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A NEW EQUIVALENT CONDITION OF THE REVERSE ORDER LAW FOR G-INVERSES OF MULTIPLE MATRIX PRODUCTS*

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Abstract. In 1999, Wei [M.Wei, Reverse order laws for generalized inverse of multiple matrix products, Linear Algebra Appl., 293 (1999), pp. 273-288] studied reverse order laws for generalized inverses of multiple matrix products and derived some necessary and sufficient conditions for

$$A_n\{1\}A_{n-1}\{1\}\cdots A_1\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}$$

by using P-SVD (Product Singular Value Decomposition). In this paper, using the maximal rank of the generalized Schur complement, a new simpler equivalent condition is obtained in terms of only the ranks of the known matrices for this inclusion.

Key words. Reverse order law, Generalized inverse, Matrix product, Maximal rank, Generalized Schur complement.

AMS subject classifications. 15A03, 15A09.

1. Introduction. Let A be an $m \times n$ matrix over the complex field. A^* and $r(A)$ denote the conjugate transpose and the rank of the matrix A, respectively. We recall that an $n\times m$ matrix G satisfying the equation $AGA = A$ is called a *{1}-inverse* or a *g-inverse* of A and is denoted by $A^{(1)}$. The set of all *{1}-inverses* of A is denoted by $A\{1\}$. We refer the reader to [1, 2] for basic results on the *g-inverse*.

The reverse order law for the generalized inverses of the multiple matrix products yields a class of interesting problems that are fundamental in the theory of generalized inverses of matrices and statistics. They have attracted considerable attention since the middle 1960s, and many interesting results have been obtained; see [3, 4, 5, 6, 7, 8].

In this paper, by applying the maximal rank of the generalized Schur complement [9], we derive a new and simple necessary and sufficient condition for the validity of the inclusion

$$A_n\{1\}A_{n-1}\{1\}\cdots A_1\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}. \tag{1.1}$$

Compared with the conditions given in [6], our condition can be easily checked and the proof is very simple.

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The following two lemmas play a key role in this paper:

LEMMA 1.1. [9] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times l}$, $C \in \mathbb{C}^{k\times n}$ *and* $D \in \mathbb{C}^{k\times l}$. *Then*

$$\max_{A^{(1)}\in A\{1\}} r(D - CA^{(1)}B) = \min\{r(C, D), r\begin{pmatrix} B \\ D \end{pmatrix}, r\begin{pmatrix} A & B \\ C & D \end{pmatrix} - r(A)\}. \tag{1.2}$$

Before presenting the next lemma, we first state the well-known Frobenius' inequality: If A, B, C are matrices such that ABC is defined, then

$$r(AB) + r(BC) \le r(B) + r(ABC). \tag{1.3}$$

LEMMA 1.2. [8] Let $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $i = 1, 2, \cdots, n$. Then

$$l_2 + l_3 + \cdots + l_n + r(A_1A_2\cdots A_n) \ge r(A_1) + r(A_2) + \cdots + r(A_n). \tag{1.4}$$

Proof. Taking $A = A_1A_2\cdots A_{i-1}$, $B = I_{l_i}$ and $C = A_i$ in (1.3), where $i = 2, 3, \cdots, n$, we obtain

$$r(A_1A_2\cdots A_{i-1}A_i) - r(A_1A_2\cdots A_{i-1}) \ge r(A_i) - l_i. \tag{1.5}$$

So we have

$$r(A_1A_2\cdots A_n) \ge \sum_{i=1}^{n} r(A_i) - \sum_{i=2}^{n} l_i. \ \square \tag{1.6}$$

2. Main result. Define the following matrix function

$$\begin{aligned} &S_{A_1,A_2,\cdots,A_n}(X_1, X_2, \cdots, X_n) \\ &= A_1A_2\cdots A_n - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_1A_1A_2\cdots A_n, \end{aligned} \tag{2.1}$$

where $X_1, X_2, \cdots, X_n$ are any matrices of appropriate sizes. In order to present the new necessary and sufficient condition for the inclusion (1.1), we first give the maximum rank of matrix $S_{A_1,A_2,\cdots,A_n}(X_1, X_2, \cdots, X_n)$ when each X_i ($i = 1, 2, \cdots, n$) varies over the set $A_i\{1\}$ of all *g-inverses* of the matrix A_i.

