Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 80-89, February 2020.

STRONGLY SELF-INVERSE WEIGHTED GRAPHS*

A. BERMAN†, N. SHAKED-MONDERER‡, AND S.K. PANDA§

Abstract. Let G be a connected, bipartite graph. Let G_{w} denote the weighted graph obtained from G by assigning weights to its edges using the positive weight function $\mathrm{w} : E(G) \to (0, \infty)$. In this article, a class $\mathcal{H}_{nmc}$ of bipartite graphs with unique perfect matchings and the family $\mathcal{W}_G$ of weight functions with weight 1 on the matching edges are considered. Then all pairs G in $\mathcal{H}_{nmc}$ and w in $\mathcal{W}_G$ such that G_{w} is strongly self-inverse are characterized.

Key words. Adjacency matrix, Weight function, Inverse graph, Strongly self-inverse, Property R, Property SR.

AMS subject classifications. 05C50, 15A18.

1. Introduction. Throughout this article, we consider finite bipartite weighted graphs. Without loss of generality, we may assume that the graphs are connected. A *perfect matching* in a graph G is a spanning forest whose components are paths on two vertices. If G has a unique perfect matching, we denote the matching by $\mathcal{M}$. We use v and v' to denote matching mates (the end vertices of a matching edge).

EXAMPLE 1.1. The paths P_2 and P_4 are bipartite graphs with a unique perfect matching. The path P_3 does not have a perfect matching, and the cycle C_4 is a bipartite graph with more than one perfect matching.

We denote by $\mathcal{H}$ the class of connected bipartite graphs with unique perfect matchings.

Let $G \in \mathcal{H}$. A *weight function* w is a function from $E(G)$, the set of edges of G, to $(0, \infty)$. We denote by G_{w} the weighted graph obtained from G by assigning weights to its edges using the weight function w. The graph G may be viewed as a weighted graph where each edge has weight 1. The adjacency matrix $A(G_{\mathrm{w}})$ of G_{w} is the square symmetric matrix whose (i, j)th entry a_{ij} is $\mathrm{w}([i, j])$ if i is adjacent to j and 0 otherwise. A weighted graph is called *nonsingular* if $A(G_{\mathrm{w}})$ is nonsingular. Furthermore, each weighted bipartite graph with a unique perfect matching is nonsingular, see [4].

A *signature* matrix is a diagonal matrix with diagonal entries from $\{1, -1\}$. Note that for any signature matrix S, we have $S^{-1} = S$. Let $G \in \mathcal{H}$. The weighted graph G_{w} is said to have an *inverse* if there is a signature matrix S such that $SA(G_{\mathrm{w}})^{-1}S$ is nonnegative. The weighted graph associated with $SA(G_{\mathrm{w}})^{-1}S$ is called the *inverse* graph of G_{w}, and it is denoted by G_{w}^{+}. The inverse of a graph was first introduced by Godsil [1]. The weighted version was introduced in [6].

In [13], the authors characterized all bipartite graphs (unweighted) with unique perfect matchings whose adjacency matrices have inverses diagonally similar to non-negative matrices, which settles an open problem of Godsil on inverses of bipartite graphs in [1]. However, the problem is still open for weighted bipartite graphs with unique perfect matchings. For more details about the graph inverse of weighted bipartite graphs

*Received by the editors on August 8, 2018. Accepted for publication on January 3, 2020. Handling Editor: Stephen J. Kirkland. Corresponding Author: Swarup Kumar Panda.

†Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel (berman@technion.ac.il).

‡The Max Stern Yezreel Valley College 19300, Israel (nomi@technion.ac.il).

§Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India (spanda@maths.iitkgp.ac.in).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 80-89, February 2020.

with unique perfect matchings, see [6, 9, 10]. The graph inverse is an interesting topic and has also been studied in [3, 5, 7, 9, 11, 12, 13, 14]

Let $G \in \mathcal{H}$, and let $\mathcal{M}$ be the unique perfect matching in G. We use $P_{\mathcal{M}}$ to denote the symmetric permutation matrix $P = [p_{ij}]$ given by the matching, where $p_{ij} = 1$ if $[i, j] \in \mathcal{M}$ and $p_{ij} = 0$, otherwise. We use $f_{\mathcal{M}}$ to denote the corresponding permutation mapping on the set of vertices of G.

