Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 334-346, October 2007

ELA

COMBINATORIAL PROPERTIES OF FOURIER-MOTZKIN ELIMINATION*

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Abstract. Fourier-Motzkin elimination is a classical method for solving linear inequalities in which one variable is eliminated in each iteration. This method is considered here as a matrix operation and properties of this operation are established. In particular, the focus is on situations where this matrix operation preserves combinatorial matrices (defined here as $(0, 1, -1)$-matrices).

Key words. Linear inequalities, Fourier-Motzkin elimination, Network matrices.

AMS subject classifications. 05C50, 15A39, 90C27.

1. Introduction.

Fourier-Motzkin elimination is a computational method that may be seen as a generalization of Gaussian elimination. The method is used for finding one, or even all, solutions to a given linear system of inequalities

$$Ax \leq b$$

where $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. Here vector inequality is to be interpreted componentwise. The solution set of the system $Ax \leq b$ is a polyhedron which we denote by P, i.e., $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. The method also finds the projection of P into certain coordinate subspaces. The idea is to eliminate one variable at the time and rewrite the system accordingly (see Section 2). The method was introduced by Fourier in 1827 [3] and his work was motivated by problems in mechanics, least squares etc. A more systematic study of the method was given in Dines [2]. The method was described in the Ph.D. thesis of T.S. Motzkin [5] and the connection to polyhedra was investigated. Kuhn [4] also described the method and used it to give a proof of Farkas' lemma. A presentation of Fourier-Motzkin elimination and its role in computations involving polyhedra (for conversions between different representations of polyhedra) is found in Ziegler [8]. For historical notes and references on linear inequalities and Fourier-Motzkin elimination we refer to Schrijver [7].

In this paper we view Fourier-Motzkin elimination as a matrix operation that transforms the given coefficient matrix into a new one, and our goal is to investigate this operation. The focus is on combinatorial aspects of the operation. This casts light on the operation of projection of polyhedra into coordinate subspaces. In Section 2 Fourier-Motzkin elimination is described and in Section 3 we introduce the mentioned matrix operation, denoted the FM operation. This operation is investigated for incidence matrices in Section 4 and for network matrices and related matrices in Section 5.

We now describe some of the notation used in this paper. If A and B are matrices of the same size, then $A \geq B$ means that the inequality holds componentwise. An all

*Received by the editors 10 March 2005. Accepted for publication 23 September 2007. Handling Editor: Bryan L. Shader.

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A publication of the International Linear Algebra Society
Volume 16, pp. 334-346, October 2007

zeros matrix (of suitable size) is denoted by O. If $A \in \mathbb{R}^{m,n}$ and $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$, then $A(I, J)$ is the submatrix of A formed by the rows in I and columns in J. If $I = \{1, 2, \ldots, m\}$ we simply write $A(:, J)$. The jth column of A is denoted by $A(:, j)$. The transpose of a matrix A is denoted by A^T. The jth unit vector (in $\mathbb{R}^n$) is denoted by e_j.

2. Fourier-Motzkin elimination. We briefly review the Fourier-Motzkin elimination method. Consider again a linear system $Ax \leq b$ where $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^m$ and let $I := \{1, 2, \ldots, m\}$. We write the system in component form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq b_2 \\ & & & & \vdots & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq b_m. \end{array} \tag{2.1}$$

Say that we want to eliminate x_1 from the system (2.1). For each i where $a_{i1} \neq 0$ we multiply the i'th inequality $\sum_{j=1}^n a_{ij}x_j \leq b_i$ by $1/|a_{i1}|$. This gives an equivalent system

$$\begin{array}{cccccccccc} x_1 & + & a'_{i2}x_2 & + & \cdots & + & a'_{in}x_n & \leq b'_i & (i \in I^+) \\ & & a_{i2}x_2 & + & \cdots & + & a_{in}x_n & \leq b_i & (i \in I^0) \\ -x_1 & + & a'_{i2}x_2 & + & \cdots & + & a'_{in}x_n & \leq b'_i & (i \in I^-) \end{array} \tag{2.2}$$

where $I^+ = \{i : a_{i1} > 0\}$, $I^0 = \{i : a_{i1} = 0\}$, $I^- = \{i : a_{i1} < 0\}$, $a'_{ij} = a_{ij}/|a_{i1}|$ and $b'_i = b_i/|a_{i1}|$. Thus, the row index set $I = \{1, 2, \ldots, m\}$ is partitioned into subsets I^+, I^0 and I^-, some of which may be empty. It follows that $x_1, x_2, \ldots, x_n$ is a solution of the original system (2.1) if and only if $x_2, x_3, \ldots, x_n$ satisfy

