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INERTIA SETS OF SEMICLIQUED GRAPHS*

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Abstract. In this paper, we investigate inertia sets of simple connected undirected graphs. The main focus is on the shape of their corresponding inertia tables, in particular whether or not they are trapezoidal. This paper introduces a special family of graphs created from any given graph, G, coined semicliqued graphs and denoted $\widetilde{K}G$. We establish the minimum rank and inertia sets of some $\widetilde{K}G$ in relation to the original graph G. For special classes of graphs, G, it can be shown that the inertia set of G is a subset of the inertia set of $\widetilde{K}G$. We provide the inertia sets for semicliqued cycles, paths, stars, complete graphs, and for a class of trees. In addition, we establish an inertia set bound for semicliqued complete bipartite graphs.

Key words. Combinatorial Matrix Theory, Graph, Inertia, Minimum Rank, Symmetric.

AMS subject classifications. 05C50, 15B57, 15A18.

1. Introduction. The inverse inertia problem seeks to find which inertias of a graph, G, can be obtained from a matrix in $\mathcal{S}(G)$, the set of real symmetric matrices corresponding to G. Motivated by past results, [1, 2, 3, 4, 9], we investigate what inertias can be attained by a special family of graphs we define as semicliqued graphs.

In [4], they introduce a family of graphs called clique-stars, denoted $K_m \vee nK_1$ or $KS_{m,n}$. This paper strongly motivated a broader study into the types of graphs investigated in our paper. These clique-stars can now be classified as a semicliqued star and their results fold nicely into the more general results for semicliqued graphs found in this paper.

The main focus of this paper is to determine the relationship of the minimum rank, inertia set, and the clique cover number of a graph and its corresponding semicliqued graphs. In addition, we investigate the shape of the inertia tables for the original graphs compared to the shapes formed from its semicliqued graphs.

2. Definitions and Notations. A *graph* is a pair $G = (V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. In this paper, each graph is connected, simple, undirected, and finite, and has a nonempty vertex set. The *order* of a graph G, denoted $|G|$, is the number of vertices of G. In this paper, we require $|G| \geq 2$. If G is a graph, then $\mathcal{S}(G)$ is the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ for which $a_{i,j} \neq 0$, $i < j$ if and only if $\{i, j\} \in E(G)$. No restrictions are placed on the diagonal entries.

Given a matrix A, the scalar λ, and a nonzero vector $\mathbf{p}$, which satisfy $A\mathbf{p} = \lambda\mathbf{p}$, λ is an *eigenvalue* of A. Given a symmetric matrix, A, the *inertia* of A is the ordered triple: $(\pi(A), \nu(A), \delta(A))$. $\pi(A)$ is the number of positive eigenvalues of A, $\nu(A)$ is the number of negative eigenvalues of A, and $\delta(A)$ is the multiplicity of 0 as an eigenvalue of A. For a given matrix, $A \in \mathcal{S}(G)$, where $S(G)$ is the set of symmetric matrices corresponding to G, the *partial inertia* of that matrix is the pair $(\pi(A), \nu(A))$. The *inertia set* for G, denoted

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$\mathcal{I}(G)$, is the set of all partial inertias for the matrices in $\mathcal{S}(G)$. This set can be viewed as a subset of the integer lattice in the plane which is called an *inertia table*. The smallest value of $r+s$, $(r,s) \in \mathcal{I}(G)$ is called the *minimum rank* of G and is denoted $mr(G)$. The *minimum rank line* of a graph G consists of all points $(\pi(A), \nu(A))$ such that $\pi(A) + \nu(A) = mr(G)$. We employ the T *notation* to describe this initial table. The T *notation* has the form:

$$T^k_{[m,n]} = \{(r,s) \in \mathbb{N}^2 | m \le r+s \le n \text{ and } k \le r \le n, k \le s \le n\}$$

for some nonnegative integers k and $m \le n$. The value m represents the minimum rank, whereas n represents the order of G and k indicates the inset from the axes. For convenience, k can be left out of the notation when $k = 0$. A graph, G, has a *trapezoidal* inertia if $\mathcal{I}(G) = T_{[m,n]}$. It should be noted that for all G, $\mathcal{I}(G) \subseteq T_{[mr(G),n]}$.

A *path* is a graph $P_n = (\{v_1, v_2, \dots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : i = 1, 2, \dots, n-1\}$. A *cycle* is a graph $C_n = (\{v_1, v_2, \dots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : 1, 2, \dots, n-1\} \cup \{v_n, v_1\}\}$. A *complete graph* is a graph $K_n = (\{v_1, v_2, \dots, v_n\}, E)$ such that $E = \{\{v_i, v_j\} : 1 \le i < j \le n\}$. A *complete bipartite graph*, denoted $K_{m,n}$, is a graph whose vertices can be partitioned into two sets, V and V', where $V = \{v_j : 1 \le j \le m\}$, $V' = \{v_j : m+1 \le j \le m+n\}$ and $E(K_{m,n}) = \{\{v_i, v_j\} | v_i \in V, v_j \in V'\}$. A *star*, denoted $K_{1,n}$, is a complete bipartite graph with $m = 1$. A *clique* in a graph is a complete subgraph. A *clique cover*, sometimes referred to as an *edge clique cover*, is a set of cliques in G such that the union of these cliques contains every edge in G. The *clique cover number*, denoted $cc(G)$, is the minimum number of cliques needed for a *clique cover* of G.

