m-NIL-CLEAN COMPANION MATRICES∗

A. CÎMPEAN†

Abstract. Companion matrices over fields of positive characteristic, \( p \), that are sums of \( m \) idempotents, \( m \geq 2 \), and a nilpotent are characterized in terms of dimension and trace of such a matrix and of \( p \).

Key words. Companion matrix, Idempotent, Nilpotent, \( m \)-nil-clean.

AMS subject classifications. 15A24, 15A83, 16U99.

1. Introduction. In [10], clean rings and clean elements in rings were introduced, in order to study some properties of direct decompositions of modules. Clean elements are sums of a unit and an idempotent element of the ring and a clean ring is such that all of its elements are clean. A particular class of clean rings was introduced by Diesl in [8]: the class of rings such that all elements are sums of a nilpotent and an idempotent. Other generalizations were considered in [6] and [4]. In the former, Chen and Sheibani considered 2-nil-clean rings, rings such that all elements are 2-nil-clean, i.e., elements that are sums of two idempotents and a nilpotent element. Weakly nil-clean rings were firstly introduced in the commutative case by Danchev and McGovern, in [7]. Breaz, Danchev and Zhou characterized weakly nil-clean rings in [4]. In the case of these rings, each element is a sum or a difference of a nilpotent and an idempotent. Moreover, in [1], the author studies elements which are sums of a nilpotent and \( m \) idempotents which commute.

It was proven in [9] that matrix rings over clean rings are clean. Matrix rings over nil-clean rings were studied in [8] and [3]. In the latter, it was proven that a matrix ring over a commutative nil-clean ring is nil-clean. Nil-clean matrices over general fields were studied in [5], where the authors study nil-clean companion matrices. We note that this can be an important step in a possible attempt to characterize general nil-clean matrices since every matrix is similar to a direct sum of companion matrices.

Using this idea we will study in this paper \( m \)-nil-clean companion matrices over fields of positive characteristic. Let \( m \geq 2 \) be an integer. Then an \( m \)-nil-clean element of a ring is an element that represents the sum of \( m \) idempotents and a nilpotent (of that ring). In Theorem 3.4, we characterize \( m \)-nil-clean companion matrices by using the dimension and the trace of such a matrix and the characteristic of the field.

Let \( \mathbb{F} \) be a field of positive characteristic, \( p \). Let \( q \) be a monic polynomial over \( \mathbb{F} \), \( q = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0 \). The companion matrix associated to \( q \) is the \( n \times n \) matrix

\[
C = C_{c_0, c_1, \ldots, c_{n-1}} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -c_0 \\
1 & 0 & \cdots & 0 & -c_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -c_{n-1}
\end{pmatrix}.
\]

We also denote \( C \) by \( C_q \).

Received by the editors on March 18, 2019. Accepted for publication on October 11, 2019. Handling Editor: Sergey Sergeev.

†Babeş-Bolyai University, Faculty of Mathematics and Computer Science, Str. Mihail Kogălniceanu 1, 400084, Cluj-Napoca, Romania (cimpeanandrada@yahoo.com).

∗Received by the editors on March 18, 2019. Accepted for publication on October 11, 2019. Handling Editor: Sergey Sergeev.
m-nil-clean Companion Matrices

2. Useful tools. Any matrix is similar to a Frobenius normal form (a direct sum of companion matrices), a matrix similar to a nilpotent is nilpotent and a matrix similar to an idempotent is idempotent. This is why in the proof of Theorem 3 of [2], one restricts without loss of generality to the case of companion matrices. The same technique is used in [6] to prove that $M_n(F_3)$ is 2-nil-clean. We are also determined to consider $m$-nil-clean companion matrices, based on the previously mentioned facts and on the fact that, if all companion matrices which appear in the Frobenius normal form of a matrix $A$ are $m$-nil-clean, then $A$ is also $m$-nil-clean.

Let $F$ be a field of positive characteristic $p$. In [5], an investigation was made for nil-clean companion matrices over $F$.