$$\begin{array}{lll} \sum_{j=2}^n a'_{kj}x_j - b'_k & \leq b'_i - \sum_{j=2}^n a'_{ij}x_j & (i \in I^+, k \in I^-) \\ \sum_{j=2}^n a_{ij}x_j & \leq b_i & (i \in I^0) \end{array} \tag{2.3}$$

and x_1 satisfies

$$\max_{k \in I^-}\left(\sum_{j=2}^n a'_{kj}x_j - b'_k\right) \leq x_1 \leq \min_{i \in I^+}\left(b'_i - \sum_{j=2}^n a'_{ij}x_j\right). \tag{2.4}$$

If I^-(resp. I^+) is empty, the first set of constraints in (2.3) vanishes and the maximum (resp. minimum) in (2.4) is interpreted as ∞ (resp. $-\infty$). If I^0 is empty and either I^- or I^+ is empty too, then we terminate: the general solution of $Ax \leq b$ is obtained by choosing $x_2, x_3, \ldots, x_n$ arbitrarily and choosing x_1 according to (2.4).

The constraint in (2.4) says that x_1 lies in a certain interval which is determined by $x_2, x_3, \ldots, x_n$. The polyhedron defined by (2.3) is the projection of P along the x_1-axis, i.e., into the space of the variables $x_2, x_3, \ldots, x_n$. One may then proceed similarly and eliminate x_2, x_3 etc. Eventually one obtains a system $l \leq x_n \leq u$.

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A publication of the International Linear Algebra Society
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If $l > u$ one concludes that $Ax \leq b$ has no solution, otherwise one may choose $x_n \in [l, u]$, and then choose x_{n-1} in an interval which depends on x_n etc. This back substitution procedure produces a solution $x = (x_1, x_2, \ldots, x_n)$ to $Ax \leq b$. Moreover, *every* solution of $Ax \leq b$ may be produced in this way. (If the system is inconsistent, this might possibly be discovered at an early stage and one terminates.)

The number of constraints may grow exponentially fast as variables are eliminated using Fourier-Motzkin elimination. Actually, a main problem in practice is that the number of inequalities becomes "too large" during the elimination process, even when redundant inequalities are removed. It is therefore of interest to know situations where the projected linear systems are not very large or, at least, have some interesting structure. These questions are discussed in the remaining part of the paper. We refer to [7] and [8] for a further discussion of Fourier-Motzkin elimination and a collection of references on this method.

3. The FM operation. Fourier-Motzkin elimination is a process that works on linear systems (of inequalities). However, it may also be viewed as an operation on matrices from which a given coefficient matrix A produces another matrix B.

Consider again a linear system (2.1) and its coefficient matrix $A \in \mathbb{R}^{m,n}$. Let B be the coefficient matrix of the new linear system (2.3), still viewed as a system in all the variables $x_1, x_2, \ldots, x_n$. B is a real $m' \times n$ matrix with $m' = |I^+| \cdot |I^-| + |I^0| \leq m^2/4$ rows. Thus, using the notation introduced in the previous section, B has a row $(0, a_{i2}, a_{i3}, \ldots, a_{in})$ for each $i \in I^0$ and a row $(0, a'_{i2} + a'_{k2}, a'_{i3} + a'_{k3}, \ldots, a'_{in} + a'_{kn})$ for each pair i, k with $i \in I^+, k \in I^-$. Such a row is simply the vector sum $A'(i, :) + A'(k, :)$ where A' is the coefficient matrix of the system (2.2). If the first column of A contains only positive entries, or only negative entries, or if A has no rows, then B becomes an empty matrix (no rows). This gives rise to a mapping, denoted by FM, which maps A into B, i.e.,

$$B = FM(A).$$

We also get the mapping FM_0 which maps A into the matrix $B_0 = B(:, \{2, 3, \ldots, n\})$, i.e., B_0 is obtained from B by deleting the first column (which is the zero vector). Thus, B_0 is the coefficient matrix of the new linear system (2.3) in the variables $x_2, x_3, \ldots, x_n$.

To be mathematically concise, we should consider FM and FM_0 as mappings on the equivalence classes of matrices under row permutations. Permutations of the rows of A or B do not play any role here; this is motivated by the fact that permutation of the inequalities of the underlying linear systems does not change the solution set.