A set of t distinct edges $\{i_1, j'_1\}$, $\{i_2, j'_2\}$,...,$\{i_t, j'_t\}$ in G, no two of which are adjacent, is said to be a *t-matching* between $\{i_1, \dots, i_t\}$ and $\{j'_1, \dots, j'_t\}$ if vertices $i_1, \dots, i_t$ are distinct, as well as vertices $j'_1, \dots, j'_t$. Such a t-matching in G is said to be *constrained* if it is the only t-matching in G between vertices $\{i_1, \dots, i_t\}$ and $\{j'_1, \dots, j'_t\}$.

Let F and G be graphs on at least two vertices, each with a vertex labeled v. Then, F $\oplus_v$ G is the graph on $|F| + |G| - 1$ vertices obtained by identifying the vertex v in F with the vertex v in G.

Let G be a graph with n vertices labeled $v_1, v_2, \dots, v_n$. A *semicliqued graph* of G, $\widetilde{K}G$, is a graph obtained from G by replacing each vertex j of G by a $K_{i_j}, i_j \ge 1$, such that for at least one $j \in \{1, 2, \dots, n\}, i_j \ge 2$ and whose edge set is

$$\bigcup_{j=1}^{n} E(K_{i_j}) \bigcup \{\{u,v\} | u \in V(K_{i_l}), v \in V(K_{i_m}), \{v_l, v_m\} \in E(G)\}.$$

In Figure 1, we have one possible semicliqued graph of G15, where $K_{i_1} = K_1$, $K_{i_2} = K_3$, $K_{i_3} = K_2$, and $K_{i_4} = K_1$.

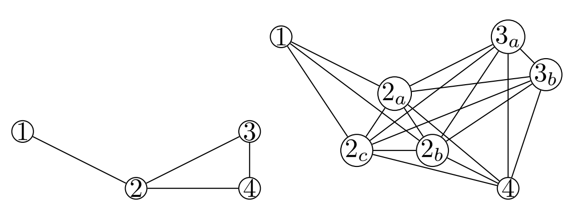

Figure 1: G15 from [10] and one of its $\widetilde{K}G15$

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3. Useful Techniques and Known Inertia Sets. This section provides the collection of previously established lemmas, theorems, observations, and techniques that are required in order to establish the main results of this paper. In Theorem 6.5 of [8], they established the following useful result, rephrased below.

THEOREM 1. *Let $A \in \mathcal{S}(G)$ and $t \in \mathbb{N}$. If there exist a constrained $t-$matching in G, then $rank(A) \geq t$.*

We proceed with a process that provides another bound for minimum rank.

Zero Forcing:

- *Color-change rule*:
  If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.
- Given a coloring of G, the *derived coloring* is the result of applying the color-change rule until no more changes are possible.
- A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.
- The *zero forcing number*, $Z(G)$, is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

With this process, we have the following bound on a graph's minimum rank:

LEMMA 2 ([9]). *For a graph G, $|G| - Z(G) \leq mr(G)$.*

We use the zero forcing number to establish a bound on the minimum rank. Next, we must examine which of the points on the minimum rank line are actually contained within the inertia of G. A common tool we implement is also discussed in [1].

To show that a given pattern has a particular inertia, say (k, l) in its inertia set, construct an explicit matrix with this inertia according to the following procedure. Let D be the $(k+l) \times (k+l)$ diagonal matrix with k +1's and l −1's on the diagonal. Then, any rank $k+l$ matrix of the form $B^T DB$ will have inertia (k, l). Given a graph G, determine a $(k+l) \times n$ matrix B such that, with respect to the indefinite inner product induced by D, columns i and j of B are orthogonal if and only if $\{i, j\}$ is not an edge of G. Choose a starting vertex (vertex 1) and assign it any vector (vector 1). Choose a second vertex and assign a vector such that vector 2 is orthogonal to vector 1 if and only if vertex 2 is nonadjacent to vertex 1. Continue so that at step p the pth vertex is assigned vector p such that for $j \in \{1, 2, \ldots, p-1\}$, vector p is orthogonal to vector j if and only if vertex p is nonadjacent to vertex j. The following example demonstrates how such matrices are constructed.