THEOREM 2.1. *Let* $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $i = 1, \cdots, n$, *and* $S_{A_1,A_2,\cdots,A_n}(X_1, X_2, \cdots, X_n)$ *be as in* (2.1). *Then*

$$\begin{aligned} &\max_{X_n,X_{n-1},\cdots,X_1} r(S_{A_1,A_2,\cdots,A_n}(X_1, X_2, \cdots, X_n)) \\ &= \min\{r(A_1A_2\cdots A_n), r(A_1A_2\cdots A_n) + \sum_{m=2}^{n} l_m - \sum_{m=1}^{n} r(A_m)\}, \end{aligned} \tag{2.2}$$

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where X_i *varies over* $A_i\{1\}$ *for* $i = 1, 2, \cdots, n$.

Proof. For $2 \le i \le n-1$ and $X_j \in A_j\{1\},\ j = i, i+1, \cdots, n$, we first prove

$$\begin{aligned}(2.3)\qquad &\max_{X_i}\ r(A_1A_2\cdots A_{i-1} - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_i)\\ &= \min\{r(A_1A_2\cdots A_{i-1})\ \ ,\ \ r(A_1A_2\cdots A_i - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1})\\ &\quad +l_i - r(A_i)\}.\end{aligned}$$

By Lemma 1.1 (with $A = A_i$, $B = I_{l_i}$, $D = A_1\cdots A_{i-1}$, $C = A_1\cdots A_nX_n\cdots X_{i+1}$), we have

$$\begin{aligned}&\max_{X_i}\ r(A_1A_2\cdots A_{i-1} - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_i)\\ &= \min\{r(A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1},\ \ A_1A_2\cdots A_{i-1}), r\begin{pmatrix} I_{l_i}\\ A_1A_2\cdots A_{i-1}\end{pmatrix},\\ &\quad r\begin{pmatrix} A_i & I_{l_i}\\ A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1} & A_1A_2\cdots A_{i-1}\end{pmatrix} - r(A_i)\}\\ &= \min\{r(A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1},\ A_1A_2\cdots A_{i-1}), r\begin{pmatrix} I_{l_i}\\ A_1A_2\cdots A_{i-1}\end{pmatrix},\\ &\quad r(A_1A_2\cdots A_i - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1}) + l_i - r(A_i)\}\\ &= \min\{r(A_1A_2\cdots A_{i-1}),\ \ r(A_1A_2\cdots A_i - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1})\\ &\quad +l_i - r(A_i)\},\end{aligned}$$

i.e., (2.3) holds, where the second equality holds as

$$\begin{aligned}&r\begin{pmatrix} A_i & I_{l_i}\\ A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1} & A_1A_2\cdots A_{i-1}\end{pmatrix}\\ &= r\begin{pmatrix} O & I_{l_i}\\ A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1} - A_1A_2\cdots A_i & O\end{pmatrix}\end{aligned}$$

and the last equality holds as

$$r(A_1A\cdots A_nX_nX_{n-1}\cdots X_{i+1},\ A_1\cdots A_{i-1}) = r(A_1\cdots A_{i-1})$$

and

$$r(A_1A_2\cdots A_{i-1}) \le r(A_{i-1}) \le l_i = r\begin{pmatrix} I_{l_i}\\ A_1A_2\cdots A_{i-1}\end{pmatrix}.$$