**Theorem 2.1.** Let $F$ be a field of positive characteristic $p$. Let $C = C_{c_0,c_1,\ldots,c_{n-1}} \in M_n(F)$ be a companion matrix. The following are equivalent:

1. $C$ is nil-clean.
2. One of the following conditions is true:
   (a) $C$ is nilpotent (i.e., $c_0 = \cdots = c_{n-1} = 0$);
   (b) $C$ is unipotent (i.e., $c_i = (-1)^i \binom{n}{n-i}$ for all $i \in \{0,\ldots,n-1\}$);
   (c) there exists an integer $k \in \{1,\ldots,p\}$ such that $-c_{n-1} = k \cdot 1$ and $n > k$.

As a consequence of this fact, the following result holds for characteristic 2:

**Corollary 2.2.** Let $F$ be a field of characteristic 2. Let $C = C_{c_1,\ldots,c_{n-1}} \in M_n(F)$ be a companion matrix. Then $C$ is nil-clean if and only if $-c_{n-1} \in \{0,1\}$.

Here is another corollary of the above theorem:

**Corollary 2.3.** Let $n \geq 3$ be a positive integer. The following are equivalent for a field $F$:

1. $F \equiv F_p$ for a prime $p < n$;
2. every companion matrix $C \in M_n(F)$ is nil-clean.

In [2] and [11], the authors use some decompositions which involve matrices that are the sum of a diagonal matrix with entries only 0 and 1 and a companion matrix. In the following, we will use a similar technique.

The following lemma will be useful while proving results on $m$-nil-clean companion matrices. We will use the notation: $\text{Lin}(\{v_1,v_2,\ldots,v_n\})$ for the subspace of a vector space $X$ generated by the set of vectors $v_1,v_2,\ldots,v_n$ of $X$.

**Lemma 2.4.** Let $F$ be a field. For every companion matrix $C_q \in M_n(F)$ and every $k \in \{1,\ldots,n\}$, there exists a companion matrix $C_q'$ such that $C_q$ and $\text{diag}(1,\ldots,1,0,\ldots,0) + C_q'$ are similar.

**Proof.** First we will prove the statement for $k \in \{1,2,\ldots,n-1\}$. Let $V$ denote the $n-$dimensional vector space of columns over $F$ and consider $C_q$ as an endomorphism $C_q : V \to V$. Denoting by $\{e_1,e_2,\ldots,e_n\}$ the standard basis of $V$, $C_q$ maps each $e_i$ to $e_{i+1}$, for each $i \in \{1,2,\ldots,n-1\}$.

Now we define $\{f_1,f_2,\ldots,f_n\}$, $f_i \in V$, $i \in \{1,2,\ldots,n-1\}$, inductively as it follows. First set $f_1 = e_1$. Assuming that $2 \leq i \leq n$ and that $f_{i-1}$ has been defined, set $f_i = C_q(f_{i-1}) - f_{i-1}$, if $i \in \{1,2,\ldots,k+1\}$ and $f_i = C_q(f_{i-1})$, if $i \in \{k+2,\ldots,n\}$
We have $e_1 = f_1$, so $e_1 \in \text{Lin}(\{f_1\})$ and $f_2 = C_q(f_1) - f_1 = C_q(e_1) - f_1 = e_2 - f_1$, so $e_2 = f_1 + f_2$ and $e_2 \in \text{Lin}(\{f_1, f_2\})$.

It is easy to see that each $f_i$ is the sum of $e_i$ and a linear combination of $e_{i-1}, e_{i-2}, \ldots, e_2, e_1$. Hence, $e_i$ is the difference of $f_i$ and a linear combination of $e_{i-1}, e_{i-2}, \ldots, e_2, e_1$. Assuming $e_1, e_2, \ldots, e_{i-1} \in \text{Lin}(\{f_1, f_2, \ldots, f_n\})$, we get $e_i$ is a linear combination of $f_1, f_2, \ldots, f_n$. Therefore, $\text{Lin}(\{e_1, e_2, \ldots, e_n\}) = \text{Lin}(\{f_1, f_2, \ldots, f_n\})$, and thus, $\{f_1, f_2, \ldots, f_n\}$ is a basis of $V$.