REMARK 3.1. Assume that $A \geq O$ and let $B = FM(A)$. Then B is obtained from A by simply deleting the rows in A corresponding to positive entries in the first column. Moreover, in (2.4) one only gets an upper bound on the variable x_1. A similar observation holds for nonpositive matrices (and one only gets a lower bound on x_1).

In order to see the full effect of Fourier-Motzkin elimination on the linear system $Ax \leq b$ we may consider the FM_0 operation applied to the augmented matrix $[A\ \ b]$.

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A publication of the International Linear Algebra Society
Volume 16, pp. 334-346, October 2007

The resulting matrix is $[B \;\; b']$ where $B = FM_0(A)$ and where $Bx \leq b'$ is the linear system (2.3).

The FM and FM_0 operations may be iterated several times.

DEFINITION 3.2. The matrix operations FM_0^k and FM^k are defined by

$$FM_0^0(A) = FM^0(A) = A$$

and for $k = 1, 2, \ldots, n-1$

$$\begin{aligned} FM_0^k(A) &= FM_0(FM_0^{k-1}(A)) \\ FM^k(A) &= \begin{bmatrix} O_k & FM_0^k(A) \end{bmatrix} \end{aligned}$$

where O_k denotes a zero matrix with k columns.

Thus, FM_0^k is the coefficient matrix of the projected linear system obtained from $Ax \leq b$ after eliminating variables $x_1, x_2, \ldots, x_k$. The matrix $FM^k(A)$ has n columns while $FM_0^k(A)$ has $n-k$ columns.

We shall focus on the FM operation for matrices with all entries being 0, -1 or 1; such matrices will be called *combinatorial matrices.* These matrices frequently arise in applications, and the corresponding linear systems consist of linear inequalities of the form

$$\sum_{j \in S_i} x_j \leq b_i + \sum_{j \in T_i} x_j$$

where S_i and T_i are disjoint index sets in $\{1, 2, \ldots, n\}$. In some applications, one may be looking for integral or $(0,1)$-vectors satisfying such combinatorial inequalities. This is a frequent theme in the area of polyhedral combinatorics.

DEFINITION 3.3. A matrix A is called *FM-combinatorial* if $FM^k(A)$ is combinatorial for $k = 0, 1, 2, \ldots, n-1$.

By Remark 3.1 it is clear that a combinatorial matrix A with entrywise positive or negative first column is trivially FM-combinatorial (since $FM(A)$ is empty).

The following result is an immediate consequence of Definition 3.3 and the mentioned projection property associated with Fourier-Motzkin elimination.

PROPOSITION 3.4. *If A is FM-combinatorial and b is integral, then the projection of the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ into the space of the variables $x_t, x_{t+1}, \ldots, x_n$ is defined by linear inequalities with $(0,1,-1)$-coefficients and integral right-hand-side ($1 \leq t \leq n$).*

This observation is a motivation for investigating classes of FM-combinatorial matrices. Actually, in polyhedral combinatorics, a very useful method is the *projection technique* where the combinatorial objects of interest are described using a linear system in some higher-dimensional space, i.e., by using additional variables. One is then interested in the structure of the linear system obtained by projecting away these additional variables. In this connection linear inequalities with $(0,1,-1)$-coefficients have a combinatorial interpretation and this explains our interest in the notion of FM-combinatorial matrices.

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The converse of the statement in Proposition 3.4 may not hold; the reason is that Fourier-Motzkin elimination may produce redundant inequalities with coefficients different from 0, ± 1. Note also that every $(0,1)$-matrix and every $(0,-1)$-matrix is trivially FM-combinatorial; this follows from Remark 3.1. Thus, interesting FM-combinatorial matrices contain both positive and negative entries.

The obvious way of checking whether a given matrix A is FM-combinatorial is to use Fourier-Motzkin elimination and calculate $FM^k(A)$ $(1 \le k \le n-1)$.

Example 3.5. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$FM_0(A) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad FM_0^2(A) = [2].$$

So A is not FM-combinatorial.

The following result is easy to establish directly from the definition of a FM-combinatorial matrix.

Proposition 3.6. *The set of FM-combinatorial matrices is closed under the following operations*

- *deletion of rows*
- *duplication of rows*
- *appending a row of zeros.*

Example 3.7. Consider the matrices

$$A_1 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

A_1 is (trivially) FM-combinatorial while A_2 is not FM-combinatorial. This shows that the set of FM-combinatorial matrices is not closed under any of the following operations: (i) multiplying a row by -1, (ii) appending a row which is a unit vector, (iii) appending a row which is the negative of some existing row, (iv) deleting a column.