When $i = n$, again by Lemma 1.1 with $A = A_n$, $B = I_{l_n}$, $D = A_1A_2\cdots A_{n-1}$ and $C = A_1A_2\cdots A_n$, we have

$$(2.4)\qquad \max_{X_n}\ r(A_1A_2\cdots A_{n-1} - A_1A_2\cdots A_nX_n)$$

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$$\begin{aligned}
&= \min\{r(A_1A_2\cdots A_n, \quad A_1A_2\cdots A_{n-1}), r\begin{pmatrix} I_{l_n} \\ A_1A_2\cdots A_{n-1}\end{pmatrix}, \\
&\quad r\begin{pmatrix} A_n & I_{l_n} \\ A_1A_2\cdots A_n & A_1A_2\cdots A_{n-1}\end{pmatrix} - r(A_n)\} \\
&= \min\{r(A_1A_2\cdots A_{n-1}), l_n - r(A_n)\}
\end{aligned}$$

in which the last equality holds since

$$r(A_1\cdots A_n,\ A_1\cdots A_{n-1}) = r(A_1\cdots A_{n-1}) \le r(A_{n-1}) \le l_n = r\begin{pmatrix} I_{l_n} \\ A_1A_2\cdots A_{n-1}\end{pmatrix}$$

and

$$r\begin{pmatrix} A_n & I_{l_n} \\ A_1A_2\cdots A_n & A_1A_2\cdots A_{n-1}\end{pmatrix} = r\begin{pmatrix} I_{l_n} \\ A_1A_2\cdots A_{n-1}\end{pmatrix}.$$

We now prove (2.2). According to Lemma 1.1 with $A = A_1$, $B = A_1A_2\cdots A_n$, $C = A_1A_2\cdots A_nX_nX_{n-1}X_2$ and $D = A_1A_2\cdots A_n$, we have

$$\begin{aligned}
&\max_{X_1}\ r(S_{A_1,A_2,\cdots,A_n}(X_1, X_2, \cdots, X_n)) \\
&= \min\{r(A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2, \quad A_1A_2\cdots A_n), r\begin{pmatrix} A_1A_2\cdots A_n \\ A_1A_2\cdots A_n\end{pmatrix}, \\
&\quad r\begin{pmatrix} A_1 & A_1A_2\cdots A_n \\ A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2 & A_1A_2\cdots A_n\end{pmatrix} - r(A_1)\} \\
&= \min\{r(A_1A_2\cdots A_n), r(A_1 - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2) + r(A_1A_2\cdots A_n) - r(A_1)\},
\end{aligned}$$

where the last equality holds as

$$r(A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2,\ A_1A_2\cdots A_n) = r(A_1A_2\cdots A_n) = r\begin{pmatrix} A_1A_2\cdots A_n \\ A_1A_2\cdots A_n\end{pmatrix}$$

and

$$\begin{aligned}
&r\begin{pmatrix} A_1 & A_1A_2\cdots A_n \\ A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2 & A_1A_2\cdots A_n\end{pmatrix} \\
&= r(A_1 - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2) + r(A_1A_2\cdots A_n).
\end{aligned}$$

Obviously $r(A_1 - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2) \le r(A_1)$, so

(2.5)
$$\begin{aligned}
&\max_{X_1}\ r(S_{A_1,A_2,\cdots,A_n}(X_1, X_2, \cdots, X_n)) \\
&= r(A_1 - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_2) + r(A_1A_2\cdots A_n) - r(A_1).
\end{aligned}$$

Combining (2.3) and (2.5), we have

$$\max_{X_2,X_1}\ r(S_{A_1,A_2,\cdots,A_n}(X_1, X_2, \cdots, X_n))$$

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$$
\begin{aligned}
&= \min\{r(A_1) + r(A_1A_2\cdots A_n) - r(A_1),\\
&\quad r(A_1A_2\cdots A_n) + r(A_1A_2 - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_3) + l_2 - r(A_2) - r(A_1)\}\\
&= \min\{r(A_1A_2\cdots A_n), r(A_1A_2\cdots A_n) + r(A_1A_2 - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_3)\\
&\quad + l_2 - r(A_2) - r(A_1)\}.
\end{aligned}
$$