EXAMPLE 3. Let $G = G41$ as in [10] and Figure 2:

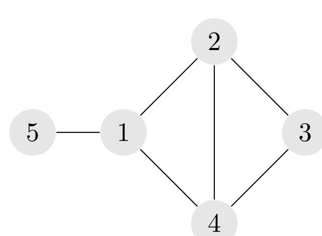

Figure 2: G41

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It is well known that G has minimum rank 3. To show the point $(2,1) \in \mathcal{I}(G)$, construct the following matrix, $A \in \mathcal{S}(G)$ by multiplying $B^T DB$ where:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 2 & 0 & 2 & 1 \\ 2 & 3 & -1 & 3 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 2 & 3 & -1 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Observe A has approximate eigenvalues 0, 0, 7.67, 1.74, and -1.42, confirming $(2,1) \in \mathcal{I}(G)$

Once we establish an actual point in the inertia set, there are many results we may use in order to determine the full inertia table of a given graph. The results listed below are of use for the purposes of this paper, but there are many more. Most are included in [3], [4], [7], and [9].

Lemma 4 (Northeast Lemma). *For a graph G with n vertices, $(p+1,q), (p,q+1) \in \mathcal{I}(G)$ if $(p,q) \in \mathcal{I}(G)$ and $p+q<n$.*

In Example 3, this lemma allows us to conclude that points such as $(3,1)$ and $(2,2)$ are contained in $\mathcal{I}(G)$. We use the notation, $S^{\nearrow}$, to indicate the set of points in S and all additional points northeast of any point $(p,q) \in S$.

Observation 5. *For a graph G, if $(p,q) \in \mathcal{I}(G)$, then $(q,p) \in \mathcal{I}(G)$.*

In Example 3, this lemma allows us to conclude that points such as $(1,2)$ and $(1,3)$ are contained in $\mathcal{I}(G)$.

In [3], they establish a direct method of finding the inertia of a given graph if it is of the form $F \oplus_v G$.

Theorem 6. *Let F and G be graphs on at least two vertices with a common vertex v and let $n = |F|+|G|-1$. Then:*

$$\mathcal{I}(F \oplus_v G) = [\mathcal{I}(F) + \mathcal{I}(G)]_n \bigcup [\mathcal{I}(F-v) + \mathcal{I}(G-v) + T^1_{[2,2]}]_n.$$

In [2], Lemma 4, the authors establish that the minimum rank of an induced subgraph is bounded above by the minimum rank of the original graph. We may extend this result to inertias in the following way.

Lemma 7. *If H is an induced subgraph of a graph G, then $\mathcal{I}(G) \subseteq \mathcal{I}(H)^{\nearrow}$.*

Note that G is an induced subgraph of any $\widetilde{K}G$. Thus, from Lemma 7, we observe the following.

Observation 8. *For a graph G, $\mathcal{I}(\widetilde{K}G) \subseteq \mathcal{I}(G)^{\nearrow}$ and $mr(G) \leq mr(\widetilde{K}G)$.*

Throughout this paper, we use Observation 8 in order to establish bounds on the minimum rank of $\widetilde{K}G$ as well as narrow our search for inertia points.

Observation 9. *For any graph G, $mr(G) \leq mr_+(G) \leq cc(G)$.*

In [4] Theorem 4.6, they prove the following very useful result.

Theorem 10. *Let G be a graph on n vertices. If $mr(G) = cc(G)$, then $\mathcal{I}(G) = T_{[mr(G),n]}$.*

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In this paper, we use Observation 9 and Theorem 10 to form a collection of semicliqued graphs which have trapezoidal inertia tables. The following well-known results establish the inertia tables for several common graphs. All are found in [4]. These inertias along with Observation 8 aid in our investigation of their corresponding cliqued and semicliqued graphs.

LEMMA 11. *For $m, n \in \mathbb{N}$ with $m \leq n$,*
$\mathcal{I}(P_n) = T_{[n-1,n]}$.
$\mathcal{I}(C_n) = T_{[n-2,n]}$.
$\mathcal{I}(K_n) = T_{[1,n]}$.
$\mathcal{I}(K_{1,n}) = T^1_{[2,n]} \cup T_{[n,n+1]}$.
$\mathcal{I}(K_{m,n}) = T^1_{[2,n]} \cup T_{[n,n+m]}$.

4. General Results for Semicliqued Graphs. The following results are derived from examining the relationship between a graph and any of its semicliqued graphs. Recall, in this paper, each graph is connected, simple, undirected, finite, and $|G| \geq 2$. We begin by establishing that the clique cover number of a semicliqued graph is equal to the clique cover number of the original graph. This establishes an upper bound for the minimum semidefinite rank of semicliqued graphs for such graphs where $cc(G)$ is known.

THEOREM 12. *For any graph, G, with $|G| \geq 2$, $cc(\widetilde{K}G) = cc(G)$.*

Proof. Let G be a graph such that $|G| = n \geq 2$ and $|E(G)| = s$. Let $|\widetilde{K}G| = p$. Observe, for any $\{j, k\} \in E(G)$, K_{i_j} and K_{i_k} form a possibly larger clique in $\widetilde{K}G$. Label this $K_{i_j+i_k}$. Let $Y_{(j,k)} = \{(E(K_{i_j+i_k}) \setminus E(K_{i_k})) \cap (E(K_{i_j+i_k}) \setminus E(K_{i_j}))\}$. Let $Y(\widetilde{K}G) = \{Y_{(a,b)} | \{a, b\} \in E(G)\}$. Observe, $|Y(\widetilde{K}G)| = s = |E(G)|$. So, there exists a bijection, $F : E(G) \to Y(\widetilde{K}G)$: $(p, q) \to Y_{(p,q)}$.