Moreover, by the definition, we have:

\[
C_q(f_1) = f_1 + f_2, \\
C_q(f_2) = f_2 + f_3, \\
\vdots \quad \vdots \quad \vdots \\
C_q(f_k) = f_k + f_{k+1}, \\
C_q(f_{k+1}) = f_{k+2}, \\
\vdots \quad \vdots \quad \vdots \\
C_q(f_{n-1}) = f_n.
\]

Let $M$ be the matrix the endomorphism $C_q$ corresponds to, with respect to the basis $B = \{f_1, f_2, \ldots, f_n\}$. Therefore,

\[
M = \begin{bmatrix}
|C_q(f_1)|_B, |C_q(f_2)|_B, \ldots, |C_q(f_k)|_B, |C_q(f_{k+1})|_B, \ldots, |C_q(f_{n-1})|_B, |C_q(f_n)|_B
\end{bmatrix}.
\]

Hence,

\[
M = \begin{bmatrix}
|f_1 + f_2|_B, |f_2 + f_3|_B, \ldots, |f_k + f_{k+1}|_B, |f_{k+2}|_B, \ldots, |f_n|_B, |C_q(f_n)|_B
\end{bmatrix}.
\]

It follows that $M = \text{diag}(1, \ldots, 1, 0, \ldots, 0) + C_q'$ for some monic polynomial $q'$ of degree $n$. So, $C_q = P^{-1}(\text{diag}(1, \ldots, 1, 0, \ldots, 0) + C_q')P$, where $P$ is the transition matrix mapping each $e_i$ to $f_i$.

As next step, we will solve the case $k = n$, that is we will prove that $C$ is similar to $I_n + C_{q'}$, where $q' = q'' + X^{n-1}$, and $q''$ is such that $\text{diag}(1, \ldots, 1, 0) + C_{q''}$ is similar to $C$. We consider the vector space of column vectors of dimension $n$ over $F$. Let $P$ be the transition matrix from canonical basis to basis $B$ defined in the first part of this lemma, taking $C_q = C$. Then $P(C_q - I_n)P^{-1} = PC_qP^{-1} - I_n = \text{diag}(1, \ldots, 1, 0) + C_{q'} - I_n = C_{q''}$, where $q'' = q'' + X^{n-1}$. Therefore, $C - I_n$ is similar to $C_{q''}$, where $q' = q'' + X^{n-1}$, and $q''$ is such that $\text{diag}(1, \ldots, 1, 0) + C_{q''}$ is similar to $C$. Hence, $C_q$ is similar to $I_n + C_{q''}$.

**Example 2.5.** For $p = 11$, $n = 6$, $k = 5$, $q = X^6 + 3X^5 - X^3 - X^2 - X - 1$ we want to see what matrix of the type $\text{diag}(1, \ldots, 1, 0, \ldots, 0) + C_{q'}$ is similar to $C_q$. We want to express the vectors in the basis $B = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ as linear combinations of vectors in the canonical basis and vectors in the canonical basis as linear combinations of vectors in basis $B$ in order to find out the transition matrix from canonical basis to basis $B$ and its inverse ($P$ and $P^{-1}$).
m-nil-clean Companion Matrices

After doing this we obtain
\[
P = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 3 & 6 \\
0 & 0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
P^{-1} = \begin{pmatrix}
1 & -1 & -1 & -1 & -1 \\
0 & 1 & -2 & 3 & -4 \\
0 & 0 & 1 & -3 & 6 \\
0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Since \( C_q = P^{-1}(\text{diag}(1,1,1,1,0) + C_{q'})P \), we have
\[
C_q = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -3
\end{pmatrix}
\sim \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 3 \\
0 & 0 & 1 & 1 & 6 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
= \text{diag}(1,1,1,1,0) + C_{q'}.
\]

3. m-nil-clean companion matrices. As a consequence of Theorem 2.1, if \( n > p \), then every \( n \times n \) companion matrix over \( \mathbb{F} \) is nil-clean, where \( \mathbb{F} \) is a field of positive characteristic \( p \). Therefore, we will assume that \( n \leq p \).