Definition 1.2. A weighted graph G_{w} is said to be *self-inverse* if there is a permutation matrix P such that $P^{-1}A(G_{\mathrm{w}}^{+})P = A(G_{\mathrm{w}})$. If $P = P_{\mathcal{M}}$, then the graph G_{w} is called *strongly self-inverse*. That is, G_{w} is strongly self-inverse if $f_{\mathcal{M}}$ is an isomorphism between G_{w} and G_{w}^{+}.

Characterizing the self-inverse bipartite graphs with unique perfect matchings is also an interesting problem which was posed by Godsil in [1]. The notion of a strongly self-inverse graph was introduced in [12]. In this article, we further study such graphs.

The following example shows that weight does matter in deciding whether a weighted graph is strongly self-inverse or not.

Example 1.3. Consider the graph G shown in Figure 1 where $\mathcal{M} = \{[1, 1'], [2, 2'], [3, 3']\}$(the solid edges are the matching edges). Consider the weight function w such that $\mathrm{w}([i, i']) = 1$ for $i = 1, 2, 3$, $\mathrm{w}([i', i+1]) = 1$ for $i = 1, 2$ and $\mathrm{w}([1', 3]) = 1/2$. Take the signature matrix S such that $S_{11} = S_{1'1'}=1$, $S_{22} = S_{2'2'} = -1$ and $S_{33} = S_{3'3'} = 1$. Then $P_{\mathcal{M}}^{-1} S A(G_{\mathrm{w}})^{-1} S P_{\mathcal{M}} = A(G_{\mathrm{w}})$. Hence, G_{w} is strongly self-inverse. If we consider the weight function $\mathrm{w} \equiv \mathbb{1}$ (assigning 1 to each edge), then G^{+} is a tree. Therefore, G is not strongly self-inverse.

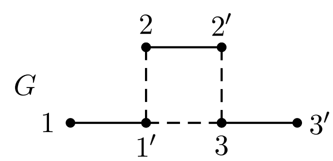

Figure 1. *Here, the solid edges are the matching edges.*

Convention: Let $G \in \mathcal{H}$. We use $\mathcal{W}_G$ to denote the set of all weight functions w such that $\mathrm{w}(e) = 1$ for each matching edge in G.

We proceed in the following way. In Section 3, we present a new subclass of graphs, generalized boxminus corona (GBC). This subclass of graphs in $\mathcal{H}$ is properly contained in $\mathcal{H}_{nmc}$ and properly contains the bipartite corona graphs. For every G in GBC we completely characterize all the weight functions w in $\mathcal{W}_G$, for which G_{w} is strongly self-inverse. This result advances knowledge in the area of graph inverse.

In Section 4, we show that the graphs in GBC are the only graphs G in $\mathcal{H}_{nmc}$ such that G_{w} is strongly self-inverse for some $\mathrm{w} \in \mathcal{W}_G$.

A weighted graph G_{w} is said to have the *strong reciprocal eigenvalue property* (property SR) if $1/\lambda \in \sigma(G_{\mathrm{w}})$ whenever $\lambda \in \sigma(G_{\mathrm{w}})$ and both have the same multiplicity. A weighted graph G_{w} is said to have the *reciprocal eigenvalue property* (property R) if $1/\lambda \in \sigma(G_{\mathrm{w}})$ whenever $\lambda \in \sigma(G_{\mathrm{w}})$. In Section 5, we relate these properties with the property of being strongly self-inverse.

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2. Preliminaries. In this section, we recall some necessary definitions and results from the literature. In particular, we shall introduce the graph class $\mathcal{H}_{nmc}$.

DEFINITION 2.1. [7] Consider a graph G with a unique perfect matching $\mathcal{M}$. A path $P = [u_1, u_2, \ldots, u_{2k}]$ is called an *alternating path* if the edges on P are alternately matching and nonmatching edges, that is, for each i, if $[u_i, u_{i+1}]$ is a matching (resp., nonmatching) edge and $[u_{i+1}, u_{i+2}] \in E(G)$, then $[u_{i+1}, u_{i+2}]$ is a nonmatching (resp., matching) edge. Let $P = [u_1, u_2, \ldots, u_{2k}]$ be an alternating path. We say P is an *mm-alternating path* (matching-matching-alternating path) if $[u_1, u_2], [u_{2k-1}, u_{2k}] \in \mathcal{M}$. We say P is an *nn-alternating path* (nonmatching-nonmatching-alternating path) if $[u_1, u_2], [u_{2k-1}, u_{2k}] \notin \mathcal{M}$.

