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m-NIL-CLEAN COMPANION MATRICES*

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Abstract. Companion matrices over fields of positive characteristic, p, that are sums of m idempotents, $m \geq 2$, and a nilpotent are characterized in terms of dimension and trace of such a matrix and of p.

Key words. Companion matrix, Idempotent, Nilpotent, m-nil-clean.

AMS subject classifications. 15A24, 15A83, 16U99.

1. Introduction.

In [10], clean rings and clean elements in rings were introduced, in order to study some properties of direct decompositions of modules. Clean elements are sums of a unit and an idempotent element of the ring and a clean ring is such that all of its elements are clean. A particular class of clean rings was introduced by Diesl in [8]: the class of rings such that all elements are sums of a nilpotent and an idempotent. Other generalizations were considered in [6] and [4]. In the former, Chen and Sheibani considered 2-nil-clean rings, rings such that all elements are 2-nil-clean, i.e., elements that are sums of two idempotents and a nilpotent element. Weakly nil-clean rings were firstly introduced in the commutative case by Danchev and McGovern, in [7]. Breaz, Danchev and Zhou characterized weakly nil-clean rings in [4]. In the case of these rings, each element is a sum or a difference of a nilpotent and an idempotent. Moreover, in [1], the author studies elements which are sums of a nilpotent and m idempotents which commute.

It was proven in [9] that matrix rings over clean rings are clean. Matrix rings over nil-clean rings were studied in [8] and [3]. In the latter, it was proven that a matrix ring over a commutative nil-clean ring is nil-clean. Nil-clean matrices over general fields were studied in [5] , where the authors study nil-clean companion matrices. We note that this can be an important step in a possible attempt to characterize general nil-clean matrices since every matrix is similar to a direct sum of companion matrices.

Using this idea we will study in this paper m-nil-clean companion matrices over fields of positive characteristic. Let $m \geq 2$ be an integer. Then an m-nil-clean element of a ring is an element that represents the sum of m idempotents and a nilpotent (of that ring). In Theorem 3.4, we characterize m-nil-clean companion matrices by using the dimension and the trace of such a matrix and the characteristic of the field.

Let $\mathbb{F}$ be a field of positive characteristic, p. Let q be a monic polynomial over $\mathbb{F}$, $q = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$. The companion matrix associated to q is the $n \times n$ matrix

$$C = C_{c_0,c_1,\ldots,c_{n-1}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}.$$

We also denote C by C_q.

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2. Useful tools. Any matrix is similar to a Frobenius normal form (a direct sum of companion matrices), a matrix similar to a nilpotent is nilpotent and a matrix similar to an idempotent is idempotent. This is why in the proof of Theorem 3 of [2], one restricts without loss of generality to the case of companion matrices. The same technique is used in [6] to prove that $M_n(\mathbb{F}_3)$ is 2-nil-clean. We are also determined to consider m-nil-clean companion matrices, based on the previously mentioned facts and on the fact that, if all companion matrices which appear in the Frobenius normal form of a matrix A are m-nil-clean, then A is also m-nil-clean.

Let $\mathbb{F}$ be a field of positive characteristic p. In [5], an investigation was made for nil-clean companion matrices over $\mathbb{F}$.

Theorem 2.1. *Let $\mathbb{F}$ be a field of positive characteristic p. Let $C = C_{c_0,c_1,\ldots,c_{n-1}} \in M_n(\mathbb{F})$ be a companion matrix. The following are equivalent:*

1. *C is nil-clean.*
2. *One of the following conditions is true:*
   (a) *C is nilpotent (i.e., $c_0 = \cdots = c_{n-1} = 0$);*
   (b) *C is unipotent (i.e., $c_i = (-1)^i\binom{n}{n-i}$ for all $i \in \{0,\ldots,n-1\}$);*
   (c) *there exists an integer $k \in \{1,\ldots,p\}$ such that $-c_{n-1} = k \cdot 1$ and $n > k$.*

As a consequence of this fact, the following result holds for characteristic 2:

Corollary 2.2. *Let $\mathbb{F}$ be a field of characteristic 2. Let $C = C_{c_0,c_1,\ldots,c_{n-1}} \in M_n(\mathbb{F})$ be a companion matrix. Then C is nil-clean if and only if $-c_{n-1} \in \{0,1\}$.*

Here is another corollary of the above theorem:

Corollary 2.3. *Let $n \geq 3$ be a positive integer. The following are equivalent for a field $\mathbb{F}$:*

1. *$\mathbb{F} \cong \mathbb{F}_p$ for a prime $p < n$;*
2. *every companion matrix $C \in M_n(\mathbb{F})$ is nil-clean.*

In [2] and [11], the authors use some decompositions which involve matrices that are the sum of a diagonal matrix with entries only 0 and 1 and a companion matrix. In the following, we will use a similar technique.

