$ has all columns nonzero, otherwise $A$ would not be $r$-regular. Thus, $A^3(1,i) = A^3(1,:)A(:,i) \neq 0$ for $i = 1, \ldots, k + 1, r + 2, \ldots, n$, and $A^3(1,:)$ would not have $r$ zero entries, a contradiction.

Subcase 2.3. Let us assume that $A^2(1,i) = 0$ for $i = 1, \ldots, r$. Then,

\[
A = \begin{bmatrix}
0 & J_{1,r-1} & 1 & 0_{1r} \\
0_{r1} & 0_{r,r-1} & R_1 & R_2 \\
J_{r1} & D_{r,r-1} & * & *
\end{bmatrix},
\]

for some blocks $R_i$, $i = 1, 2$. Taking into account Lemma 2.7, all columns of $R_2$ are nonzero, which implies that $A^2(1,i) \neq 0$ for $i = r + 2, \ldots, n$. If $R_1 = 0$, then

\[
A = \begin{bmatrix}
0 & J_{1r} & 0 \\
0 & 0 & J_{rr} \\
J_{r1} & C_{rr} & 0
\end{bmatrix},
\]

and $exp(A) = 5$. If $R_1$ is nonzero, then, $M(A^2)(1,:) = [0_{1r} J_{1,r+1}]$ and $A^3(1,:) = A^2(1,:)A$ has at most one nonzero entry, which is a contradiction. (Note that the last row of $[R_1R_2]$ has exactly one zero entry.)

Subcase 2.4. Assume that $A^2(1,i) = 0$ for $i = 2, \ldots, r + 1$. Then,

\[
A = \begin{bmatrix}
0 & J_{1r} & 0 \\
* & 0 & * \\
* & D_{rr} & *
\end{bmatrix}.
\]

Note that, by Lemma 2.4, $A^2(1,:)$ has at least $r$ nonzero entries.

- Let us assume that $A^2(1,:)$ has exactly $r$ nonzero entries. If $M(A^2)(1,:) = [0_{1,r+1} J_{1r}]$, then

\[
A = \begin{bmatrix}
0 & J_{1r} & 0 \\
0 & 0 & J_{rr} \\
J_{r1} & C_{rr} & 0
\end{bmatrix}; \quad \text{(4.5)}
\]

if $M(A^2)(1,:) = [1_{r,1} J_{1,r-1}]$, then

\[
A = \begin{bmatrix}
0 & J_{1r} & 0 & 0 \\
J_{r1} & 0 & 0 & J_{r,r-1} \\
0 & C_{rr} & J_{r1} & 0
\end{bmatrix}. \quad \text{(4.6)}
\]

A straightforward computation shows that in both cases $exp(A) = 5$. 
• Let us assume that $A^2(1,:)$ has exactly $r+1$ nonzero entries. Then, $M(A^2)(1,:)[1 \ 0_{1r} \ J_{1r}]$ and $A$ has the form

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ R_1 & 0 & R_2 \\ R_3 & D_{rr} & R_4 \end{bmatrix},$$

where $R_1$ and $R_2$ are $r$-by-1 and $r$-by-$r$ matrices, respectively, with all columns nonzero. Notice also that, since not all rows of $D_{rr}$ sum $r$, either $R_3$ or some column in $R_4$ is nonzero. A calculation shows that $A^3(1,i) \neq 0$ for $i = 2, \ldots, r+1$. Moreover, there is another nonzero entry in $A^3(1,:)$.

If $A^3(1,:) = [J_{1,r+1} \ 0_{1r}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ 0 & 0 & J_{rr} \\ J_{r1} & C_{rr} & 0 \end{bmatrix};$$

if $A^3(1,:) = [0 \ J_{1,r+1} \ 0_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ J_{r1} & 0 & 0 & J_{r,r-1} \\ 0 & C_{rr} & J_{r1} & 0 \end{bmatrix}.$$
If \( M(A^2)(1,:) = [1 \ 0_{1,r-1} \ 1 \ 0_{1,r-1}] \), then

\[
A = \begin{bmatrix}
0 & J_{1,r-1} & 1 & 0 & 0_{1,r-1} \\
* & 0_{r,r-1} & * & 0_{1} & * \\
* & D_{r,r-1} & * & J_{r} & *
\end{bmatrix}
\]

