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ON GENERALIZED INVERSES OF BANDED MATRICES*

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Abstract. Bounds for the ranks of upper-right submatrices of a generalized inverse of a strictly lower k-banded matrix are obtained. It is shown that such ranks can be exactly predicted under some conditions. The proof uses the Nullity Theorem and bordering technique for generalized inverse.

Key words. Generalized inverse, Banded matrix, Rank.

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1. Introduction and preliminaries.

We consider matrices over the complex field. If A is an $m \times n$ matrix and if $S \subset \{1, 2, \ldots, m\}$ and $T \subset \{1, 2, \ldots, n\}$ are nonempty subsets, then $A(S, T)$ will denote the submatrix indexed by rows in S and columns in T. For $u_1 \leq u_2$, the set $\{u_1, u_1 + 1, \ldots, u_2\}$ will be denoted by $u_1 : u_2$.

Let A and B be matrices of order $m \times n$ and $n \times m$ respectively. If $S \subset \{1, 2, \ldots, m\}$ and $T \subset \{1, 2, \ldots, n\}$ are nonempty proper subsets, then submatrices $A(S, T)$ and $B(T', S')$ are said to be complementary. Here $S' = \{1, 2, \ldots, m\} \setminus S$ and $T' = \{1, 2, \ldots, n\} \setminus T$.

Let A be an $n \times n$ nonsingular matrix and let $B = A^{-1}$. If $b_{ij} = 0$, then, in view of the cofactor formula for the inverse of a matrix, the $(n-1) \times (n-1)$ submatrix of A which is complementary to b_{ij} must be singular. In fact, something more is true. The rank of that submatrix of A is precisely $n-2$. The proof of this fact is a simple exercise using the Jacobi identity and it may be worthwhile to recall it here. (If A is a nonsingular $n \times n$ matrix with $B = A^{-1}$, and if $S \subset \{1, 2, \ldots, n\}$ and $T \subset \{1, 2, \ldots, n\}$ are nonempty proper subsets, then the Jacobi identity asserts that $|A(S, T)||B| = |B(T', S')|$.) For convenience, suppose $b_{11} = 0$. Since B is nonsingular, there exist r and s such that $b_{1r} \neq 0$ and $b_{s1} \neq 0$. Then the rank of $B(\{1, s\}, \{1, r\})$ equals 2. By the Jacobi identity, the complementary submatrix of A must be nonsingular. Therefore, the rank of $A(2 : n, 2 : n)$ is $n-2$.

If A is an $m \times n$ matrix of rank r, then the row nullity $\eta(A)$ of A is defined to be $m-r$. The observation contained in the preceding paragraph is extended in the *Nullity Theorem* due to Gustafson [6] and, independently, due to Fiedler and Markham [5].

THEOREM 1.1. (The Nullity Theorem) *Let A be a nonsingular $n \times n$ matrix. Then any submatrix of A and its complementary submatrix in A^{-1} have the same row nullity.*

Theorem 1.1 has received considerable attention recently due to its application to predicting ranks of submatrices of inverses of structured matrices, see, for example, [2, 8, 9], and the references contained therein.

If A is rectangular, or square and singular, then it is natural to consider complementary submatrices of A and of a generalized inverse ($\{1\}$-inverse) of A. Recall that

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if A is an $m \times n$ matrix, then an $n \times m$ matrix is said to be a generalized inverse of A if $AXA = A$. We assume familiarity with basic concepts in the theory of generalized inverses, see [3, 4]. As usual, we denote the Moore-Penrose inverse of A as A^+.

An early extension of Theorem 1.1 to the Moore-Penrose inverse was given by Robinson [7], who obtained bounds for the difference between the row nullity of a submatrix of A and that of the complementary submatrix of A^+. These bounds were shown to be true for an arbitrary generalized inverse in [1] and are stated next.