We contend that, for $2 \leq i \leq n-1$,

$$
\begin{aligned}
&\max_{X_i,X_{i-1},\cdots,X_1} r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n))\\
&\quad = \min\{r(A_1A_2\cdots A_n), r(A_1A_2\cdots A_n)\\
&\qquad + r(A_1A_2\cdots A_i - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1}) + \sum_{m=2}^{i} l_m - \sum_{m=1}^{i} r(A_m)\}.
\end{aligned}
\tag{2.6}
$$

We proceed by induction on i. For $i = 2$, the equality relation (2.6) has been proved. Assume that (2.6) is true for $i-1$ ($i \geq 3$), that is,

$$
\begin{aligned}
&\max_{X_{i-1},X_{i-2},\cdots,X_1} r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n))\\
&\quad = \min\{r(A_1A_2\cdots A_n), r(A_1A_2\cdots A_n)\\
&\qquad + r(A_1A_2\cdots A_{i-1} - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_i) + \sum_{m=2}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m)\}.
\end{aligned}
\tag{2.7}
$$

We now prove that (2.6) is also true for i. By (2.3) and (2.7), we have

$$
\begin{aligned}
&\max_{X_i,X_{i-1},\cdots,X_1} r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n))\\
&= \min\{r(A_1A_2\cdots A_n),\\
&\quad r(A_1A_2\cdots A_n) + r(A_1A_2\cdots A_{i-1}) + \sum_{m=2}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m),\\
&\quad r(A_1A_2\cdots A_n) + r(A_1A_2\cdots A_i - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1})\\
&\quad + l_i - r(A_i) + \sum_{m=2}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m)\}.
\end{aligned}
$$

From Lemma 1.2 we know

$$
l_2 + l_3 + \cdots + l_{i-1} + r(A_1A_2\cdots A_{i-1}) \geq r(A_1) + r(A_2) + \cdots + r(A_{i-1});
$$

thus

$$
\begin{aligned}
&\max_{X_i,X_{i-1},\cdots,X_1} r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n))\\
&= \min\{r(A_1A_2\cdots A_n),\\
&\quad r(A_1A_2\cdots A_n) + r(A_1A_2\cdots A_i - A_1A_2\cdots A_nX_nX_{n-1}\cdots X_{i+1})\\
&\quad + \sum_{m=2}^{i} l_m - \sum_{m=1}^{i} r(A_m)\}.
\end{aligned}
$$

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In particular, when $i = n-1$, we have

$$\begin{aligned}\max_{X_{n-1},X_{n-2},\cdots,X_1} & r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n)) \\ = \min\{ & r(A_1A_2\cdots A_n), \quad r(A_1A_2\cdots A_n) \\ & + r(A_1A_2\cdots A_{n-1} - A_1A_2\cdots A_nX_n) + \sum_{m=2}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m)\}.\end{aligned} \tag{2.8}$$

On account of (2.4) and (2.8), it is seen that

$$\begin{aligned}\max_{X_n,X_{n-1},\cdots,X_1} & r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n)) \\ = \min\{ & r(A_1A_2\cdots A_n), \\ & r(A_1A_2\cdots A_n) + r(A_1A_2\cdots A_{n-1}) + \sum_{m=2}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m), \\ & r(A_1A_2\cdots A_n) + l_n - r(A_n) + \sum_{m=2}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m)\}.\end{aligned}$$

Noting that

$$r(A_1A_2\cdots A_{n-1}) + \sum_{m=2}^{n-1} l_m \geq \sum_{m=1}^{n-1} r(A_m),$$

we finally have

$$\begin{aligned}\max_{X_n,X_{n-1},\cdots,X_1} & r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n)) \\ = \min\{ & r(A_1A_2\cdots A_n), r(A_1A_2\cdots A_n) + \sum_{m=2}^{n} l_m - \sum_{m=1}^{n} r(A_m)\}. \ \square\end{aligned}$$

Since the inclusion (1.1) holds if and only if

$$\max_{X_n,X_{n-1},\cdots,X_1} r(S_{A_1,A_2,\cdots,A_n}(X_1,X_2,\cdots,X_n)) = 0,$$

by Theorem 2.1 we can immediately obtain the following result:

THEOREM 2.2. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1,2,\cdots,n$. Then the inclusion* (1.1) *holds if and only if*

$$\min\{r(A_1A_2\cdots A_n), r(A_1A_2\cdots A_n) + \sum_{m=2}^{n} l_m - \sum_{m=1}^{n} r(A_m)\} = 0,$$

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that is,

$$A_1 A_2 \cdots A_n = O$$

or

$$r(A_1 A_2 \cdots A_n) + \sum_{m=2}^{n} l_m - \sum_{m=1}^{n} r(A_m) = 0.$$

To end this paper, we now recover the equivalent condition established in [6] (by applying the multiple P-SVD [10], Product Singular Value Decomposition) for the inclusion (1.1).

THEOREM 2.3. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \cdots, n$. Then the following statements are equivalent.*

(1) $A_n\{1\}A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_n)\{1\}$;

(2) [6] $(2a')$ $A_1 A_2 \cdots A_n = O$ *or* $(2b')$ $r(A_1 A_2 \cdots A_n) > 0$
and for $i = 1, 2, \cdots, n-1$,
$r(A_1 A_2 \cdots A_i) + r(A_{i+1}) = l_{i+1} + r(A_1 A_2 \cdots A_{i+1})$;

(3) $(2a)$ $A_1 A_2 \cdots A_n = O$ *or* $(2b)$ $r(A_1 A_2 \cdots A_n) + \sum_{m=2}^{n} l_m - \sum_{m=1}^{n} r(A_m) = 0.$

Proof. It suffices to prove that $(2b')$ is equivalent to $(2b)$. The implication $(2b') \Rightarrow (2b)$ is easy. Now we show $(2b) \Rightarrow (2b')$.

We first prove by induction (on i) the following identities:

$$r(A_1 A_2 \cdots A_{n-i}) = \sum_{m=1}^{n-i} r(A_m) - \sum_{m=2}^{n-i} l_m \quad , \quad i = 0, 1, 2, \cdots, n-2. \tag{2.9}$$

When $i = 0$, it reduces to identity (2b). Assume that the identity (2.9) holds for $i-1$, $1 \le i \le n-2$, i.e.,

$$r(A_1 A_2 \cdots A_{n-i+1}) = \sum_{m=1}^{n-i+1} r(A_m) - \sum_{m=2}^{n-i+1} l_m. \tag{2.10}$$

We now prove that the identity (2.9) is also true for i. Based on Lemma 1.2, we know

$$r(A_1 A_2 \cdots A_{n-i+1}) + l_{n-i+1} \ge r(A_1 A_2 \cdots A_{n-i}) + r(A_{n-i+1}). \tag{2.11}$$

Hence from (2.10) and (2.11), we have

$$l_{n-i+1} + \sum_{m=1}^{n-i+1} r(A_m) - \sum_{m=2}^{n-i+1} l_m \ge r(A_1 A_2 \cdots A_{n-i}) + r(A_{n-i+1}),$$

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that is,

$$\sum_{m=1}^{n-i} r(A_m) \geq \sum_{m=2}^{n-i} l_m + r(A_1A_2\cdots A_{n-i}). \tag{2.12}$$

On the other hand, by Lemma 1.2 again, we know

$$\sum_{m=1}^{n-i} r(A_m) \leq \sum_{m=2}^{n-i} l_m + r(A_1A_2\cdots A_{n-i}). \tag{2.13}$$

Thus from (2.12) and (2.13), we get

$$\sum_{m=1}^{n-i} r(A_m) = \sum_{m=2}^{n-i} l_m + r(A_1A_2\cdots A_{n-i}).$$

This implies that the identity in (2.9) holds for $i = 0, 1, \cdots, n-2$. From (2.9) it is easy to check that the identity

$$r(A_1A_2\cdots A_i) + r(A_{i+1}) = l_{i+1} + r(A_1A_2\cdots A_{i+1})$$

holds for $i = 1, 2, \cdots, n-1$. $\square$

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