and \( M(A^3)(1,2:r+2) = J_{1,r+1} \). Because \( A^3(1,:) \) has at least \( r \) zero entries, it follows that \( M(A^3)(1,:) = [0 \ J_{1,r+1} \ 0_{1,r-1}] \). Since \( A^2(1,r+1) \neq 0 \), then \( A^3(1,i) = A^2(1,:)A(:,i) = 0 \) implies \( A(r+1,i) = 0 \). Thus, \( A(r+1,i) = 0 \), for \( i = 1, \ldots, r, r+2, \ldots, n \), and the \( (r+1) \)-th row of \( A \) would have at least \( 2r \) entries equal to 0, which contradicts the fact that \( A \) is \( r \)-regular.

Notice that, according to the proof of Theorem 4.4, the only “types” of matrices in \( P_{2r+1,r} \) (up to a permutation similarity) that achieve maximum exponent are

\[
A_1 := \begin{bmatrix}
0 & 0 & J_{rr} \\
J_{1r} & 0 & 0 \\
C_{rr} & J_{r1} & 0
\end{bmatrix}
\]

and

\[
A_2 := \begin{bmatrix}
0 & 0 & J_{rr} \\
C_{rr} & J_{1r} & 0 \\
J_{rr} & 0 & 0
\end{bmatrix}.
\]

Clearly, if \( C_{rr} \) is chosen equal to \( T_{r,r}^{r,r-1} \), then \( A_1 = E_{2r+1,r} \).

Note that the matrix \( A_2 \) has nonzero trace and has maximum exponent among the matrices in \( P_{2r+1,r} \). However, Corollary 2.6 shows that, for most combinations of \( n \) and \( r, u_{nr} \) is not attained by matrices with nonzero trace. In particular, this is true if \( n = gr + c, \) with \( 0 < c < r \) and \( g > r + \sqrt{r^2 - 4r + 5} + 2c \), as \( 2n - 4r + 6 < g^2 + 1 \) and, by Theorem 4.3, \( u_{nr} \geq g^2 + 1 \).

4.2. The case in which \( n \) is a multiple of \( r \). Suppose that \( n = gr \), for some positive integer \( g \geq 2 \). Denote by \( E_{nr} \) the \( n \times n \) matrix given by

\[
E_{nr} = H_{2r,r}, \quad \text{if } n = 2r,
\]

\[
E_{nr} = \begin{bmatrix}
0 & J_{rr} \\
H_{2r,r} & 0
\end{bmatrix}, \quad \text{if } n = 3r,
\]

\[
E_{nr} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & J_{rr} \\
J_{rr} & 0 & \cdots & 0 & 0 & 0 \\
0 & J_{rr} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & J_{rr} & 0 & 0 \\
0 & 0 & \cdots & 0 & H_{2r,r} & 0
\end{bmatrix}, \quad \text{if } n = gr, \text{ with } g \geq 4,
\]
where

\[
H_{2r,r} = \begin{bmatrix}
J_{r-1,r-1} & J_{r-1,1} & 0_{r-1,1} & 0_{r-1,r-1} \\
J_{1,r-1} & 0 & 1 & 0_{1,r-1} \\
0_{1,r-1} & 1 & 0 & J_{1,r-1} \\
0_{r-1,r-1} & 0_{r-1,1} & J_{r-1,1} & J_{r-1,r-1}
\end{bmatrix}.
\]

We will show that \( u_{2r,r} = \exp(E_{2r,2}) \). Taking into account the result of some numerical experiments, we also conjecture that, when \( n = gr \) for some \( g \geq 3 \), the matrices \( E_{nr} \) achieve the maximum exponent in the set \( P_{nr} \). This conjecture is also reinforced by the following observation. Let us say that the exponent of an \( n \)-by-\( n \) \( r \)-regular matrix \( A \) is infinite if \( A \) is not primitive. Given \( n = gr \), with \( g \geq 3 \), consider the following cyclic matrix:

\[
P_1 = \begin{bmatrix}
0 & 0 & \cdots & 0 & J_{rr} \\
J_{rr} & 0 & \cdots & 0 & 0 \\
0 & J_{rr} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & J_{rr} & 0
\end{bmatrix}
\]

which is irreducible but not primitive and, therefore, has infinite exponent. In [3] it was proven that given two \( n \)-by-\( n \) \( r \)-regular matrices \( A \) and \( B \), then \( B \) can be gotten from \( A \) by a sequence of interchanges on 2-by-2 submatrices of \( A \):

\[
L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

The matrix \( E_{nr} \) we have constructed has been obtained by applying just one of these interchanges to \( P_1 \). Notice, however, that not any arbitrary interchange in \( P_1 \) produces a matrix with maximum exponent.