THEOREM 1.2. *Let A be an $m \times n$ matrix of rank r and let X be a generalized inverse of A. If $S \subset \{1, 2, \ldots, m\}$ and $T \subset \{1, 2, \ldots, n\}$ are nonempty proper subsets, then*

$$-(m - r) \leq \eta(X(T', S')) - \eta(A(S, T)) \leq n - r. \tag{1.1}$$

We now introduce some more definitions. The $m \times n$ matrix A is called lower k-banded if $a_{ij} = 0$ for $j - i > k$ and strictly lower k-banded if, in addition, $a_{ij} \neq 0$ for $j - i = k$. We assume $1 - m \leq k \leq n - 1$. Note that if A is a square $n \times n$ matrix, then it is lower Hessenberg if and only if it is lower 1-banded.

In the next section, we employ Theorem 1.2 to get information about ranks of certain submatrices of generalized inverses of a strictly lower k-banded matrix. In Section 3, we use bordering technique to show that such ranks can be exactly predicted under some conditions.

2. Submatrices of generalized inverses of a banded matrix. If A is a strictly lower k-banded matrix, then the rank of an upper-right submatrix of A is completely determined in some cases. For example, consider the 6×7 matrix A which is strictly lower 3-banded:

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \times & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \times & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \times & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Here a dot indicates an arbitrary entry, while a $\times$ indicates a nonzero entry. Note that $\mathit{rank}\, A(1:3, 4:7) = 3$, $\mathit{rank}\, A(1:2, 3:7) = 2$ etc., while the rank of $A(1:5, 2:7)$ cannot be determined completely, except for the fact that it must be at least 4. Evidently, there are certain submatrices which are not "upper-right" and whose rank is determined, such as $\mathit{rank}\, A(2:3, 5:6) = 2$. However, we will work with only upper-right matrices for convenience. Observe that the complementary submatrix of an upper-right submatrix is also an upper-right submatrix.

In the next result, we summarize the situations in which the rank of an upper-right submatrix of a strictly lower k-banded matrix is completely determined.

THEOREM 2.1. *Let A be an $m \times n$ strictly lower k-banded matrix, $1 - m \leq k \leq n - 1$, and let $1 \leq u \leq m$, $1 \leq v \leq n$.*

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(i) Suppose $k \geq 0$. Then

$$rank\, A(1:u, v:n) = \begin{cases} u & \text{if} \quad u \leq n-k,\ v \leq k+1 \\ 0 & \text{if} \quad u \leq n-k,\ v > u+k \\ u-v+k+1 & \text{if} \quad u \leq n-k,\ k+1 < v \leq u+k \\ n-v+1 & \text{if} \quad n-k \leq u,\ v \geq k+1 \end{cases}.$$

(ii) Suppose $k < 0$. Then

$$rank\, A(1:u, v:n) = \begin{cases} 0 & \text{if} \quad u < v-k \\ u-v+k+1 & \text{if} \quad v-k \leq u < n+k \\ n-v+1 & \text{if} \quad u-k \geq n \end{cases}.$$

Proof. If $u \leq n-k$ and $v \leq k+1$, then $A(1:u, v:n)$ has the form

$$A(1:u,v:n) = \begin{array}{c c} & \begin{array}{ccccccc} v & \cdots & k+1 & \cdots & u+k & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ u \end{array} & \begin{pmatrix} \cdot & \cdot & \times & 0 & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdot & \ddots & \cdots & \cdots & 0 \\ \vdots & & & & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \times & \cdots & 0 \end{pmatrix} \end{array}.$$

Note that $A(1:u, k+1:u+k)$ is a submatrix of $A(1:u, v:n)$ which is clearly nonsingular. Therefore, $A(1:u, v:n)$ has rank u. We illustrate one more case. Suppose $k < 0$ and let $v-k \leq u < n+k$. Then $A(1:u, v:n)$ has the form

$$A(1:u,v:n) = \begin{array}{c c} & \begin{array}{ccccc} v & \cdots & u+k & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ v-k \\ \vdots \\ u \end{array} & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \times & \cdots & 0 & \cdots & 0 \\ \cdot & \ddots & \cdot & \cdot & \vdots \\ \cdot & \cdot & \times & & 0 \end{pmatrix} \end{array}.$$