Moreover, the set of FM-combinatorial matrices is not closed under column permutations.

A natural problem is to find interesting classes of FM-combinatorial matrices. Another problem is to investigate when the columns of a matrix may be permuted so as to get a FM-combinatorial matrix. The remainder of the paper is focused on these two problems. In particular, we study incidence matrices and network matrices and interpret Fourier-Motzkin elimination combinatorially for these matrix classes.

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A publication of the International Linear Algebra Society
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4. Incidence matrices. Consider a digraph $D = (V, E)$ with vertex set V and arc set E. For notational simplicity we assume that $V = \{1, 2, \dots, m\}$. Let A be the *incidence matrix* of D. The rows (columns) of A correspond to the vertices (arcs) of D: the column corresponding to the arc (i, j) has a 1 in row i, a -1 in row j, and zeros in the remaining rows. For $i \in V$ we define $\delta^+(i)$ ($\delta^-(i)$) as the set of arcs in E that leaves (enters) vertex i. More generally, $\delta^+(S)$ ($\delta^-(S)$) is the set of arcs (i, j) where $i \in S$ and $j \notin S$ ($i \notin S$, $j \in S$). A basic operation in a digraph is *contraction* of an arc: it creates a new graph by deleting an arc (and its parallel arcs) and identifying the two end vertices of that arc. We let χ^S denote the incidence vector of a set $S \subseteq E$, so $\chi^S_e = 1$ if $e \in S$ and $\chi^S_e = 0$ otherwise.

The following theorem says that the FM operation on an incidence matrix corresponds to contraction in the digraph.

THEOREM 4.1. *Let A be the incidence matrix of a digraph D and let e be the arc corresponding to the first column of A. Let $B = FM(A)$. Then every column of B corresponding to e or its parallel arcs is the zero vector, and the submatrix consisting of the remaining columns of B is the incidence matrix of the digraph D' obtained from D by contracting e.*

Proof. The rows of A correspond to the vertex set V: the row corresponding to vertex k is $\chi^{\delta^+(k)} - \chi^{\delta^-(k)}$. Let $e = (i, j)$ be the arc corresponding to the first column of A. Consider $B = FM(A)$. Each row in A corresponding to a vertex $k \notin \{i, j\}$ satisfies $a_{k1} = 0$ and it gives rise to a similar row of B. The matrix B has only one more row, and it is obtained by summing the two rows in A corresponding to i and j (as $a_{i1} = 1$ and $a_{j1} = -1$). This row becomes

$$(\chi^{\delta^+(i)} - \chi^{\delta^-(i)}) + (\chi^{\delta^+(j)} - \chi^{\delta^-(j)}) = \chi^{\delta^+(\{i,j\})} - \chi^{\delta^-(\{i,j\})}$$

and this vector has component zero for e and its parallel arcs. This shows that B is the incidence matrix of the digraph D'. $\square$

A direct consequence of the previous theorem is that incidence matrices form a FM-combinatorial matrix class.

COROLLARY 4.2. *Incidence matrices of directed graphs are FM-combinatorial.*

A larger matrix class, containing the incidence matrices, is discussed next. Consider a $(0, \pm 1)$-matrix A of size $m \times n$ whose row index set I may be partitioned into two sets I_1 and I_2 so that each column equals one of the vectors

$$\begin{array}{lll} 1. & O, & \\ 2. & \pm e_i & \text{where } i \in I, \\ 3. & e_i + e_k & \text{where } i \in I_1,\ k \in I_2, \\ 4. & e_i - e_j & \text{where } i, j \text{ are distinct and both lie in } I_1 \text{ or in } I_2. \end{array} \tag{4.1}$$

Let $\mathcal{M}$ be the class of all such matrices. It is known that every matrix $A \in \mathcal{M}$ is totally unimodular (TU), i.e., every minor is -1, 0 or 1, see [1]. One may view $A \in \mathcal{M}$ as the incidence matrix of a "mixed graph" D. Here a *mixed graph* consists of vertices and arcs, but it may have both directed and undirected arcs, and even arcs with only one end, or a zero arc with no end! The matrix A corresponds to a mixed graph D:

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A publication of the International Linear Algebra Society
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the rows of A correspond to the vertices and the columns to the arcs. There are four types of arcs corresponding to the four column types in (4.1): a zero arc (type 1) with no end vertices, a semi-arc (type 2) with a single end vertex, an undirected arc (type 3) and, finally, a directed arc (type 4). We say that a semi-arc corresponding to the column e_i ($-e_i$) *leaves* (*enters*) vertex i. Figure 4.1 shows a mixed graph with two directed arcs, two semi-arcs (to the right in the figure) and one undirected arc. (Zero arcs cannot be visualized, but they may be counted.)