The following lemma will be useful while proving results on m-nil-clean companion matrices. We will use the notation: $\mathrm{Lin}(\{v_1, v_2, \ldots, v_n\})$ for the subspace of a vector space X generated by the set of vectors $v_1, v_2, \ldots, v_n$ of X.

Lemma 2.4. *Let $\mathbb{F}$ be a field. For every companion matrix $C_q \in M_n(\mathbb{F})$ and every $k \in \{1,\ldots,n\}$, there exists a companion matrix $C_{q'}$ such that C_q and $\mathrm{diag}(\underbrace{1,\ldots,1}_{k\text{-times}},0,\ldots,0) + C_{q'}$ are similar.*

Proof. First we will prove the statement for $k \in \{1,2,\ldots,n-1\}$. Let V denote the $n-$dimensional vector space of columns over $\mathbb{F}$ and consider C_q as an endomorphism $C_q : V \to V$. Denoting by $\{e_1, e_2, \ldots, e_n\}$ the standard basis of V, C_q maps each e_i to e_{i+1}, for each $i \in \{1,2,\ldots,n-1\}$.

Now we define $\{f_1, f_2, \ldots, f_n\}$, $f_i \in V$, $i \in \{1,2,\ldots,n-1\}$, inductively as it follows. First set $f_1 = e_1$. Assuming that $2 \leq i \leq n$ and that f_{i-1} has been defined, set $f_i = C_q(f_{i-1}) - f_{i-1}$, if $i \in \{1,2,\ldots,k+1\}$ and $f_i = C_q(f_{i-1})$, if $i \in \{k+2,\ldots,n\}$

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We have $e_1 = f_1$, so $e_1 \in \mathrm{Lin}(\{f_1\})$ and $f_2 = C_q(f_1) - f_1 = C_q(e_1) - f_1 = e_2 - f_1$, so $e_2 = f_1 + f_2$ and $e_2 \in \mathrm{Lin}(\{f_1, f_2\})$

It is easy to see that each f_i is the sum of e_i and a linear combination of $e_{i-1}, e_{i-2}, \ldots, e_2, e_1$. Hence, e_i is the difference of f_i and a linear combination of $e_{i-1}, e_{i-2}, \ldots, e_2, e_1$. Assuming $e_1, e_2, \ldots, e_{i-1} \in \mathrm{Lin}(\{f_1, f_2, \ldots, f_n\})$, we get e_i is a linear combination of $f_1, f_2, \ldots, f_n$. Therefore, $\mathrm{Lin}(\{e_1, e_2, \ldots, e_n\}) = \mathrm{Lin}(\{f_1, f_2, \ldots, f_n\})$, and thus, $\{f_1, f_2, \ldots, f_n\}$ is a basis of V.

Moreover, by the definition, we have:

$$\begin{aligned} C_q(f_1) &= f_1 + f_2, \\ C_q(f_2) &= f_2 + f_3, \\ \vdots \quad & \quad \vdots \\ C_q(f_k) &= f_k + f_{k+1}, \\ C_q(f_{k+1}) &= f_{k+2}, \\ \vdots \quad & \quad \vdots \\ C_q(f_{n-1}) &= f_n. \end{aligned}$$

Let M be the matrix the endomorphism C_q corresponds to, with respect to the basis $B = \{f_1, f_2, \ldots, f_n\}$. Therefore,

$$M = \left[[C_q(f_1)]_B, [C_q(f_2)]_B, \ldots, [C_q(f_k)]_B, [C_q(f_{k+1})]_B, \ldots, [C_q(f_{n-1})]_B, [C_q(f_n)]_B\right].$$

Hence,

$$M = \left[[f_1 + f_2]_B, [f_2 + f_3]_B, \ldots, [f_k + f_{k+1}]_B, [f_{k+2}]_B, \ldots, [f_n]_B, [C_q(f_n)]_B\right].$$

It follows that $M = \mathrm{diag}(\underbrace{1, \ldots, 1}_{k\text{-times}}, 0, \ldots, 0) + C_{q'}$ for some monic polynomial q' of degree n. So, $C_q = P^{-1}(\mathrm{diag}(\underbrace{1, \ldots, 1}_{k\text{-times}}, 0, \ldots, 0) + C_{q'})P$, where P is the transition matrix mapping each e_i to f_i.