In particular, our conjecture implies that \( u_{nr} < \lfloor \frac{n}{r} \rfloor^2 + 1 \). It is worth to point out that Shen [6] proved that \( u_{n2} < \lfloor \frac{n}{2} \rfloor^2 + 1 \).

Next we show that, if \( n = 2r \), then \( u_{nr} = \frac{n(n-r)}{2r^2} + 2 = 3 \).

**Theorem 4.5.** Let \( r \geq 2 \). Then, \( u_{2r,r} = 3 \).

**Proof.** Let \( A \in P_{2r,r} \) and suppose that \( \exp(A) > 3 \). Then, there must exist a zero entry in \( A^3 \). Without loss of generality, we can assume that \( A^3(1,i) = 0 \) for some \( i \in \{1,\ldots,n\} \). Applying Lemma 2.3, we deduce that there must be at least \( r \) zero entries in the first row of \( A^3 \). Without loss of generality, we can assume that one of the next cases holds.
Case 1. Suppose that \( A(1,:) = [J_{1r} \ 0_{1r}] \). Then, for \( A \) to have exponent larger than 3, \( M(A^2)(1,:) = [J_{1r} \ 0_{1r}] \). Taking into account the position of the zeros in the first row of \( A^2 \), we deduce that

\[
A = \begin{bmatrix}
J_{rr} & 0_{rr} \\
0_{rr} & J_{rr}
\end{bmatrix},
\]

which is a reducible matrix.

Case 2. Suppose that \( A(1,:) = [0 \ J_{1r} \ 0_{1,r-1}] \). If \( A^2(1,1) = 0 \) or \( A^2(1,i) = 0 \) for some \( i \geq r+1 \), then \( A \) would not be \( r \)-regular. Therefore, for \( A \) to have exponent larger than 3, \( M(A^2)(1,:) = [1 \ 0_{1r} \ J_{1,r-1}] \). Then,

\[
A = \begin{bmatrix}
0 & J_{1r} & 0_{1,r-1} \\
J_{r1} & 0_{rr} & J_{r,r-1} \\
0_{r-1,1} & J_{r-1,r} & 0_{r-1,r-1}
\end{bmatrix},
\]

which is reducible.

In both cases, we get a contradiction. Thus, for any \( A \in P_{2r,r} \), \( \text{exp}(A) \leq 3 \). Since \( E_{2r,r}^2 \) is not positive, then \( \text{exp}(E_{2r,r}) = 3 = u_{2r,r} \).

Next we give the exponent of the matrices \( E_{nr} \) when \( n = gr \) for some positive integer \( g \geq 3 \). Before we prove the result, we include a preliminary result.

Let \( a_1, a_2, \ldots, a_p \) be positive integers such that \( \gcd(a_1, \ldots, a_p) = 1 \). The Frobenius-Schur index, \( \phi(a_1, \ldots, a_p) \), is the smallest integer such that the equation \( x_1a_1 + \cdots + x_pa_p = l \) has a solution in nonnegative integers \( x_1, x_2, \ldots, x_p \) for all \( l \geq \phi(a_1, \ldots, a_p) \). The following result is due to Brauer in 1942.

**Proposition 4.6.** [1] Let \( y \) be a positive integer. Then

\[
\phi(y, y+1, \ldots, y+j-1) = y \left\lfloor \frac{y+j-3}{j-1} \right\rfloor.
\]

**Lemma 4.7.** Let \( y > 1 \) be a positive integer. Then,

\[
\phi(y, y+1, y+2) = \begin{cases} 
\frac{1}{2}y^2, & \text{if } y \text{ is even} \\
\frac{1}{2}(y-1)y, & \text{if } y \text{ is odd}.
\end{cases}
\]