Secondly, for \( n = 1 \) the only nilpotent of \( M_n(\mathbb{F}) \) is \( (0) \) and the only idempotents of \( M_n(\mathbb{F}) \) are \( (0) \) and \( (1) \). Therefore, \( C \in M_1(\mathbb{F}) \) is m-nil-clean if and only if \( C \in \{ (0), (1), (2), \ldots, (m) \} \). Hence, we will not refer to the case \( n = 1 \), so we will assume \( n > 1 \) from now on.

**Lemma 3.1.** Let \( m \geq 2 \) be an integer. Let \( \mathbb{F} \) be a field of positive characteristic \( p \). Let \( A \in M_n(\mathbb{F}) \) be a (not necessarily companion) matrix, for which there exists the decomposition \( A = E_1 + E_2 + \cdots + E_m + N \), with \( k_i = \text{rank}(E_i) \), \( E_i \) idempotent, \( i \in \{1,2,\ldots,m\} \) and \( N \) is nilpotent. Then there is an integer \( c \) such that \( \text{trace}(A) = c \cdot 1 \), and \( c = k_1 + k_2 + \cdots + k_m \) (mod \( p \)) and each \( k_i \) is a natural number less than or equal to \( n \).

**Proof.** If \( A = E_1 + E_2 + \cdots + E_m + N \), \( E_i \) idempotent, \( i \in \{1,\ldots,m\} \) and \( N \) is nilpotent, then \( \text{trace}(A) = \text{trace}(E_1) + \text{trace}(E_2) + \cdots + \text{trace}(E_m) + \text{trace}(N) \). It follows that \( \text{trace}(A) = \text{trace}(E_1) + \text{trace}(E_2) + \cdots + \text{trace}(E_m) \). Moreover, it is known that if \( E \) is an idempotent, then \( \text{trace}(E) = \text{rank}(E) \cdot 1 \). So \( \text{trace}(A) = (k_1 + k_2 + \cdots + k_m) \cdot 1 \), and \( k_i \leq n, i \in \{1,2,\ldots,m\} \).

**Lemma 3.2.** Let \( \mathbb{F} \) be a field of positive characteristic \( p \), \( 1 < n \leq p \). If \( -c_{n-1} = c \cdot 1 \) and \( c \in \{ 2,3,\ldots,2n-2,2n-1 \} \), then \( C = C_{c_0,c_1,\ldots,c_{n-1}} \in M_n(\mathbb{F}) \) is 2-nil-clean.

**Proof.** Since \( c \in \{ 2,3,\ldots,2n-1 \} \), there exist \( k \in \{ 1,2,\ldots,n \} \) and \( l \in \{ 1,2,\ldots,n-1 \} \) such that \( c \cdot 1 = (k + l) \cdot 1 \). Let \( C = C_q \), where \( q = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0 \). Then, by Lemma 2.4, there exists \( q' = X^n + d_{n-1}X^{n-1} + \cdots + d_1X + d_0 \), \( q' \in \mathbb{F}[X] \) such that
\[
C_q \sim \text{diag}(1,\ldots,1,0,\ldots,0) + C_{d_0,d_1,\ldots,d_{n-1}}.
\]

Hence, it follows \(-c_{n-1} = k - d_{n-1} \) and so \(-d_{n-1} = l \cdot 1 \), and we know \( n > l \) and \( l \neq 0 \). Now, by Theorem 2.1, it follows that there exist an idempotent matrix \( E \) and a nilpotent matrix \( N \) such that \( C_{d_0,d_1,\ldots,d_{n-1}} = E + N \). Using this and equation 3.1 we have \( C_q \sim \text{diag}(1,\ldots,1,0,\ldots,0) + E + N \). We know
that $C_q = P^{-1} \left( \text{diag}(1, \ldots, 1, 0, \ldots, 0) + C_{d_0, d_1, \ldots, d_{n-1}} \right) P$, where $P$ is the transition matrix mapping each $e_i$ of canonical basis to $f_i$ of the basis met in Lemma 2.4. Now we have

$$C_q = P^{-1} \left( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \right) P + P^{-1} EP + P^{-1} NP. \quad (3.2)$$

Since a matrix similar to an idempotent is idempotent and a matrix similar to a nilpotent is nilpotent, we have actually showed the fact that $C_q$ is $2$-nil-clean.