As next step, we will solve the case $k = n$, that is we will prove that C is similar to $I_n + C_{q'}$, where $q' = q'' + X^{n-1}$, and q'' is such that $\mathrm{diag}(\underbrace{1, \ldots, 1}_{n-1\text{-times}}, 0) + C_{q''}$ is similar to C. We consider the vector space of column vectors of dimension n over $\mathbb{F}$. Let P be the transition matrix from canonical basis to basis B defined in the first part of this lemma, taking $C_q = C$. Then $P(C_q - I_n)P^{-1} = PC_qP^{-1} - I_n = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n-1\text{-times}}, 0) + C_{q''} - I_n = C_{q'}$, where $q' = q'' + X^{n-1}$. Therefore, $C - I_n$ is similar to $C_{q'}$, where $q' = q'' + X^{n-1}$, and q'' is such that $\mathrm{diag}(\underbrace{1, \ldots, 1}_{n-1\text{-times}}, 0) + C_{q''}$ is similar to C. Hence, C_q is similar to $I_n + C_{q'}$. $\square$

Example 2.5. For $p = 11$, $n = 6$, $k = 5$, $q = X^6 + 3X^5 - X^3 - X^2 - X - 1$ we want to see what matrix of the type $\mathrm{diag}(\underbrace{1, \ldots, 1}_{k\text{-times}}, 0, \ldots, 0) + C_{q'}$ is similar to C_q. We want to express the vectors in the basis $B = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ as linear combinations of vectors in the canonical basis and vectors in the canonical basis as linear combinations of vectors in basis B in order to find out the transition matrix from canonical basis to basis B and its inverse (P and P^{-1}).

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After doing this we obtain

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & -1 \\ 0 & 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & -3 & 6 & 1 \\ 0 & 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $C_q = P^{-1}(\operatorname{diag}(1,1,1,1,1,0) + C_{q'})P$, we have

$$C_q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} = \operatorname{diag}(1,1,1,1,1,0) + C_{q'}.$$

3. m-nil-clean companion matrices. As a consequence of Theorem 2.1, if $n > p$, then every $n \times n$ companion matrix over $\mathbb{F}$ is nil-clean, where $\mathbb{F}$ is a field of positive characteristic p. Therefore, we will assume that $n \leq p$.

Secondly, for $n = 1$ the only nilpotent of $M_n(\mathbb{F})$ is (0) and the only idempotents of $M_n(\mathbb{F})$ are (0) and (1). Therefore, $C \in M_1(\mathbb{F})$ is m-nil-clean if and only if $C \in \{(0), (1), (2), \dots, (m)\}$. Hence, we will not refer to the case $n = 1$, so we will assume $n > 1$ from now on.

LEMMA 3.1. *Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive characteristic p. Let $A \in \mathbb{M}_n(\mathbb{F})$ be a (not necessarily companion) matrix, for which there exists the decomposition $A = E_1 + E_2 + \cdots + E_m + N$, with $k_i = \operatorname{rank}(E_i)$, E_i idempotent, $i \in \{1, 2, \dots, m\}$ and N is nilpotent. Then there is an integer c such that $\operatorname{trace}(A) = c \cdot 1$, and $c = k_1 + k_2 + \cdots + k_m$ (mod p) and each k_i is a natural number less than or equal to n.*

Proof. If $A = E_1 + E_2 + \cdots + E_m + N$, E_i idempotent, $i \in \{1, \dots, m\}$ and N is nilpotent, then $\operatorname{trace}(A) = \operatorname{trace}(E_1) + \operatorname{trace}(E_2) + \cdots + \operatorname{trace}(E_m) + \operatorname{trace}(N)$. It follows that $\operatorname{trace}(A) = \operatorname{trace}(E_1) + \operatorname{trace}(E_2) + \cdots + \operatorname{trace}(E_m)$. Moreover, it is known that if E is an idempotent, then $\operatorname{trace}(E) = \operatorname{rank}(E) \cdot 1$. So $\operatorname{trace}(A) = (k_1 + k_2 + \cdots + k_m) \cdot 1$, and $k_i \leq n$, $i \in \{1, 2, \dots, m\}$. ☐

LEMMA 3.2. *Let $\mathbb{F}$ be a field of positive characteristic p, $1 < n \leq p$. If $-c_{n-1} = c \cdot 1$ and $c \in \{2, 3, \dots, 2n-2, 2n-1\}$, then $C = C_{c_0, c_1, \dots, c_{n-1}} \in M_n(\mathbb{F})$ is 2-nil-clean.*