DEFINITION 2.2. [7] Let G be a connected graph with a unique perfect matching $\mathcal{M}$, and let $[u, v]$ be a nonmatching edge in G. An *extension* at $[u, v]$ is an nn-alternating u-v-path other than $[u, v]$. An extension at $[u, v]$ is called *even type* (resp., *odd type*) if the number of nonmatching edges on that extension is even (resp., odd).

The following definition classifies the nonmatching edges of a graph in $\mathcal{H}$.

DEFINITION 2.3. [7] Let $G \in \mathcal{H}$, and let $[u, v]$ be a nonmatching edge in G. The nonmatching edge $[u, v]$ is said to be an *odd type* edge if either there are no extensions at $[u, v]$ or each extension at $[u, v]$ is odd type. We say $[u, v]$ is an *even type* edge if each extension at $[u, v]$ is even type. We say $[u, v]$ is *mixed type* if it has an even type extension and an odd type extension. Let $\mathcal{E}$ be the set of all even type edges in G.

DEFINITION 2.4. [7] We denote by $\mathcal{H}_{nmc}$ the class of graphs G in $\mathcal{H}$ such that

i) G has no mixed type edges,
ii) G satisfies the condition 'C: *the extensions at two distinct even type edges never have an odd type edge in common*'.

Here 'nmc' is an abbreviation of 'no mixed type edges and condition C'.

EXAMPLE 2.5. The following is an example of a graph in $\mathcal{H}_{nmc}$. Since all the extensions in G are even type, there are no mixed type edges. The graph G has exactly two even type edges which are e and f. The extensions at e and f never have an odd type edge in common. So, $G \in \mathcal{H}_{nmc}$.

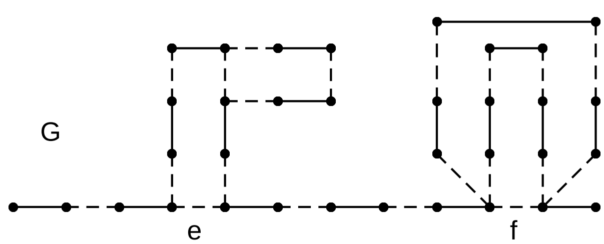

FIGURE 2. *Here, the solid edges are the matching edges.*

To proceed further we need the following definition.

DEFINITION 2.6. [10] For a path P in G_{w}, the weight $\mathrm{w}(P)$ of P is the number $\mathrm{w}(P) = \prod_{e \in E(P)} \mathrm{w}(e)$. Let $G \in \mathcal{H}$, $\mathrm{w} \in \mathcal{W}_G$, and let e be a nonmatching edge in G. We define $W(e) = \sum_{Q(e)} \mathrm{w}(Q(e))$, where the sum is

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 80-89, February 2020.

taken over all extensions $Q(e)$ at e. That is, $W(e)$ is the sum of the weights of all extensions at e.

To proceed further we need the following known result.

THEOREM 2.7. [10, Theorem 2.14] *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. Then the inverse G^+_{w} exists if and only if $(G - \mathcal{E})/\mathcal{M}$ is (connected) bipartite and $\mathrm{w}(e) \leq W(e)$ for each $e \in \mathcal{E}$.*

Let $G \in \mathcal{H}$, and suppose that u and v are two distinct vertices in G. An mm-alternating u-v-path is a *minimal path*, if it does not contain any even type extensions (of any nonmatching edge in G), see [7].

LEMMA 2.8. [7] *Let $G \in \mathcal{H}_{nmc}$. Let $P(i,j)$ be an mm-alternating i-j-path. Then there exists a unique minimal i-j-path $P_m(i,j)$ and a set F of even type edges on $P_m(i,j)$ such that $P(i,j)$ is obtained from $P_m(i,j)$ by replacing each edge $f \in F$ with an even type extension Q_f at f.*

REMARK 2.9. In particular, if $G \in \mathcal{H}_{nmc}$ and $[u,v]$ is an even type edge in G, then $[u',u,v,v']$ is the only minimal u'-v'-path in G, since every other u'-v'-path contains an even type extension at $[u,v]$.