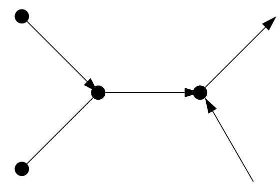

FIG. 4.1. *A mixed graph.*

Let $S \subseteq I$ be a vertex subset in D. The operation of *deleting* S from the mixed graph D produces a new mixed graph with vertex set $I \setminus S$ and with arcs obtained from those of D by removing the end vertices in S. Thus, this operation may produce semi-arcs or even zero arcs. The following theorem shows that the incidence matrix of every mixed graph and its transpose both are FM-combinatorial. Moreover, the proof contains a combinatorial interpretation of the elimination of an arc or a vertex as certain operations on the mixed graph.

THEOREM 4.3. *Let $A \in \mathcal{M}$. Then both A and A^T are FM-combinatorial.*

Proof. Let $A \in \mathcal{M}$ and let $B = FM_0(A)$. We show that $B \in \mathcal{M}$ by finding a new mixed graph D' with B as its incidence matrix. Consider the first column of A, and its corresponding arc e, and distinguish between the four possible cases in (4.1).

1. $A(:,1) = O$. Then B is the incidence matrix of the mixed graph obtained from D by deleting the zero arc e.
2. $A(:,1) = \pm e_i$. Then B is the incidence matrix of the mixed graph obtained from D by deleting the arc e and the vertex i.
3. $A(:,1) = e_i + e_k$ *where* $i \in I_1$, $k \in I_2$. Then B is the incidence matrix of the mixed graph obtained from D by deleting the arc e the vertices i and j.
4. $A(:,1) = e_i - e_k$ *where* i, j *are distinct and both lie in* I_1. (The case when both lie in I_2 is treated similarly.) Then B is the incidence matrix of D' obtained from D by deleting e and contracting the vertices i and j into a new single vertex. Each arc parallel to e becomes a zero arc and arcs with exactly one vertex among i and j arc incident to the new vertex (with similar direction). Since both i and j lie in I_1, the matrix B satisfies all properties of (4.1).

Thus, $B = FM(A) \in \mathcal{M}$ and it follows by induction that A is FM-combinatorial.

Next, consider $C = A^T$ where $A \in \mathcal{M}$ and let v be the vertex (of the mixed graph) corresponding to the first column of C. Let $B = FM_0(C)$. Then B is the incidence matrix of the mixed graph D' obtained from D by deleting v and its incident arcs and adding some new arcs. These new arcs are described in Figure 4.2: a new arc is indicated in italic for each combination of two old arcs with opposite signs incident

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 334-346, October 2007

to v. Note that parallel arcs may arise.

	undirected arc $[v, w]$	directed arc (v, w)	semi-arc leaving v
semi-arc entering v	*semi-arc leaving w*	*semi-arc entering w*	*zero arc*
directed arc (u, v)	*new undirected arc $[u, w]$*	*directed arc (u, w)*	*semi-arc leaving u*

FIG. 4.2. *New arcs in the elimination of v.*

Again, by induction, it follows that A^T is FM-combinatorial. □

In particular, if A is the incidence matrix of a directed graph and we apply Fourier-Motzkin elimination to A^T, then elimination of the vertex v corresponds to deleting v and incident arcs from the digraph D and adding an arc (u, w) whenever (u, v) and (v, w) are arcs in D, see Figure 4.3. Thus, $FM_0(A^T)$ is the transpose of the incidence matrix of this new digraph.

FIG. 4.3. *Elimination of v.*

COROLLARY 4.4. *Let A be the incidence matrix of a digraph D. Then the matrix $\begin{bmatrix} A \\ -A \end{bmatrix}$ is FM-combinatorial.*

Proof. Let $B = FM(A)$ so B has the form described in Theorem 4.1. Then we see that

$$FM(\begin{bmatrix} A \\ -A \end{bmatrix}) = \begin{bmatrix} B \\ -B \\ O \end{bmatrix}$$

where the zero matrix has two rows. The result now follows by induction. □

5. Network matrices and extensions. In this section we consider the FM operation in connection with network matrices and some related matrices. In particular we are concerned with the role that column permutations play for the FM operation. We refer to Brualdi and Ryser [1] or Schrijver [7] for a discussion of network matrices. Note that every matrix in the class $\mathcal{M}$ (see Section 4) is a network matrix.