Suppose that $cc(G) = u$. Denote a set of complete graphs which minimally cover G as $\{K_{a_1}, \ldots, K_{a_u}\}$. Observe, since K_{a_v} is a complete graph which covers vertices $v_1, \ldots, v_q \in V(G)$, $K_{i_{v_1}+\cdots+i_{v_q}}$ covers $K_{i_{v_1}}, \ldots,$ $K_{i_{v_q}}$ and all of their mutually incident edges in $\widetilde{K}G$. Therefore, the clique cover for $\widetilde{K}G$ does not require more cliques than the clique cover of G. Hence, $cc(\widetilde{K}G) \leq cc(G)$.

If $cc(\widetilde{K}G) < cc(G)$, let $B = \{B_1, B_2, \ldots, B_m\}$ be a minimal clique cover of $\widetilde{K}G$ where $m < u$. Since, for any $\{j, k\} \in E(G)$, the subgraph induced by $V(K_{i_j}) \cup V(K_{i_k})$ forms clique in $\widetilde{K}G$. Again, labeling this $K_{i_j+i_k}$, any such minimum clique cover has at least two such K_{i_j} and K_{i_k} in each B_z. Let $t_z = \{v_j \in V(G) | K_{i_j} \subseteq B_z\}$ with $1 \leq z \leq m$. Then, $K_{|t_z|}$ covers the subgraph of G induced by t_z. If there exists an edge $\{c, d\} \in E(G)$ that is not covered by one of these $K_{|t_z|}$, then $Y_{(c,d)}$ is not covered by any of the B_z's. A contradiction. Hence, $K_{|t_1|}, K_{|t_2|}, \ldots, K_{|t_m|}$ covers G. So, $K_{a_1}, K_{a_2}, \ldots, K_{a_u}$ is not a minimal clique cover of G. Thus, implying $cc(\widetilde{K}G) \geq cc(G)$. Hence, $cc(G) = cc(\widetilde{K}G)$ ◻

The next lemma establishes a sufficient condition for a trapezoidal inertia table of any semicliqued graph. This result follows directly from Theorems 12 and 10.

LEMMA 13. *Let $\widetilde{K}G$ have order p. If $mr(G) = cc(G)$, then $\mathcal{I}(\widetilde{K}G) = T_{[mr(G),p]}$.*

Proof. Let G be a connected simple graph with $|G| = n \geq 2$ and $|\widetilde{K}G| = p$. Suppose $mr(G) = cc(G)$. By observation 8, $mr(G) \leq mr(\widetilde{K}G)$. By observation 9, $mr(\widetilde{K}G) \leq cc(\widetilde{K}G)$ which implies $cc(G) = mr(G) \leq mr(\widetilde{K}G) \leq cc(\widetilde{K}G)$. Theorem 12 states that $cc(\widetilde{K}G) = cc(G)$, hence $cc(G) = mr(G) \leq mr(\widetilde{K}G) \leq cc(\widetilde{K}G) = cc(G)$, giving $mr(\widetilde{K}G) = cc(\widetilde{K}G)$. Thus, by Theorem 10, $\mathcal{I}(\widetilde{K}G) = T_{[mr(G),p]}$. ◻

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The final theorem of this section establishes that all points that lie on the axes in the inertia table of G are also contained in the inertia table of $\widetilde{K}G$.

THEOREM 14. *If* $(m, 0) \in \mathcal{I}(G)$*, then* $(m, 0) \in \mathcal{I}(\widetilde{K}G)$*.*

Proof. Let G be a graph on n vertices such that $(m, 0) \in \mathcal{I}(G)$. For $A \in S(G)$ with inertia $(m, 0)$, there is an $m \times n$ matrix B with orthogonal rows and a diagonal matrix D such that $A = B^T DB$ where D has m positive eigenvalues. Now consider $\widetilde{K}G$, a semicliqued graph for G with order p. Let $u \in V(G)$ and K_{i_u} denote the clique in $\widetilde{K}G$ corresponding to u. Construct an $m \times p$ matrix, C, such that the column corresponding to u in B is repeated i_u times in C. Observe any relationships of orthogonality between columns j and k within B hold for corresponding i_j and i_k columns in C. Thus, $F \in \widetilde{K}G$ is formed by multiplying $C^T DC = F \in S(\widetilde{K}G)$. Hence, $(m, 0) \in \mathcal{I}(\widetilde{K}G)$. ◻

An immediate consequence of Theorem 14 and Observation 5 is that $(0, m)$ is also contained within the inertia set of $\widetilde{K}G$. Observe, this provides us with an upper bound for $mr(\widetilde{K}G)$. That is if $(m, 0) \in \mathcal{I}(G)$ then $mr(\widetilde{K}G) \leq m$.