The proof of the following important observation is a part of the proof of Theorem 2.14 in [10].

THEOREM 2.10. *Let $G \in \mathcal{H}_{nmc}$. Then*

$$A(G_{\mathrm{w}})^{-1}_{ij} = \sum_{P(i,j) \in \mathcal{P}^M(i,j)} (-1)^{\|\mathcal{O}(P(i,j))\|} \mathrm{w}(P(i,j)) \prod_{e \in \mathcal{E}(P(i,j))} \left[\frac{W(e)}{\mathrm{w}(e)} - 1\right],$$

where $\mathcal{P}^M(i,j)$ is the set of minimal i-j-paths in G, $\mathcal{E}(P(i,j))$ and $\mathcal{O}(P(i,j))$ are the set of even type and odd type edges in $P(i,j)$, respectively and $\|\mathcal{O}(P(i,j))\|$ denotes the number of edges in $P(i,j)$.

3. A class of strongly self-inverse weighted graphs. In this section, we define a class of graphs $G \in \mathcal{H}_{nmc}$ and then characterize for these graphs the weight functions $\mathrm{w} \in \mathcal{W}_G$ such that G_{w} is strongly self-inverse. Our class includes the boxminus corona graphs defined in [8].

A graph G is called a *corona* graph, if it is obtained from some other graph H by adding a new pendent vertex at each vertex of H, and we denote it by $G = H \circ K_1$, see [2]. A corona graph G is said to be a *bipartite corona* graph if H is bipartite. For example, the path P_4 is a corona tree as it can be obtained from P_2 by adding a new pendent vertex at each vertex of P_2. Let $R_{2n}(u,v)$ denote the graph on $2n$ ($n \geq 2$) vertices consisting of $n-1$ rectangles with a common base $[u,v]$. For an example see Figure 3.

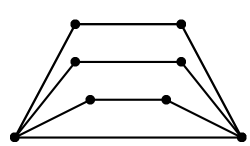

FIGURE 3. R_8.

DEFINITION 3.1. [8] Let $H \circ K_1$ be a corona graph where H is a bipartite graph. Let S be a subset of nonmatching edges of $H \circ K_1$ such that each cycle in H has an even number of edges from S. Let G be the graph created from $H \circ K_1$ by replacing each $[u,v] \in S$ with an R_{2k} graph. We call G a *generalized boxminus corona* graph. If we replace each $[u,v] \in S$ with a copy of R_6 we get boxminus corona graph, for boxminus corona graph, see [8, Definition 27]. In Figure 4, $S = \{[1,2],[3,4],[5,6]\}$.

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A publication of the International Linear Algebra Society
Volume 36, pp. 80-89, February 2020.

FIGURE 4. *Example of a generalized boxminus corona graph.*

Let $H_g = \{G \in \mathcal{H} |\ \ G/\mathcal{M}\ \ is\ \ bipartite\}$, where $G/\mathcal{M}$ is the graph obtained from G by contracting the matching edges, see [8]. The following example says that the class GBC neither contains the class $\mathcal{H}_g$ nor is contained in the class $\mathcal{H}_g$.

REMARK 3.2. 1. Let G be a generalized boxminus corona graph. It is clear from Definition 3.1 that G has a perfect matching $\mathcal{M}$ consisting of the pendent edges and the middle edge of each extension of the edges in S. Thus, each edge in S is an even type edge of G. Moreover, these are the only even type edges in G, since in the corona graph $H \circ K_1$, the pendent edges are the unique perfect matching edges, and hence, no nonmatching edge has an extension in $H \circ K_1$. So there is no odd type extension in G and the extensions at two distinct even type edges never have an odd type edge in common. Clearly then $G \in \mathcal{H}_{nmc}$.

2. Since each matching edge in G is either a pendent edge or an internal edge of an even type extension, there is no minimal path in G of length at least 5.

To proceed further we need the following known result.