Let $T = (V, E)$ be a directed tree, i.e., a directed graph where the corresponding undirected graph is a tree. Moreover, let $D = (V, F)$ be a directed graph on the same vertex set V. For each arc $(u, v) \in F$ we let $P_{u,v}$ denote (the arc set of) the unique path in T from u to v. Moreover, $P^+_{u,v}$ ($P^-_{u,v}$) consists of the arcs in $P_{u,v}$ that are directed from u to v (from v to u). The following *network matrix* A is associated with the pair (D, T). The rows and columns of A are associated with F and E, respectively. The row corresponding to the arc $(u, v) \in F$ has a 1 in each column corresponding

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to an arc $e \in P^+_{u,v}$, a -1 in each column corresponding to an arc $e \in P^-_{u,v}$, and it contains zeros elsewhere. Thus, this row equals $\chi^{P^+_{u,v}} - \chi^{P^-_{u,v}}$ which we consider as the signed incidence vector of the path $P_{u,v}$.

Consider a pair (D,T) as above and the associated network matrix A. Let v be a leaf of T (i.e., it has degree 1) and let e be the incident arc. The digraph obtained from T by deleting v (and e) is denoted by $T' = (V', E')$, so $V' = V \setminus \{v\}$ and $E' = E \setminus \{e\}$. Moreover, define $D' = (V', F')$ where

$$F' = (F \setminus (\delta^+_D(v) \cup \delta^-_D(v))) \cup \{(u,w) : (u,v), (v,w) \in F\}.$$

Thus, D' is obtained from D by deleting v and incident arcs and introducing new arcs of the form (u,w) for each pair $(u,v),(v,w) \in F$. We say that the pair (T', D') is obtained from (D,T) by *elimination of* v.

Theorem 5.1. *Consider a directed tree $T = (V,E)$ and a digraph $D = (V,F)$ as above with associated network matrix A. Let v be a leaf of T and assume that the incident arc e corresponds to the first column of A. Let (T',D') be obtained from (D,T) by elimination of v. Then $B = FM_0(A)$ is the network matrix of (T',D') (using the same ordering of arcs in T' as in T).*

Proof. Let as usual $D = (V,F)$. The nonzeros in the first column of A are in those rows corresponding to arcs of the form (u,v) or (v,w) in F. Moreover, the first column contains entries 1 and -1 in a pair of rows that correspond to a pair (u,v), (v,w) with $(u,v),(v,w) \in F$. Such a pair gives rise to a row R in $B = FM_0(A)$ which is the sum of the corresponding rows in A. These two rows in A are the signed incidence vectors of the two paths $P_{u,v}$ and $P_{v,w}$ in the tree T. But then there is a vertex s such that $P_{u,v} = P_{u,s} \cup P_{s,v}$ and $P_{v,w} = P_{v,s} \cup P_{s,w}$; this is due to the fact that T is a tree. It follows that the row R is the signed incidence vector of the (u,w)-path $P_{u,s} \cup P_{s,w}$. The remaining rows of the matrix B are the signed incidence vectors of paths $P_{u,w}$ where $(u,w) \in F$ and $u,w \neq v$. Therefore B is the network matrix of (T',D'). □

If we delete a leaf from a tree T we obtain a new tree where we can delete a new leaf etc. until we are left with a single vertex. This induces an ordering of the arcs of T, namely the order in which they are deleted. We call such an ordering of the arcs in T an *arc elimination ordering* in T. Every nontrivial tree has several arc elimination orderings.

Corollary 5.2. *Let A be a network matrix associated with (D,T) where the columns of A correspond to an arc elimination ordering in the tree T. Then A is FM-combinatorial.*

Proof. Since the columns of A are ordered according to an arc elimination ordering in T, Theorem 5.1 gives that $FM(A)$ is the network matrix associated with (T',D'), $FM^2(A)$ is the network matrix associated with $((T')',(D')')$ etc. So, by induction, A is FM-combinatorial. □

The following example illustrates what can happen if the columns of a network matrix do not correspond to an arc elimination ordering in T.