Moreover, there are nonnegative integers \( a, b, c \) satisfying \( \phi(y, y+1, y+2) - 2 = ay + b(y+1) + c(y+2) \) if and only if \( y \) is even. If \( y \) is odd, there are nonnegative integers \( a, b, c \) satisfying \( \phi(y, y+1, y+2) - 3 = ay + b(y+1) + c(y+2) \).
Proof. The first claim follows from Proposition 4.6. Now we show the second claim. Clearly, if $y$ is even, $\phi(y, y + 1, y + 2) - 2 = \left(\frac{n}{2} - 1\right)(y + 2)$ can be written as $ay + b(y + 1) + c(y + 2)$ for some nonnegative numbers $a, b, c$. If $y$ is odd

$$\phi(y, y + 1, y + 2) - 3 = \frac{1}{2}(y - 1)y - 3 = \left(\frac{y - 1}{2} - 1\right)(y + 2).$$

which implies that $\phi(y, y + 1, y + 2) - 3$ can be written as $ay + b(y + 1) + c(y + 2)$ for some nonnegative integers $a, b, c$. To see that there are no nonnegative integers $a, b, c$ such that

$$\phi(y, y + 1, y + 2) - 2 = ay + b(y + 1) + c(y + 2),$$

notice that the largest number of the form $ay + b(y + 1) + c(y + 2)$, for some nonnegative integers $a, b, c$, smaller than $\phi(y, y + 1, y + 2)$ is $\left(\frac{n}{2} - 1\right)(y + 2)$ and

$$\left(\frac{y - 1}{2} - 1\right)(y + 2) < \left(\frac{y - 1}{2} - 1\right)(y + 2) + 3 - 2 = \phi(y, y + 1, y + 2) - 2. \quad \Box$$

**Theorem 4.8.** Let $n = gr$, with $g \geq 3$ and $r \geq 2$. Then,

$$\exp(E_{nr}) = \begin{cases} 
\frac{n(n-r)}{2r^2} + 2, & \text{if } \frac{n}{r} \text{ is even} \\
\frac{1}{2} \left(\frac{n}{r}\right)^2 + 1, & \text{if } \frac{n}{r} \text{ is odd.}
\end{cases}$$

Proof. Consider the digraph $G$ associated with $E_{nr}$. We group the vertices of $G$ in the following way: for $i = 1, \ldots, g$, we call block $B_i$ the set of vertices from $(g - i)r + 1$ to $(g - i + 1)r$. For convenience, we denote the vertices $n - 3r + 1, \ldots, n - 2r$ in $B_3$ by $w_1, \ldots, w_r$, resp; the vertices $n - 2r + 1, \ldots, n - r$ in $B_2$ by $v_1, \ldots, v_r$, resp., and the vertices $n - r + 1, \ldots, n$ in $B_1$ by $u_1, \ldots, u_r$, resp. Let $B'_1 = \{u_2, \ldots, u_r\}$, $B'_2 = \{v_2, \ldots, v_{r-1}\}$ and $B'_3 = \{w_1, \ldots, w_{r-1}\}$. Note that $B'_2$ is empty if $r = 2$. The digraph $G$ is given in Figure 4.1.

A directed edge in this graph from a set $S_1$ to a set $S_2$ means that there is an arc from each vertex in $S_1$ to each vertex in $S_2$.

Let $G'$ be the subgraph of $G$ induced by the vertices in $B_1 \cup B_2 \cup B_3$. The following table gives the possible lengths of a walk in $G'$ from a vertex in $B_1$ to a vertex in $B_3$.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Possible lengths</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>any vertex in $B'_3$</td>
<td>2, 3</td>
</tr>
<tr>
<td>$u_1$</td>
<td>$w_r$</td>
<td>1, 2 (if $r &gt; 2$), 3</td>
</tr>
<tr>
<td>any vertex in $B'_3$</td>
<td>any vertex in $B'_3$</td>
<td>2, 3</td>
</tr>
<tr>
<td>any vertex in $B'_1$</td>
<td>$w_r$</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

Table 1.
Thus, for any \( i \in \{1, \ldots, g\} \setminus \{2\} \), any walk in \( G \) from a vertex \( u \in B_i \) to a vertex \( v \in B_i \) has length \( t \) if and only if
\[
t = a \left[ \frac{1}{2} (g - 2) + 1 \right] + b \left[ \frac{1}{2} (g - 2) + 2 \right] + c \left[ \frac{1}{2} (g - 2) + 3 \right],
\]
for some nonnegative integers \( a, b, c \), with \( b + c > 0 \) if either \( u \in B'_1 \) or \( v \in B'_3 \).