**Lemma 3.3.** Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive characteristic $p$, and $1 < n \leq p$. If $-c_{n-1} = c \cdot 1$ and $c \in \{m, m+1, \ldots, mn-1\}$, then $C = C_{c_0, c_1, \ldots, c_{n-1}} \in M_n(\mathbb{F})$ is $m$-nil-clean.

**Proof.** This lemma’s statement is proved for $m = 2$ in Lemma 3.2.

Now assume that the lemma’s statement is true for $m \geq 2$. We will prove that it is true also for $m + 1$. Let

$$c \in \{m + 1, m + 2, \ldots, (m+1)n-1\}.$$ 

We will prove that $C$ is $(m+1)$-nil-clean.

Since $c \in \{m + 1, m + 2, \ldots, (m+1)n-1\}$, it follows that there exist $k \in \{1, 2, \ldots, n\}$ and $l \in \{m, m+1, \ldots, mn-1\}$ such that $c = (k+l) \cdot 1$. Let $C = C_q$, $q = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$. Then by Lemma 2.4, there exists $q' = X^n + d_{n-1}X^{n-1} + \cdots + d_1X + d_0$, $q' \in \mathbb{F}[X]$, such that

$$C_q \sim \text{diag}(1, \ldots, 1, 0, \ldots, 0) + C_{d_0, d_1, \ldots, d_{n-1}}. \quad (3.3)$$

It follows $-c_{n-1} = k - d_{n-1}$ and so $-d_{n-1} = l \cdot 1$. We know that $l \in \{m, m+1, \ldots, mn-1\}$, so by hypothesis induction we obtain that there exist idempotent matrices $E_1, E_2, \ldots, E_m$, and the nilpotent matrix $N$, such that $C_{d_0, d_1, \ldots, d_{n-1}} = E_1 + E_2 + \cdots + E_m + N$. Using this and equation 3.3 we have

$$C_q \sim \text{diag}(1, \ldots, 1, 0, \ldots, 0) + E_1 + E_2 + \cdots + E_m + N.$$

We know that there exists an invertible matrix $P$ such that

$$C_q = P^{-1} \left( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \right) P + P^{-1} E_1 P + P^{-1} E_2 P + \cdots + P^{-1} E_m P + P^{-1} NP.$$

Therefore,

$$C_q = P^{-1} \left( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \right) P + P^{-1} E_1 P + P^{-1} E_2 P + \cdots + P^{-1} E_m P + P^{-1} NP.$$

Since a matrix similar to an idempotent is idempotent and a matrix similar to a nilpotent is nilpotent, we have actually showed the fact that $C_q$ is $(m+1)$-nil-clean.

Having the case $p = 2$ solved in Corollary 2.2 ($C$ is nil-clean in this case, so is also $m$-nil-clean), we can assume from now on that $p$ is odd.

**Theorem 3.4.** Let $m \geq 2$ be an integer. Let $\mathbb{F}$ be a field of positive odd characteristic $p$, and $1 < n \leq p$. Let $C = C_{c_0, c_1, \ldots, c_{n-1}}$ be a companion matrix. Let $c$ be the smallest nonnegative integer such that $-c_{n-1} = c \cdot 1$.

The following hold:

1. If $c = 0$ and $mn - 1 < p$, then $C$ is $m$-nil-clean if and only if $C$ is nilpotent or $C - (m-1)I_n$ is unipotent.
2. If \( c = t, 1 \leq t \leq m \), then \( C \) is \( t \)-nil-clean (1-nil-clean is just nil-clean), so is \( m \)-nil-clean.
3. If \( c \in \{m, m + 1, \ldots, mn - 2, mn - 1\} \), then \( C \) is \( m \)-nil-clean.
4. If \( mn - 2 \geq p \), then \( C \) is \( m \)-nil-clean.
5. Assume that \( mn - 2 < p \).
   (a) If \( c = mn \) and \( p = mn - 1 \), then \( C \) is nil-clean, so is \( m \)-nil-clean.
   (b) If \( c = mn \) and \( p = mn \), then \( C \) is \( m \)-nil-clean if and only if \( C \) is nilpotent or \( C - (m - 1)I_n \) is an unipotent matrix.
   (c) If \( c = mn, p > mn \), then \( C \) is \( m \)-nil-clean if and only if \( C - (m - 1)I_n \) is an unipotent matrix.
   (d) If \( c > mn \), then \( C \) is not \( m \)-nil-clean.