Example 5.3. Consider the tree T in Figure 5.1, and let the arcs in D be (v_1,v_6), (v_5,v_2), (v_2,v_6), (v_5,v_1). Let A be the network matrix where rows correspond to the

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 334-346, October 2007

arcs in D ordered as above and columns correspond to $e_1, e_2, \ldots, e_6$ (in this order), so

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & 0 \end{bmatrix}.$$

Then

$$FM_0(A) = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad FM_0^2(A) = \begin{bmatrix} 0 & -2 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}.$$

Let now A' be obtained from A by permuting columns so that the first column in A is the last one in A'. Then

$$A' = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & -1 \end{bmatrix}.$$

and

$$FM_0(A') = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{bmatrix}, \; FM_0^2(A') = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

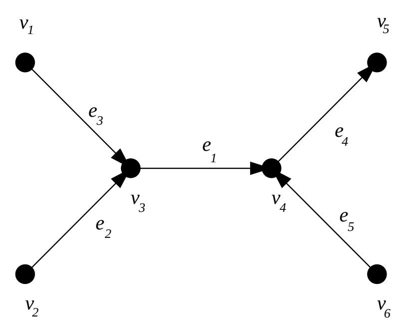

FIG. 5.1. *The tree T.*

When Fourier-Motzkin elimination is used to calculate projections of polyhedra into coordinate subspaces, one often has the freedom to choose the order in which the variables are eliminated. Motivated by this we say that a matrix A is *permuted FM-combinatorial* if the columns of A may be permuted to obtain an FM-combinatorial matrix. If a matrix A is permuted FM-combinatorial, then one would also like to find a permutation matrix P so that AP is FM-combinatorial.

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We now study a large class of permuted FM-combinatorial matrices related to network matrices. Consider a partitioned matrix A of the following form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1p} \\ O & A_{22} & A_{23} & \cdots & A_{2p} \\ O & O & A_{33} & \cdots & A_{3p} \\ \vdots & \vdots & & \ddots & \vdots \\ O & O & & \cdots & A_{pp} \end{bmatrix} \tag{5.1}$$

where (i) all matrices A_{ij} are $(0,1,-1)$-matrices, (ii) the first column in A_{ii} is either e or $-e$ where e is an all ones vector ($i \leq p-1$), (iii) A_{ij} is arbitrary ($1 \leq i < j \leq p$), (iv) A_{pp} is a network matrix. We call B a *generalized network matrix* if its rows and columns may be permuted to obtain a matrix of the specified from in (5.1), i.e., if $A = PBQ$ has the form (5.1) for suitable permutation matrices P and Q.

The following algorithm may be used to test if a matrix is a generalized network matrix.

Algorithm 1.

Input: a $(0,1,-1)$-matrix A.

1. *Start with the matrix A and perform the following operation recursively: test if the matrix contains a nonnegative or nonpositive column. If so, delete this column and the rows in which it contains its nonzeros.*
2. *Test if the remaining matrix is a network matrix.*

We refer to [6] for a detailed description of an efficient (polynomial time) algorithm for recognizing a network matrix.

THEOREM 5.4. *Each generalized network matrix is a permuted FM-combinatorial matrix. Moreover, Algorithm 1 decides in polynomial time whether a given matrix A is a generalized network matrix. The algorithm also finds the desired column permutation that takes A into an FM-combinatorial matrix.*

Proof. Assume that B is a generalized network matrix, so $A = PBQ$ has the form (5.1) for suitable permutation matrices P and Q. We may here assume that the columns corresponding to the network matrix A_{pp} are ordered according to an arc elimination ordering in the associated tree T. We now apply Fourier-Motzkin elimination to the matrix A. Since the first column of A_{11} is either e or $-e$, the resulting matrix $A' = FM_0(A)$ is obtained from A by deleting the first block row and the first column. Then A' contains $k-1$ leading columns that are equal to the zero vector, where k is the number of columns in A_{11}. The FM_0 operation simply deletes these $k-1$ zero columns so the resulting matrix is the same as the one obtained from A by deleting the first block row and the first block column. We proceed similarly with the block A_{22}, and after $p-1$ iterations, we are left with the matrix A_{pp}. Then, by Corollary 5.2, it follows that A is FM-combinatorial. This proves that B is a permuted FM-combinatorial matrix.

We now prove that Algorithm 1 correctly determines whether a given matrix A is a generalized network matrix. Let C denote the matrix obtained by Algorithm 1 after completion of Step 1.