LEMMA 3.3. [7] *Let $G \in \mathcal{H}_{nmc}$, and let $[u, v]$ be a nonmatching edge in G. Let $Q(u, v)$ be an extension at $[u, v]$. Then each nonmatching edge on $Q(u, v)$ is odd type.*

LEMMA 3.4. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. Let $e = [u, v]$ be a nonmatching edge in G. Then*

$$A(G_{\mathrm{w}})^{-1}_{u',v'} = \begin{cases} W(e) - \mathrm{w}(e) & \text{if } [u,v] \text{ is even type,} \\ -W(e) - \mathrm{w}(e) & \text{if } [u,v] \text{ is odd type.} \end{cases}$$

Proof. We first assume that $[u, v]$ is even type. Then there is exactly one minimal u'-v'-path in G_{w}, which is $P = [u', u, v, v']$ and this path has exactly one even type edge which is $[u, v]$ and no odd type edges. Thus, $\mathrm{w}(P) = \mathrm{w}(e)$. By Theorem 2.10,

$$A(G_{\mathrm{w}})^{-1}_{u',v'} = \mathrm{w}(P) \left[\frac{W(e)}{\mathrm{w}(e)} - 1 \right] = W(e) - \mathrm{w}(e).$$

We now assume that $[u, v]$ is odd type. Let $Q_1(u, v), Q_2(u, v), \ldots, Q_k(u, v)$ be the odd type extensions at $[u, v]$. By using Lemma 3.3, each nonmatching edge in $Q_i(u, v)$ is odd type for $i = 1, \ldots, k$. Then the paths $P_i = [u', Q_i(u, v), v']$ for $i = 1, \ldots, k$ and $Q = [u', u, v, v']$ are the only mm-alternating u'-v'-paths which

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A publication of the International Linear Algebra Society
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are minimal u'-v'-paths. Since $Q_i(u,v)$ contains an odd number of odd type edges, each minimal u'-v'-path contains an odd number odd type edges. By using Theorem 2.10,

$$\begin{aligned} A(G_{\mathrm{w}})^{-1}_{u',v'} &= \sum_{P(u',v')\in\mathcal{P}^M(u',v')} -\mathrm{w}(P) \\ &= -\sum_{i=1}^{k}\mathrm{w}(P_i) - \mathrm{w}(Q) \\ &= -W(e) - \mathrm{w}(e). \end{aligned}$$

The proof is complete. ∎

Lemma 3.5. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. Assume that G_{w} is strongly self-inverse. Then $\mathrm{w}(e) = \frac{W(e)}{2}$ for each even type edge e in G.*

Proof. Since G_{w} is strongly self-inverse, $f_{\mathcal{M}}$ is an isomorphism from G_{w} to G^+_{w}. Let $e = [u,v]$ be an even type edge in G_{w}. Then $[f_{\mathcal{M}}(u), f_{\mathcal{M}}(v)] = [u',v']$ is an edge in G^+_{w} and the weights of the edges $e = [u,v]$ in G_{w} and $[u',v']$ in G^+_{w} are same. That is,

$$\mathrm{w}([u,v]) = \mathrm{w}([u',v']) = A(G^+_{\mathrm{w}})_{u',v'} = |A(G_{\mathrm{w}})^{-1}_{u',v'}|. \tag{3.1}$$

By Lemma 3.4 and Theorem 2.7,

$$A(G_{\mathrm{w}})^{-1}_{u',v'} = W(e) - \mathrm{w}(e) \geq 0. \tag{3.2}$$

Combining (3.1) and (3.2), we get $\mathrm{w}(e) = W(e) - \mathrm{w}(e)$. So, $\mathrm{w}(e) = \frac{W(e)}{2}$. ∎

Let $G \in \mathcal{H}_{nmc}$, and let $\mathcal{E}$ be the set of all even type edges. Then by $(G-\mathcal{E})/\mathcal{M}$ denotes the graph obtained by deleting all the even type edges and then contracting each matching edge to a single vertex, see [7].

Theorem 3.6. *Let G be a generalized boxminus corona graph, and let $\mathrm{w} \in \mathcal{W}_G$. Then G_{w} is strongly self-inverse if and only if $\mathrm{w}(e) = \frac{W(e)}{2}$ for each even type edge e in G.*

Proof. We first assume that G_{w} is strongly self-inverse. Since $G \in \mathcal{H}_{nmc}$, by Lemma 3.5, $\mathrm{w}(e) = \frac{W(e)}{2}$ for each even type edge e in G.