5. Trapezoidal Semicliqued Inertia Tables.

5.1. Semicliqued Common Graphs. The result in Section 4 are now utilized to derive the following inertias of the semicliqued graphs constructed from common graphs. The following results establish the inertias for semicliqued graphs of K_n, P_n, $K_{1,n}$, and C_n. We end this section with a bound on the inertia table for semicliqued graphs $\widetilde{K}K_{m,n}$.

COROLLARY 15. $\mathcal{I}(\widetilde{K}K_n) = T_{[1,p]}$.

Proof. Let K_n be a complete graph on n vertices. Let $\widetilde{K}K_n$ be a semicliqued graph of K_n with order p. Observe, $mr(K_n) = cc(K_n) = 1$, so by Lemma 13, $\mathcal{I}(\widetilde{K}K_n) = T_{[1,p]}$. ◻

An alternative to the above proof is to observe $\widetilde{K}K_n$ is just isomorphic to K_p. Then by Lemma 11, $\mathcal{I}(\widetilde{K}K_n) = T_{[1,p]}$.

COROLLARY 16. $\mathcal{I}(\widetilde{K}P_n) = T_{[n-1,p]}$.

Proof. Let P_n, be a path on n vertices and let $\widetilde{K}P_n$ be a semicliqued graph of P_n with order p. Observe, $mr(P_n) = cc(P_n) = n - 1$ so by Lemma 13, $\mathcal{I}(\widetilde{K}P_n) = T_{[n-1,p]}$. ◻

THEOREM 17. $\mathcal{I}(\widetilde{K}C_n) = T_{[n-2,p]}$.

Proof. Let C_n be a cycle on n vertices. Note, $\mathcal{I}(C_n) = T_{[n-2,n]}$. Let $\widetilde{K}C_n$ be a semicliqued graph of order p. To show the point $(k, j) \in \mathcal{I}(\widetilde{K}C_n)$, where $k + j = n - 2$ for all $k \geq j$, create a diagonal matrix, D, with k positive ones and j negative ones on the diagonal. Construct an additional matrix, B, as found in Appendix A, under the following conditions: $b \neq 1$ and $b^2 \neq k - j + 1$. Lastly if $j = 1$, the expressions in row $k + 1$ must be implemented in the construction of B. In addition, the leading ones in each row correspond to each K_{i_j} in $\widetilde{K}C_n$. Multiplying $B^T DB = A$ where $A \in \mathcal{S}(\widetilde{K}C_n)$ with k positive and j negative eigenvalues. Thus, $(k, j) \in \mathcal{I}(\widetilde{K}C_n)$ and by Lemma 5, $(j, k) \in \mathcal{I}(\widetilde{K}C_n)$. Utilizing Lemma 4, $T_{[n-2,p]} \subseteq \mathcal{I}(\widetilde{K}C_n)$. Since C_n is a induced subgraph of $\widetilde{K}C_n$, by Lemma 7, $\mathcal{I}(\widetilde{K}C_n) \subseteq \mathcal{I}(C_n)^{\nearrow}$, which implies that $\mathcal{I}(\widetilde{K}C_n) \subset T_{[n-2,p]}$. Hence, $\mathcal{I}(\widetilde{K}C_n) = T_{[n-2,p]}$. ◻

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The following theorem establishes a lower bound for the minimum rank for semicliqued graphs of complete bipartite graphs. This in turn provides a bound for the inertia set of a $\widetilde{K}K_{m,n}$.

Theorem 18. *Let $\widetilde{K}K_{m,n}$ be a semicliqued graph of $K_{m,n}$ where $n \geq m$. Let S be the set of cliques of order 2 or more in the partite set in $\widetilde{K}K_{m,n}$ corresponding to m and R be the set of cliques of order 2 or more in the other partite set in $\widetilde{K}K_{m,n}$. If $|R| \geq m$, then $mr(\widetilde{K}K_{m,n}) \geq |R|$ and $\mathcal{I}(\widetilde{K}K_{m,n}) \subseteq T^1_{[|R|,n]} \bigcup T_{[n,p]}$. Otherwise, $mr(\widetilde{K}K_{m,n}) \geq |S|$ and $\mathcal{I}(\widetilde{K}K_{m,n}) \subseteq T^1_{[|S|,n]} \bigcup T_{[n,p]}$.*

Proof. Let $G = K_{m,n}$ where $n \geq m$. Label one partite set of vertices $1, 2, \ldots, m$ and the remaining partite set of vertices $m+1, m+2, \ldots, m+n$. Let $\widetilde{K}K_{m,n}$ be a semicliqued graph of G with order p. Let $K_{i_1}, K_{i_2}, \ldots, K_{i_m}$ correspond to vertices $1, 2, \ldots, m$ and $S = \{K_{i_j} | 1 \leq j \leq m \text{ and } |K_{i_j}| > 1\}$. Let $K_{i_{m+1}}, K_{i_{m+2}}, \ldots, K_{i_{m+n}}$ correspond to vertices $m+1, m+2, \ldots, m+n$ and $R = \{K_{i_j} | m+1 \leq j \leq m+n$ and $|K_{i_j}| > 1\}$. We proceed by finding a constrained matchings for $\widetilde{K}K_{m,n}$.