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A publication of the International Linear Algebra Society
Volume 16, pp. 334-346, October 2007

Claim: this matrix C is unique *in the sense that it does not depend on the order in which the columns were deleted in Step 1 of the algorithm.*

Proof of claim: Consider two possible sequences of column deletions, say $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ and $\pi' = (\pi'_1, \pi'_2, \ldots, \pi'_s)$. This means that, in the first (second) sequence, column π_i (π'_i) is deleted in the ith step. We prove that $r = s$ and that π' is a permutation of π. If $\pi = \pi'$, we are done. Otherwise, let i be smallest possible such that $\pi'_i \neq \pi_i$. This means that, in the ith step in the π' sequence, it would be feasible to delete column π_i, although we decided to delete π'_i instead. This possibility of deleting column π_i (in connection with the π' sequence) remains, and therefore there exists a $j > i$ such that $\pi'_j = \pi_i$. By iterating this argument we see that π' must contain all the numbers $\pi_i, \pi_{i+1}, \ldots, \pi_r$ (it is possible to delete these columns in this order). Similarly, we also see that π must contain all the numbers $\pi'_i, \pi'_{i+1}, \ldots, \pi'_s$. It follows that $r = s$ and that π' is a permutation of π.

So, all possible column deletions are permutations of each other. Finally, the set of rows deleted clearly does not depend on the order in which the columns are deleted, but only on which *set* of columns that is deleted. This shows the uniqueness of C, and the claim follows.

By the claim the matrix C after Step 1 is unique. Moreover, A is a generalized network matrix if and only if C is a network matrix, and this is determined in Step 2 of the algorithm (using the polynomial time algorithm described in [6]). This proves the theorem. $\square$

We now turn our attention to a class of $(0, 1, -1)$-matrices associated with paths in directed graphs. A *path incidence matrix* is a matrix where each row is the signed incidence vector of a path in an underlying fixed digraph D. So each such path consists of an arc sequence connecting the initial vertex to the terminal vertex, and the incidence vector contains a $+1$ or a -1 for all these arcs where the sign depends on the direction in which the arc is traversed.

Every network matrix is a path incidence matrix (where the underlying digraph D is a tree), and by Corollary 5.2, these matrices are permuted FM-combinatorial. Moreover, it is possible to construct examples of path incidence matrices that are *not* permuted FM-combinatorial, see below. This property (permuted FM-combinatorial) seems to reflect a complicated interplay between the selection of paths and the structure of the underlying digraph D.

Our final result characterizes the digraphs D for which *all* path incidence matrices are permuted FM-combinatorial.

Theorem 5.5. *Consider a digraph D and the corresponding undirected graph G. Then every path incidence matrix associated with D is permuted FM-combinatorial if and only if G is acyclic.*

Proof. Let A be a path incidence matrix in a digraph D where the corresponding undirected graph G is acyclic. Then G decomposes into a set of disjoint trees $T_1, T_2, \ldots, T_k$. By suitable permutations of the rows and columns of A we obtain a matrix which is the direct sum of path incidence matrices corresponding to the trees $T_1, T_2, \ldots, T_k$. Furthermore, we may order the columns of this matrix according to an arc elimination ordering in each T_i. It now follows from Corollary 5.2 that

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 334-346, October 2007

this permuted matrix A is FM-combinatorial, so the original matrix A is permuted FM-combinatorial.

Assume next that G (the undirected graph associated with D) contains a cycle. Let $e_1, e_2, \ldots, e_k$ be the arcs in this cycle (ordered consecutively). For simplicity of presentation we assume that all arcs in the cycle have the same direction (so the tail of e_i is the head of e_{i+1} for each i); the general case where the directions vary can be treated similarly. For each $i \leq k$ consider the three paths $P_i^1 = (e_{i-1}, e_i)$, $P_i^2 = (e_i, e_{i+1})$, and $P_i^3 = (e_{i-1}, e_i, e_{i+1})$, as well as the three paths that are the opposite of P_i^1, P_i^2 and P_i^3 (e.g., the opposite path of P_i^1 is (e_i, e_{i-1}). Let A be the path incidence matrix corresponding to these $6k$ paths. This matrix may contain some columns that are equal to the zero vector; they correspond to arcs outside C. Consider a arbitrary ordering of the columns of A and apply Fourier-Motzkin elimination based on this ordering. Whenever we meet a column which is the zero vector, that column will simply be deleted. Eventually we come to the first column corresponding to an arc in C, say this is e_i. Let B be the matrix obtained after eliminating e_i. Then B contains the following submatrix corresponding to the columns e_{i-1} and e_{i+1} and suitable rows

$$B^* = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The entries in the remaining columns of B in these rows are all zero. Therefore, eventually we have to eliminate either e_{i-1} or e_{i+1} and, in either case, we see from B^* that we will get an entry which is equal to 2 or -2. This proves that the original matrix A is not permuted FM-combinatorial. ◻

Acknowledgment. The author would like to thank the referee for several valuable comments. Also, thanks to Kristina R. Dahl for interesting discussions in connection with Theorem 5.4.

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