We now prove the converse. Since $\mathrm{w}(e) = \frac{W(e)}{2} \leq W(e)$, for all $e \in \mathcal{E}$, to show that G^+_{w} exists, it is enough to show that $(G-\mathcal{E})/\mathcal{M}$ is bipartite, by Theorem 2.7. Towards that, note that $\mathcal{E} = S$ and suppose that $(G-S)/\mathcal{M}$ is not bipartite. Then it has an odd cycle, say, Γ. Notice that each $R_{2k}(u,v)$ is reduced to a $K_{2,k-1}$ in $(G-S)/\mathcal{M}$ and Γ contains either exactly 2 consecutive edges or no edges at all from each $K_{2,k-1}$. Since Γ is an odd cycle, it follows that Γ contains an odd number of edges apart from the edges from the $K_{2,k-1}$'s. Consider the cycle Γ' obtained from Γ by doing the following. For each pair of consecutive edges that Γ contains from a $K_{2,k-1}$ corresponding to an edge $[u,v] \in S$, replace these two edges by $[u,v]$. Then Γ' is a cycle in H. By the definition of G, Γ' has an even number of edges from S and does not contain any matching edges. So Γ has an even number of even type edges and an odd number of odd type edges. Hence, the length of Γ' is odd, contradicting the bipartiteness of H. Hence, G^+_{w} exists.

We now show that $f_{\mathcal{M}}$ is an isomorphism from G_{w} to G^+_{w}. Let $[u,u']$ be a matching edge in G_{w}. There is exactly one minimal u-u'-path in G_{w}, which is $[u,u']$. By using Theorem 2.10, $A(G^+_{\mathrm{w}})_{u,u'} = \mathrm{w}([u,u']) = 1$. Hence, $[u,u']$ is an edge in G^+_{w} with weight 1.

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A publication of the International Linear Algebra Society
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Let $e = [u, v]$ be a nonmatching edge in G_{w}. Since there are no odd type extensions in G_{w}, the path $[u', u, v, v']$ is the only minimal u'-v'-path in G_{w}. By Lemma 3.4, $A(G^+_{\mathrm{w}})_{u',v'} = |A(G_{\mathrm{w}})^{-1}_{u',v'}| = \mathrm{w}(e)$ if $[u, v]$ is an odd type edge (since there are no odd type extensions, $W(e) = 0$) and $A(G^+_{\mathrm{w}})_{u',v'} = |A(G_{\mathrm{w}})^{-1}_{u',v'}| = W(e) - \mathrm{w}(e)$ if e is an even type edge. Combined with the assumption that $\mathrm{w}(e) = \frac{W(e)}{2}$ for each even type edge e in G_{w} we get $A(G^+_{\mathrm{w}})_{u',v'} = \frac{W(e)}{2} = \mathrm{w}(e)$ if $e = [u, v]$ is an even type edge in G_{w}. That is, for each edge $[u, v]$ in G_{w} there is an edge $[u', v']$ in G^+_{w} and its weight in G^+_{w} is $\mathrm{w}([u, v])$. Hence, G_{w} is isomorphic to a subgraph of G^+_{w} via $f_{\mathcal{M}}$.

Suppose that G_{w} is not isomorphic to G^+_{w} via $f_{\mathcal{M}}$. Then we have an edge $[x, y]$ in G^+_{w} such that $[f^{-1}_{\mathcal{M}}(x), f^{-1}_{\mathcal{M}}(y)] = [x', y']$ is not in G_{w}. By Theorem 2.10, there is a minimal x-y-path in G_{w}, say $P(x, y)$. The length of this minimal path $P(x, y)$ is at least 5, otherwise $P(x, y) = [x, x', y', y]$ and the edge $[x', y']$ is in G_{w}. By Remark 3.2, G_{w} does not have a minimal path of length 5. Hence, G_{w} is strongly self-inverse. □