Suppose $|R| \geq m$. Then for each of the $K_{i_j} \in R$ match, two vertices contained in that K_{i_j}. This creates a constrained $|R|$-matching in $\widetilde{K}K_{m,n}$. Then by Theorem 1 $mr(\widetilde{K}K_{m,n}) \geq |R|$ and by Lemma 7, $\mathcal{I}(\widetilde{K}K_{m,n}) \subseteq T^1_{[|R|,n]} \bigcup T_{[n,p]}$.

Suppose $|R| < m$. Then for each of the $K_{i_j} \in S$ match, two vertices contained in that K_{i_j}. This creates a constrained $|S|$-matching in $\widetilde{K}K_{m,n}$. Then by Theorem 1 $mr(\widetilde{K}K_{m,n}) \geq |S|$ and by Lemma 7, $\mathcal{I}(\widetilde{K}K_{m,n}) \subseteq T^1_{[|S|,n]} \bigcup T_{[n,p]}$. ∎

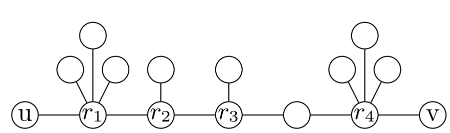
It should be noted when $m = 1$ this is a "clique-star" (now a semicliqued star) as defined in [4]. In that paper, they established that $mr(\widetilde{K}K_{1,n}) = 2$ and $\mathcal{I}(\widetilde{K}K_{1,n}) = T^1_{[2,n]} \bigcup T_{[n,p]}$. We may observe that Theorem 18 confirms that the bump of $T^1_{[2,n]}$ has the potential of reducing toward the trapezoidal region of $T_{[n,p]}$ as the set T is increased.

Corollary 19. *Let $\widetilde{K}K_{1,n}$ have order p where all of the pendent vertices correspond to $|K_{i_j}| > 1$ then $\mathcal{I}(\widetilde{K}K_{1,n}) = T_{[n,p]}$.*

Proof. Let $\widetilde{K}K_{1,n}$ be a semicliqued graph of $K_{1,n}$ with order p where all of the pendent vertices correspond to $|K_{i_j}| > 1$. Note $\mathcal{I}(K_{1,n}) = T^1_{[2,n]} \cup T_{[n,n+1]}$ and by Lemmas 4 and 7, $\mathcal{I}(\widetilde{K}K_{1,n}) \subseteq \mathcal{I}(K_{1,n})^{\nearrow} = T^1_{[2,n]} \cup T_{[n,n+1]}{}^{\nearrow}$. From Theorem 18, we know $mr(\widetilde{K}K_{1,n}) \geq n$ and $\mathcal{I}(\widetilde{K}K_{1,n}) \subseteq T_{[n,p]}$. Observe that $cc(\widetilde{K}K_{1,n}) = n$. Thus, $n \leq mr(\widetilde{K}K_{1,n}) \leq cc(\widetilde{K}K_{1,n}) = n$. Hence, by Theorem 10, $\mathcal{I}(\widetilde{K}K_{1,n}) = T_{[n,p]}$. ∎

5.2. Inertia Tables of uv-Starpaths. In this paper, a tree, T, is classified as a uv-starpath if it can be written as a series of consecutive adjoinings of the centers of stars with vertices, r_i, in the uv path, a longest path in the tree. We can represent such a tree as $T = P_k \bigoplus_{r_i} K_{1,n_i}$ where P_k corresponds to the uv-path, $1 \leq n_i$, and each K_{1,n_i} is joined at their center.

Consider the tree, T, below:



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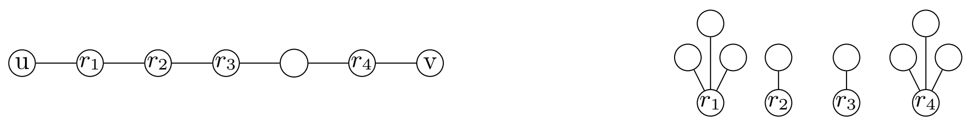
Note that this tree, T, is a uv-starpath because we can identify the uv path and the individual stars that were adjoined to this uv path. We can write $T = P_7 \oplus_{r_1} K_{1,3} \oplus_{r_2} K_{1,1} \oplus_{r_3} K_{1,1} \oplus_{r_4} K_{1,3}$. Below we can see the path and the stars that were adjoined to it.



It should be noted that no stars are adjoined to the endpoints of the uv path. If there was a star adjoined to one of the endpoints, you would get a tree such as the one on the left:

However, the initially identified uv path is no longer the longest path in the tree, so it can be redrawn to the tree on the right. This tree now has a new endpoint in the uv path. We proceed by utilizing zero forcing and clique covers. With these, we relate the zero forcing number of a semi-cliqued uv-starpath to the clique cover number of that semi-cliqued uv-starpath.