Taking into account Lemma 4.7, the smallest nonnegative integer \( t_0 \) such that, for any \( t \geq t_0 \), (4.8) holds for some nonnegative integers \( a, b, c \) is
\[
t_0 = \begin{cases} 
\frac{1}{2} (g - 1)^2, & \text{if } g \text{ is odd} \\
\frac{1}{2} (g - 2) (g - 1), & \text{if } g \text{ is even}.
\end{cases}
\]

We will show that, if \( g \) is odd, any two vertices \( u, v \) in \( G \) are connected by a walk of length \( t_0 + g \) but not of length \( t_0 + g - 1 \); if \( g \) is even, any two vertices \( u, v \) in \( G \) are connected by a walk of length \( t_0 + g + 1 \) but not of length \( t_0 + g \). Denote by \( d(u, v) \) the distance from the vertex \( u \) to the vertex \( v \). Clearly, \( d(u, v) \leq g \).

If \( u, v \in B_i \) for some \( i \in \{1, \ldots, g\} \setminus \{2\} \), with \( u = u_1 \) if \( i = 1 \), and \( v = w_r \) if \( i = 3 \), then, for any \( t \geq t_0 \), there is a walk of length \( t \) from \( u \) to \( v \).

Suppose that \( u, v \in B_2 \). Clearly, there is a walk of length 1 from \( u \) to some vertex

**Fig. 4.1.**
in $B_3$. Also, there is a vertex $v'$ in $B_1$ such that there is a walk of length 1 from $v'$ to $v$. Taking into account these observations, and the fact that, for $t \geq t_0$, there is a walk of length $t$ from any vertex in $B_3$ to $w_r$, it follows that there is a walk of length $t + (g - 2) + 2 = t + g$ from $u$ to $v$.

Suppose that $u \in B_1 \setminus v \in B_1$. Notice that there is a walk of length $g$ from $u$ to $u_1$. Since, for $t \geq t_0$, there is a walk of length $t$ from $u_1$ to $v$, it follows that there is a walk of length $t + g$ from $u$ to $v$.

Let $u \in B_3$ and $v \in B_1 \setminus v \in B_1$. Then, there is a walk of length $g$ from $w_r$ to $v$. Since, for $t \geq t_0$, there is a walk of length $t$ from $u$ to $w_r$, then there is a walk of length $t + g$ from $u$ to $v$.

Now suppose that $u \in B_i$ and $v \in B_j$, with $i \neq j$.

Suppose that $u \notin B_1 \cup B_2$. Let $w = u$ if $i \neq 3$, and $w = w_r$ otherwise. Then, for $t \geq t_0$, since $g - d(w, v) > 0$, $t + g - d(w, v) \geq t_0$ and there is a walk of length $t + g - d(w, v)$ from $u$ to $w$. This implies that there is a walk of length $t + g$ from $u$ to $v$.

Suppose that $u \in B_1 \setminus v \notin B_2 \cup B_3$. Note that $d(w_r, v) \leq g - 2$. Also, there is a walk of length 2 from $u$ to $w_r$. As, for $t \geq t_0$, $w_r$ lies on a closed walk of length $t + g - d(w_r, v) - 2$, then there is a walk of length $2 + (t + g - d(w_r, v) - 2) + d(w_r, v) = t + g$ from $u$ to $v$.

Suppose that $u \in B_2$ and $v \notin B_3 \setminus B_3$. Then $d(w_r, v) \leq g - 1$. As, for $t \geq t_0$, there is a walk of length $t + g - d(w_r, v) - 1$ from any vertex in $B_3$ to $w_r$, then there is a walk of length $1 + (t + g - d(w_r, v) - 1) + d(w_r, v) = t + g$ from $u$ to $v$.

We have shown that, for any $t \geq t_0$, there is a walk of length $t + g$ from $u$ to $v$, unless either $u \in B_2$ and $v \in B_3$, or $u \in B_1$ and $v \in B_2 \cup B_3$.

In order to determine the exponent of $E_{nr}$, we now consider two cases, depending on the parity of $g$.