Proof. Since $c \in \{2, 3, \dots, 2n-1\}$, there exist $k \in \{1, 2, \dots, n\}$ and $l \in \{1, 2, \dots, n-1\}$ such that $c \cdot 1 = (k + l) \cdot 1$. Let $C = C_q$, where $q = X^n + c_{n-1}X^{n-1} + \cdots + c_1 X + c_0$. Then, by Lemma 2.4, there exists $q' = X^n + d_{n-1}X^{n-1} + \cdots + d_1 X + d_0$, $q' \in \mathbb{F}[X]$ such that

$$C_q \sim \operatorname{diag}(\underbrace{1, \dots, 1}_{k\text{-times}}, 0, \dots, 0) + C_{d_0, d_1, \dots, d_{n-1}}. \tag{3.1}$$

Hence, it follows $-c_{n-1} = k - d_{n-1}$ and so $-d_{n-1} = l \cdot 1$, and we know $n > l$ and $l \neq 0$. Now, by Theorem 2.1, it follows that there exist an idempotent matrix E and a nilpotent matrix N such that $C_{d_0, d_1, \dots, d_{n-1}} = E + N$. Using this and equation 3.1 we have $C_q \sim \operatorname{diag}(1, \dots, 1, 0, \dots, 0) + E + N$. We know

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that $C_q = P^{-1}(\mathrm{diag}(\underbrace{1,\dots,1}_{k\text{-times}},0,\dots,0) + C_{d_0,d_1,\dots,d_{n-1}})P$, where P is the transition matrix mapping each e_i of canonical basis to f_i of the basis met in Lemma 2.4. Now we have

$$C_q = P^{-1}\mathrm{diag}(1,\dots,1,0,\dots,0)P + P^{-1}EP + P^{-1}NP. \tag{3.2}$$

Since a matrix similar to an idempotent is an idempotent and a matrix similar to a nilpotent is nilpotent, we have actually showed the fact that C_q is 2-nil-clean. ∎

Lemma 3.3. *Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive characteristic p, and $1 < n \leq p$. If $-c_{n-1} = c \cdot 1$ and $c \in \{m, m+1, \dots, mn-1\}$, then $C = C_{c_0,c_1,\dots,c_{n-1}} \in \mathbb{M}_n(\mathbb{F})$ is m-nil-clean.*

Proof. This lemma's statement is proved for $m = 2$ in Lemma 3.2.

Now assume that the lemma's statement is true for $m \geq 2$. We will prove that it is true also for $m+1$. Let $c \in \{m+1, m+2, \dots, (m+1)n-1\}$. We will prove that C is $(m+1)-$nil-clean.

Since $c \in \{m+1, m+2, \dots, (m+1)n-1\}$, it follows that there exist $k \in \{1,2,\dots,n\}$ and $l \in \{m, m+1, \dots, mn-1\}$ such that $c \cdot 1 = (k+l) \cdot 1$. Let $C = C_q$, $q = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$. Then by Lemma 2.4, there exists $q' = X^n + d_{n-1}X^{n-1} + \cdots + d_1X + d_0$, $q' \in \mathbb{F}[X]$, such that

$$C_q \sim \mathrm{diag}(\underbrace{1,\dots,1}_{k\text{-times}},0,\dots,0) + C_{d_0,d_1,\dots,d_{n-1}}. \tag{3.3}$$

It follows $-c_{n-1} = k - d_{n-1}$ and so $-d_{n-1} = l \cdot 1$. We know that $l \in \{m, m+1, \dots, mn-1\}$, so by hypothesis induction we obtain that there exist idempotent matrices $E_1, E_2, \dots, E_m$, and the nilpotent matrix N, such that $C_{d_0,d_1,\dots,d_{n-1}} = E_1 + E_2 + \cdots + E_m + N$. Using this and equation 3.3 we have

$$C_q \sim \mathrm{diag}(\underbrace{1,\dots,1}_{k\text{-times}},0,\dots,0) + E_1 + E_2 + \cdots + E_m + N.$$

We know that there exists an invertible matrix P such that

$$C_q = P^{-1}(\mathrm{diag}(\underbrace{1,\dots,1}_{k\text{-times}},0,\dots,0) + C_{d_0,d_1,\dots,d_{n-1}})P.$$

Therefore,

$$C_q = P^{-1}(\mathrm{diag}(\underbrace{1,\dots,1}_{k\text{-times}},0,\dots,0))P + P^{-1}E_1P + P^{-1}E_2P + \cdots + P^{-1}E_mP + P^{-1}NP.$$

Since a matrix similar to an idempotent is idempotent and a matrix similar to a nilpotent is nilpotent, we have actually showed the fact that C_q is $(m+1)-$nil-clean. ∎

Having the case $p = 2$ solved in Corollary 2.2 (C is nil-clean in this case, so is also m-nil-clean), we can assume from now on that p is odd.