THEOREM 20. *Let T be a uv-starpath with longest path, P_j, and q stars adjoined to vertices on P_j. Let $\widetilde{K}T$ be a semi-cliqued graph of T with all star pendents cliqued. Then $\widetilde{K}T$ is trapezoidal.*

Proof. Let T be a uv-starpath with longest path, P_j, and q stars adjoined to vertices on P_j. Label the pendents of the adjoined star r_1 as $1, 2, \dots, s$. Label the pendents of the adjoined star r_2 as $s{+}1, s{+}2, \dots, s{+}t$. Proceed in this manner until the last adjoined star r_q has had its pendents labeled $l - x$ through l. Now label the vertices of the path P_j as $l+1, l+2, \dots, l+j$. Finally, label the vertices r_1 as $l+c_1$, r_2 as $l+c_2$, $\dots$, and r_q as $l+c_q$. Observe that $|V(T)| = j + l$ and $cc(T) = j + l - 1$. Create $\widetilde{K}T$ by cliqueing up all the pendents of the stars. Label these cliques $K_{i_1}, K_{i_2}, K_{i_3}, \dots, K_{i_l}$ with $i_m \geq 2$ for all $m \in 1, 2, 3, \dots, l$. So $|\widetilde{K}T| = j + i_1 + i_2 + \cdots + i_l$. Color $1 + (i_1 - 1) + (i_2 - 1) + \cdots + (i_l - 1)$ vertices black such that each of the $(i_m - 1)$ black vertices lie within the cliqued star K_{i_m} and the remaining black vertex on $l + 1$. Label the white vertex w_m and a black vertex b_m in each K_{i_m} for all $m \in 1, 2, 3, \dots, l$. Then the following chain occurs: $[l+1 \to l+2 \to \dots \to l+c_1, b_1 \to w_1, b_2 \to w_2, \dots, b_s \to w_s, l+c_1 \to l+c_1+1 \to \dots \to l+c_2, b_{s+1} \to w_{s+1}, \dots, b_{s+t} \to w_{s+t}, l+c_2 \to \dots \to l+c_q, b_{l-x} \to w_{l-x}, \dots, b_l \to w_l, l+c_q \to \dots \to l+j]$. This creates a zero forcing set of size $1 + (i_1 - 1) + (i_2 - 1) + \cdots + (i_l - 1)$. So $|\widetilde{K}T| - |Z(\widetilde{K}T)| \geq j + i_1 + i_2 + \cdots + i_l - (1 + (i_1 - 1) + (i_2 - 1) + \cdots + (i_l - 1)) = j + l - 1$. Thus, $j + l - 1 \leq |\widetilde{K}T| - |Z(\widetilde{K}T)| \leq mr(\widetilde{K}T) \leq cc(\widetilde{K}T) = cc(T) = j + l - 1$. Therefore, $mr(\widetilde{K}T) = j + l - 1$. Thus, since $mr(\widetilde{K}T) = cc(\widetilde{K}T) = j + l - 1$, then $\widetilde{K}T$ is trapezoidal. ∎

6. Nontrapezoidal Inertia Tables for Semicliqued Graphs. In the investigation of semicliqued graphs, for every connected simple graphs of 6 or fewer vertices that we checked, when every K_{i_j} has $i_j \geq 2$ (*fully cliqued*), then $\mathcal{I}(\widetilde{K}G)$ is trapezoidal. Most of the analysis was a direct implementation of Lemma 13, the remaining were confirmed by hand with the assistance of Maple. We also observed several occurrences where the inertia became trapezoidal before the graph was fully cliqued. We have yet to confirm but it also appears that a fully cliqued complete bipartite graph has trapezoidal inertia. A natural question arises, is this observation true for any graph G regardless of the shape of the initial inertia table? The answer turns out to be no and an example may be found with just seven vertices.

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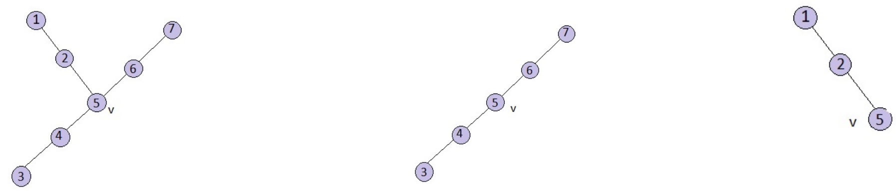
Consider the following graph, $S(2,2,2)$:



$S(2,2,2)$ P_5 P_3

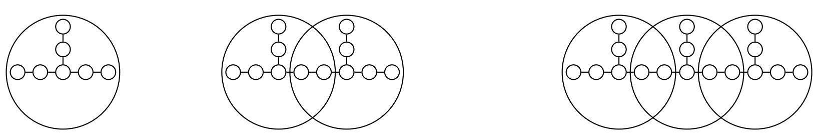
Observe that $S(2,2,2) = P_3 \oplus_v P_5$. Using the inertia equation in Theorem 6 along with the known inertias of paths from Lemma 11, the inertia for $S(2,2,2)$ is $T^1_{[5,6]} \cup T_{[6,7]}$.