**Case 1.** Suppose that $g$ is odd. Notice that every walk in $G$ from $v_1$ to $v_r$ of length $t > g$ contains a subgraph which is a walk of length $t - g$ from a vertex in $B_3$ to a vertex in $B_3$. Because there is no walk of length $t_0 - 1$ from a vertex in $B_3$ to a vertex in $B_3$, then there is no walk of length $t_0 + g - 1$ from $v_1$ to $v_r$.

We have already proven that there is a walk of length $t_0 + g$ from any vertex $u$ to any vertex $v$, unless either $u \in B_2$ and $v \in B_3$, or $u \in B_1$ and $v \in B_2 \cup B_3$, in which cases there is a walk of length $s_1$ from $u$ to some vertex in $B_3$ and there is a walk of length $s_2$ from some vertex in $B_1$ to $v$, with $s_1 + s_2 = 4$. By Lemma 4.7, there are
nonnegative integers $a, b, c$ such that

$$t_0 - 2 = \frac{1}{2}(g - 1)^2 - 2 = a(g - 1) + bg + c(g + 1).$$

Thus, from any vertex in $B_3$, there is a walk to $w_r$ of length $t_0 - 2$, which implies that there is a walk of length $(t_0 - 2) + (g - 2) + 4 = t_0 + g$ from $u$ to $v$. Therefore,

$$\exp(E_{u,r}) = t_0 + g = \frac{1}{2}(g^2 + 1) = \frac{1}{2} \left( \left( \frac{n}{r} \right)^2 + 1 \right).$$

**Case 2.** Suppose that $g$ is even. First, consider the case $u \in B_1$ and $v \in B_3$. Clearly, there is a walk of length 3 from $u$ to $w_r$; also, there is a walk of length 3 from some vertex in $B_1$ to $v$. Taking into account Lemma 4.7, $w_r$ lies on a closed walk of length $t_0 - 3$, which implies that there is a walk of length $(t_0 - 3) + (g - 2) + 6 = t_0 + g + 1$ from $u$ to $v$.

Now suppose that either $u \in B_2$ and $v \in B_3$, or $u \in B_1$ and $v \in B_2$. Then, there is a walk of length $s_1$ from $u$ to some vertex in $B_3$ and there is a walk of length $s_2$ from some vertex in $B_1$ to $v$, with $s_1 + s_2 = 3$. As, from any vertex in $B_3$, there is a walk of length $t_0$ to $w_r$, then there is a walk of length $t_0 + (g - 2) + 3 = t_0 + g + 1$ from $u$ to $v$.

Now we show that there are two vertices not connected by a walk of length $t_0 + g$. Note that $t_0 + g > g + 2$. Also, every walk of length $t > g + 2$ from $u \in B_1$ to $v_r$ contains a subgraph which is a walk of length $t - g - 1$ or $t - g - 2$ from a vertex in $B_3$ to a vertex in $B_3$. By Lemma 4.7, for $k \in \{1, 2\}$, there are no nonnegative integers such that $t_0 - k = a(g - 1) + bg + c(g + 1)$. So, there is no walk of length $t_0 + g$ from $u \in B_1$ to $v_r$.

Thus,

$$\exp(E_{u,r}) = t_0 + g + 1 = \frac{1}{2}(g^2 - g) + 2 = \frac{n(n-r)}{2r^2} + 2. \Box$$

If $n = r$, the only matrix in $P_{r,r}$ is $J_n$ which has exponent 1. Note that $n/r = 1$ is odd and $u_{rr} = \frac{1}{2} \left( \left( \frac{n}{r} \right)^2 + 1 \right) = 1$. If $n = 2r$, by Theorem 4.5, $u_{nr} = \frac{n(n-r)}{2r^2} + 2 = 3$. If $n = gr$, with $g \geq 3$ and $r \geq 2$, it follows from Theorem 4.8 that $u_{nr} \geq \exp(E_{nr})$. We conjecture that in this case the equality also holds. Note that $\exp(E_{nr}) < \left[ \frac{2}{r} \right]^2 + 1$.

**Conjecture 1.** Let $n = gr$ with $g \geq 1$ and $r \geq 2$. Then,

$$u_{nr} = \begin{cases} \frac{n(n-r)}{2r^2} + 2, & \text{if } \frac{n}{r} \text{ is even} \\ \frac{1}{2} \left( \left( \frac{n}{r} \right)^2 + 1 \right), & \text{if } \frac{n}{r} \text{ is odd.} \end{cases}$$
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