Theorem 3.4. *Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive odd characteristic p, and $1 < n \leq p$. Let $C = C_{c_0,c_1,\dots,c_{n-1}}$ be a companion matrix. Let c be the smallest nonnegative integer such that $-c_{n-1} = c \cdot 1$.*

The following hold:

1. *If $c = 0$ and $mn - 1 < p$, then C is m-nil-clean if and only if C is nilpotent or $C - (m-1)I_n$ is unipotent.*

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2. *If $c = t$, $1 \le t \le m$, then C is $t-$nil-clean ($1-$nil-clean is just nil-clean), so is m-nil-clean.*
3. *If $c \in \{m, m+1, \dots, mn-2, mn-1\}$, then C is m-nil-clean.*
4. *If $mn - 2 \ge p$, then C is m-nil-clean.*
5. *Assume that $mn - 2 < p$.*
   (a) *If $c = mn$ and $p = mn - 1$, then C is nil-clean, so is m-nil-clean.*
   (b) *If $c = mn$ and $p = mn$, then C is m-nil-clean if and only if C is nilpotent or $C - (m-1)I_n$ is an unipotent matrix.*
   (c) *If $c = mn$, $p > mn$, then C is m-nil-clean if and only if $C - (m-1)I_n$ is an unipotent matrix.*
   (d) *If $c > mn$, then C is not m-nil-clean.*

Proof.

1. If $c = 0$ and $mn - 1 < p$, then $c = 0$ and $mn \le p$.
   If $mn < p$, then $c = (\underbrace{0 + \cdots + 0}_{m\text{-times}})$ is the only form of c as sum, modulo p of m positive integers less or equal to n, and since C is m-nil-clean it follows that the idempotents in the m-nil-clean decomposition of C have rank zero. Therefore, they are O_n, and hence, C is nilpotent.
   If $mn = p$, then $c = (\underbrace{0 + \cdots + 0}_{m\text{-times}})$ and $c = (\underbrace{n + \cdots + n}_{m\text{-times}})$ are the only forms of c as a sum modulo p of m positive integers less or equal to n, and, since C is m-nil-clean, it follows C is nilpotent or $C - (m-1)I_n$ is unipotent.
   It is obvious that if C is nilpotent or $C - (m-1)I_n$ is unipotent, then C is m-nil-clean.
2. If $c = 1$ and $n > 1$, then we have by Theorem 2.1 that C is nil-clean. If $c = t$, $2 \le t \le m$, by Lemma 3.3, we have that C is $t-$nil-clean.
3. This is Lemma 3.3.
4. This is a direct consequence of 3.
5. (a) If $c = mn$ and $p = mn - 1$, it follows that $-c_{n-1} = 1 \cdot 1$, so C is nil-clean.
   (b) Since $c = mn$, $p = mn$ it follows that $c = 0$ and $mn - 1 < p$, so by 1. we have that C is m-nil-clean if and only if C is nilpotent or $C - (m-1)I_n$ is unipotent.
   (c) Since $c = mn$, $p > mn$, it follows that $-c_{n-1} = mn = (\underbrace{n + n + \cdots + n}_{m\text{-times}})(mod p)$, that is the only decomposition as a sum of m integers between 0 and n. Hence, the idempotents in the m-nil-clean decomposition of C are idempotent units, which means they are I_n, so $C - (m-1)I_n$ is unipotent. Conversely, if $C - (m-1)I_n$ is unipotent, then it is obvious that C is m-nil-clean.
   (d) If $c > mn$, then c cannot be written modulo p as sum of m integers less or equal to n, but by Lemma 3.1, we have that C is not m-nil-clean. ∎

Remark 3.5. In Theorem 3.4, one of the conclusions was that $C - (m-1)I_n$ is unipotent. We can prove that $C - (m-1)I_n$ is similar to a companion matrix, so one can say more, the companion matrix with whom $C - (m-1)I_n$ is similar can be only the unipotent companion matrix of type $n \times n$.

Let us prove that $C - (m-1)I_n$ is similar to a companion matrix. To begin with, we have by Lemma 2.4, case $k = n$, that there exists the monic polynomial q_1 of degree n such that $C - I_n \sim C_{q_1}$. But using again Lemma 2.4, case $k = n$, we get $C_{q_1} - I_n \sim C_{q_2}$. Therefore, one can obtain $C - 2I_n \sim C_{q_2}$. Repeating these steps we find that there exists the sequence of monic polynomials $(q_i)_{1 \le i \le m-1}$ such that $C - iI_n \sim C_{q_i}$, for $i \in \{1, 2, \dots, m-1\}$.

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