Remark 3.7. There are generalized boxminus corona graphs which are self-inverse for some weight functions but not strongly self-inverse. For example consider the graph G shown in Figure 5 with the weight function w such that $\mathrm{w}([1, 3]) = 1/3$, $\mathrm{w}([4, 6]) = 2/3$ and the rest of the edges weight are equal to 1. By Theorem 2.10, we get the inverse graph G^+_{w} where $\mathrm{w}([1', 3']) = 2/3$, $\mathrm{w}([4'.6']) = 1/3$ and the rest of the edges weight are equal to 1. The underlying graph G' of G^+_{w} is shown in Figure 5. One can easily check that the following function $f : V(G_{\mathrm{w}}) \longrightarrow V(G^+_{\mathrm{w}})$ defined by $f(1) = 6', f(2) = 5, f(3) = 4', f(4) = 3', f(5) = 2, f(6) = 1', f(1') = 6, f(2') = 5', f(3') = 4, f(4') = 3, f(5') = 2'$ and $f(6') = 1$ is an isomorphism from G_{w} to G^+_{w}. Hence, the graph G_{w} is self-inverse. By Theorem 3.6, G_{w} is not strongly self-inverse.

Figure 5.

This suggests the following open problem.

Problem 3.8. *Let G be a generalized boxminus corona graph. Characterize all weight functions* w *such that G_{w} is a self-inverse graph.*

4. Strongly self-inverse graphs in $\mathcal{H}_{nmc}$. In this section, we show that generalized boxminus corona graphs are the only graphs in $\mathcal{H}_{nmc}$ that are strongly self-inverse for some weight $\mathrm{w} \in \mathcal{W}_G$. The following is a necessary condition for a weighted graph in $\mathcal{H}_{nmc}$ to be strongly self-inverse. To proceed further we need the following known result.

Lemma 4.1. [7, Lemma 27] *Let $G \in \mathcal{H}_{nmc}$, and let $(G - \mathcal{E})/\mathcal{M}$ be bipartite. Then, if one path from a vertex u to a vertex v contains an odd number of odd type edges, then each path from u to v must contain an odd number of odd type edges.*

Lemma 4.2. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. If G_{w} is strongly self-inverse, then G has no minimal path of*

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A publication of the International Linear Algebra Society
Volume 36, pp. 80-89, February 2020.

length 5.

Proof. Suppose that G has a minimal path of length 5, say, $[1,1',2,2',3,3']$. By using Lemma 3.5 and Theorem 2.10, we have

$$A(G_{\mathrm{w}})^{-1}_{1,3'} = \sum_{P(1,3')\in\mathcal{P}^M(1,3')} (-1)^{\|\mathcal{O}(P(1,3'))\|}\mathrm{w}(P(1,3')).$$

Since G_{w} is strongly self-inverse, by Theorem 2.7, $(G-\mathcal{E})/\mathcal{M}$ is bipartite. Then by using Lemma 4.1, each 1-3′-minimal path has an even number of odd type edges. Hence, $A(G^+_{\mathrm{w}})_{1,3'} > 0$. Therefore, $[1,3'] \in E(G^+_{\mathrm{w}})$. Since $f_{\mathcal{M}}$ is an isomorphism from G_{w} to G^+_{w}, we have $[1',3] \in E(G_{\mathrm{w}})$. Then $[1',2,2',3]$ is an even type extension at $[1',3]$ which contradicts the assumption that $[1,1',2,2',3,3']$ is a minimal path. ∎

The following example shows that the converse of the above lemma is not true.

Example 4.3. Consider the graph G shown in Figure 5. The graph G is generalized boxminus corona and it has no minimal path of length 5. The graph G has exactly two even type edges $[1,3]$ and $[4,6]$. Consider the weight function w such that $\mathrm{w}([1,3]) = 1/3$, $\mathrm{w}([4,6]) = 1/3$ and the rest of the edges weight are equal to 1. It is clear that $\mathrm{w}([1,3]) \neq \frac{W([1,3])}{2}$ and $\mathrm{w}([4,6]) \neq \frac{W([4,6])}{2}$. By Theorem 3.6, G_{w} is not strongly self-inverse.

Lemma 4.4. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. Assume that G_{w} is strongly self-inverse. Then the following are true.*

1. *There are no odd type extensions in G.*
2. *The length of each even type extension in G is* 3.
3. *The degree of each internal vertex of an extension is* 2.