What follows is a realization of a $\widetilde{K}S(2,2,2)$ where every K_{i_j} has $i_j \geq 2$:

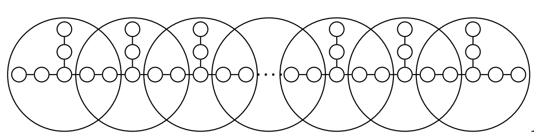
$$A = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&1&1&0&0&1&1&0&0 \\ 0&0&1&1&1&1&1&1&0&0&1&1&0&0 \\ 0&0&0&0&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{bmatrix}$$

Matrix A produces the spectrum, $\{-2,0,0,0,0,0,0,0,0,0,2,4,4,6\}$. This set corresponds to the partial inertia $(4,1)$ on the inertia table of $\widetilde{K}S(2,2,2)$. Thus, we have established that the $\widetilde{K}S(2,2,2)$ inertia table can achieve the same minimum rank as $S(2,2,2)$, namely 5. Observe if $\mathcal{I}(\widetilde{K}S(2,2,2))$ could achieve every point on the minimum rank line of 5, then $(5,0) \in \mathcal{I}(\widetilde{K}S(2,2,2))$. This would imply $\mathcal{I}(\widetilde{K}S(2,2,2)) \nsubseteq \mathcal{I}(S(2,2,2))^{\nearrow}$, contradicting observation 8. Since $\mathcal{I}(S(2,2,2))$ does not contain every point on the minimum rank line of 5, neither can $\mathcal{I}(\widetilde{K}S(2,2,2))$. Thus, $\mathcal{I}(\widetilde{K}S(2,2,2))$ is nontrapezoidal.

Naturally, the next question we seek to answer is if this is the only such graph that has this property? Consider the following construction of a family of trees we denoted N_k:



N_1 N_2 N_3



N_k

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Note k is the number of total $S(2,2,2)$s in the N_k. It can be confirmed using Theorem 6 that when each K_{i_j} has $i_j \geq 2$, $\mathcal{I}(\widetilde{K}N_k)$ is nontrapezoidal.

7. Conclusion. Within this paper, we discovered various properties of semicliqued graphs. As a result, we established the inertia tables of semicliqued common graphs. Many of these results utilize properties of the induced subgraph, G, to establish properties of $\widetilde{K}G$. The most substantial result being $cc(G) = cc(\widetilde{K}G)$.

In the investigation of uv-starpaths, we discovered exactly how many cliques are required and where to place those cliques to achieve a trapezoidal inertia. It would be interesting to find a more formal method for where cliques should be placed in any nontrapezoidal graph in order for its semicliqued graph to be trapezoidal. In general, discovering a minimum number of cliques that create a trapezoidal inertia table would be of interest as well.

Lastly, we found a family of trees, N_k, which seem to resist becoming trapezoidal regardless of how many cliques are placed in $\widetilde{K}N_k$. Are these the only ones? What underlying properties of these graphs is causing this and how may we use that to find more such graphs? We observed that this family of graph's have the same minimum rank regardless of how many cliques are used to form $\widetilde{K}G$. Thus, a natural question would be what other graphs maintain their minimum rank for any $\widetilde{K}G$? We also observed specific examples of the graphs that have the property if $mr(G) = mr(\widetilde{K}G)$ then $\mathcal{I}(\widetilde{K}G) = \mathcal{I}(G)^{\nearrow}$. Is this true anytime $mr(G) = mr(\widetilde{K}G)$?

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Appendix A
B matrix for Theorem 17

$$
\begin{array}{c|cccccccccccccccccccccccccccccc|ccc}
 & \leftarrow & i_1 & \rightarrow & \leftarrow & i_2 & \rightarrow & \leftarrow & i_3 & \rightarrow & \leftarrow & \cdots & \leftarrow & i_k & \rightarrow & \leftarrow & i_{k+1} & \rightarrow & \leftarrow & i_{k+2} & \rightarrow & \cdots & \cdots & \leftarrow & i_{k+j-2} & & \rightarrow & \leftarrow & i_{k+j-1} & & \rightarrow & \leftarrow & i_{k+j} & \rightarrow \\
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & (-1)^{k-1} & \cdots & (-1)^{k-1} \\
2 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & (-1)^{k-2} & \cdots & (-1)^{k-2} \\
3 & \vdots & \vdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & 1 & 0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & \vdots & -1 \\
k & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & 1 & b & \cdots & b & 0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 \\
\hline
k+1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & b & \cdots & b \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \cdots & 0 & b & \cdots & b & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & (-1)^{j-(j-1)} & \cdots & (-1)^{j-(j-1)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
n-3 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & (-1)^{j-2} & \vdots & (-1)^{j-2} \\
n-2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & (-1)^{j-1} & \cdots & (-1)^{j-1}
\end{array}
$$