Proof.

1. If \( c = 0 \) and \( mn - 1 < p \), then \( c = 0 \) and \( mn \leq p \).
   If \( mn < p \), then \( c = (0 + \cdots + 0) \) is the only form of \( c \) as sum, modulo \( p \) of \( m \) positive integers less or equal to \( n \), and since \( C \) is \( m \)-nil-clean it follows that the idempotents in the \( m \)-nil-clean decomposition of \( C \) have rank zero. Therefore, they are \( O_n \), and hence, \( C \) is nilpotent.
   If \( mn = p \), then \( c = (0 + \cdots + 0) \) and \( c = (n + \cdots + n) \) are the only forms of \( c \) as a sum modulo \( p \) of \( m \) positive integers less or equal to \( n \), and, since \( C \) is \( m \)-nil-clean, it follows \( C \) is nilpotent or \( C - (m - 1)I_n \) is unipotent.
   It is obvious that if \( C \) is nilpotent or \( C - (m - 1)I_n \) is unipotent, then \( C \) is \( m \)-nil-clean.
2. If \( c = 1 \) and \( n > 1 \), then we have by Theorem 2.1 that \( C \) is nil-clean. If \( c = t, 2 \leq t \leq m \), by Lemma 3.3, we have that \( C \) is \( t \)-nil-clean.
3. This is Lemma 3.3.
4. This is a direct consequence of 3.
5. (a) If \( c = mn \) and \( p = mn - 1 \), it follows that \( -c_{n-1} = 1 \cdot 1 \), so \( C \) is nil-clean.
   (b) Since \( c = mn, p = mn \) it follows that \( c = 0 \) and \( mn - 1 < p \), so by 1, we have that \( C \) is \( m \)-nil-clean if and only if \( C \) is nilpotent or \( C - (m - 1)I_n \) is unipotent.
   (c) Since \( c = mn, p > mn \), it follows that \( -c_{n-1} = mn = (n + n + \cdots + n)(mod p) \), that is the only decomposition as a sum of \( m \) integers between 0 and \( n \). Hence, the idempotents in the \( m \)-nil-clean decomposition of \( C \) are idempotent units, which means they are \( I_n \), so \( C - (m - 1)I_n \) is unipotent. Conversely, if \( C - (m - 1)I_n \) is unipotent, then it is obvious that \( C \) is \( m \)-nil-clean.
   (d) If \( c > mn \), then \( c \) cannot be written modulo \( p \) as sum of \( m \) integers less or equal to \( n \), but by Lemma 3.1, we have that \( C \) is not \( m \)-nil-clean. 

Remark 3.5. In Theorem 3.4, one of the conclusions was that \( C - (m - 1)I_n \) is unipotent. We can prove that \( C - (m - 1)I_n \) is similar to a companion matrix, so one can say more, the companion matrix with whom \( C - (m - 1)I_n \) is similar can be only the unipotent companion matrix of type \( n \times n \).

Let us prove that \( C - (m - 1)I_n \) is similar to a companion matrix. To begin with, we have by Lemma 2.4, case \( k = n \), that there exists the monic polynomial \( q_1 \) of degree \( n \) such that \( C - I_n \sim C_{q_1} \). But using again Lemma 2.4, case \( k = n \), we get \( C_{q_1} - I_n \sim C_{q_2} \). Therefore, one can obtain \( C - 2I_n \sim C_{q_2} \). Repeating these steps we find that there exists the sequence of monic polynomials \( (q_i)_{1 \leq i \leq m-1} \) such that \( C - iI_n \sim C_{q_i} \), for \( i \in \{1, 2, \ldots, m-1\} \).
REFERENCES