Proof. The proof is similar to the proof of Lemmas 17, 23 and 24 in [8] . ∎

Lemma 4.5. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. Assume that G_{w} is strongly self-inverse, and let G' be the graph obtained from G by deleting all the even type extensions while keeping the endvertices of the extensions. Then G' is a corona graph of a bipartite graph and each cycle in G' has an even number of even type edges in G.*

Proof. Let us assume that G' is not a corona graph. Then there is a matching edge $[v,v']$ in G' which is not a leaf, that is, $d_{G'(v)}, d_{G'(v')} \geq 2$. So, we can find a path $[u,v,v',w]$ in G', and hence, the path $[u',u,v,v',w,w']$ is an mm-alternating u'-w'-path of length 5 in G. By Lemma 4.2, this path is not minimal. Hence, the path $[u,v,v',w]$ is an even type extension at $[u,w]$. But this is impossible, as G' was obtained by deleting all the even type extensions, so $[v,v']$ is a leaf in G'. Hence, G' must be a corona graph.

Let Γ be a cycle in G'. Since G' is a bipartite corona graph, Γ has an even number of edges. By construction of G', all the edges of Γ are odd type, but some of them may be even type edges in G. By Theorem 2.7, the cycle Γ in G has an even number of odd type edges. Since G is bipartite, the number of even type edges in Γ should also be even. ∎

The following is our main theorem of this article.

Theorem 4.6. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. Then the following are equivalent.*

1. *G_{w} is strongly self-inverse.*
2. *G is a generalized boxminus corona graph and $\mathrm{w}(e) = \frac{W(e)}{2}$ for each even type edge e in G.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 80-89, February 2020.

Proof. [2 $\Rightarrow$ 1] It follows by Theorem 3.6.

[1 $\Rightarrow$ 2] By Lemma 4.4, there is no odd type extension, each even type extension has length 3, and its internal vertices have degree 2 in G. By Lemma 4.5, G is obtained by adding even type extensions to a set S of edges in a bipartite corona graph. Thus, G is a generalized boxminus corona graph, and by Theorem 3.6 $\mathrm{w}(e) = \frac{W(e)}{2}$ for each even type edge e. ∎

5. Application. In this section, we supply a result as an application of Theorem 4.6. This result is also a generalization of two known theorems. The spectrum $\sigma(G_{\mathrm{w}})$ of G_{w} is defined as the multiset of eigenvalues of $A(G_{\mathrm{w}})$. The largest eigenvalue of G_{w} is called the *spectral radius* of G_{w} and it is denoted by ρ.

In [6], the authors proved the following theorem.

THEOREM 5.1. *Let $G \in \mathcal{H}$ such that $G/\mathcal{M}$ is bipartite, and let $\mathrm{w} \in \mathcal{W}_G$. Then the following are equivalent.*

1. *$1/\rho$ is the smallest positive eigenvalue of G_{w}.*
2. *G_{w} is strongly self-inverse.*
3. *G_{w} has property R.*
4. *G_{w} has property SR.*
5. *G is a bipartite corona graph.*

In [8], the authors proved the following theorem.

THEOREM 5.2. *Let $G \in \mathcal{H}_{nmc}$, and let each even type edge be strict. Assume that G^+ exists. Then the following are equivalent.*

1. *$1/\rho$ is the smallest positive eigenvalue of G.*
2. *G is strongly self-inverse.*
3. *G has property R.*
4. *G has property SR.*
5. *G is a boxminus corona graph.*

The following theorem is a generalization of Theorems 5.1 and 5.2.

THEOREM 5.3. *Let $G \in \mathcal{H}_{nmc}$ and let $\mathrm{w} \in \mathcal{W}_G$ such that $\mathrm{w}(e) \leq \frac{W(e)}{2}$ for each $e \in \mathcal{E}$. Assume that G^+_{w} exists. Then the following are equivalent.*

1. *$1/\rho$ is the smallest positive eigenvalue of G_{w}.*
2. *G_{w} is strongly self-inverse.*
3. *G_{w} has property SR.*
4. *G_{w} has property R.*
5. *G is a generalized boxminus corona graph and $\mathrm{w}(e) = \frac{W(e)}{2}$.*

Acknowledgements. This research was partially supported by the Joint NSFC-ISF Research Program (jointly funded by the National Natural Science Foundation of China and the Israel Science Foundation, grant no. 11561141001 and grant no. 2219/15, and the National Natural Science Foundation of China, grant no. 11531001). The authors sincerely thank the referees and the editors for their valuable suggestions.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 36, pp. 80-89, February 2020.

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