Again $A(v-k:u, v:u+k)$ is a nonsingular submatrix of $A(1:u, v:n)$ of order $u-v+k+1$. It is also evident that any square submatrix of $A(1:u, v:n)$ of order greater than $u-v+k+1$ is singular. Therefore, the rank of $A(1:u, v:n)$ is $u-v+k+1$. The rest of the cases are proved similarly. ☐

Let A be an $m \times n$ strictly lower k-banded matrix and let X be a generalized inverse of A. We may use Theorems 1.2 and 2.1 to get bounds for the rank of an upper-right submatrix of X. We give some such bounds as illustration in the next result.

THEOREM 2.2. *Let A be an $m \times n$ strictly lower k-banded matrix and let X be a generalized inverse of A.*

(i) If $u \leq n-k$ and $k+1 < v \leq u+k$, then

$$-n+r+k \leq rank\, X(1:v-1, u+1:m) \leq k+m-r.$$

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(ii) If $u \leq n-k$ and $v > u+k$, then

$$-n+r+v-1-u \leq rank\, X(1:v-1, u+1:m) \leq m-r+v-1-u.$$

Proof. By Theorem 1.2,

$$-(m-r) \leq \eta(X(1:v-1, u+1:m)) - \eta(A(1:u, v:n)) \leq n-r. \tag{2.1}$$

Since $u \leq n-k$ and $k+1 < v \leq u+k$, by Theorem 2.1,

$$rank\, A(1:u, v:n) = u-v+k+1. \tag{2.2}$$

Therefore,

$$\eta(A(1:u, v:n)) = v-k-1. \tag{2.3}$$

It follows from (2.1) and (2.3) that

$$-m+r-v-k+1 \leq \eta(X(1:v-1, u+1:m)) \leq n-r+v-k-1. \tag{2.4}$$

Since $\eta(X(1:v-1, u+1:m)) = v-1-rank\, X(1:v-1, u+1:m)$, (i) follows from (2.4). The proof of (ii) is similar. ☐

We remark that the upper bound in Theorem 2.2 (i), which clearly holds even if A is lower k-banded and not strictly lower k-banded, is the central result in [2]. Furthermore, in the same paper, the lower bound in Theorem 2.2 (ii) has been given as a conjecture (see [2, p. 165]).

We also observe that if A is a tridiagonal $n \times n$ matrix and if X is a generalized inverse of A, then by Theorem 2.2 (i), the rank of any submatrix of X above the main diagonal is at most 2.

3. Generalized inverses of Hessenberg matrices. Let A be a strictly lower k-banded matrix and let X be a generalized inverse of A. In Theorem 2.2 we obtained upper and lower bounds for the rank of an upper-right submatrix of X. We now show that with some further conditions the rank of an upper-right submatrix of X can be predicted exactly.

We begin by reproducing the following example from [2] which motivates the result in this section. Consider the tridiagonal matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and its Moore-Penrose inverse

$$T^{+} = \frac{1}{3}\begin{bmatrix} 0 & 2 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & 2 & 0 \end{bmatrix}.$$

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By Theorem 2.2, $rank\, T^{+}(1:2,3:5)$ is in the interval $[0,2]$. However, here the rank is equal to 2 and we show that this can be predicted precisely. The proof is based on bordering technique for generalized inverses and the observation that T admits right and left null vectors without a zero coordinate.

In this section, by a Hessenberg matrix we mean a square, strictly lower 1-banded matrix. Similar statements can be proved for a strictly lower k-banded matrix, $k > 1$.

THEOREM 3.1. *Let A be an $n \times n$ Hessenberg matrix with $rank\, A = n-1$. Let w and z be $n \times 1$ vectors with no zero coordinate. Let X be the reflexive generalized inverse of A which satisfies $Xw = 0$ and $z^*X = 0$. Then for $1 \le u \le n-1$, $2 \le v \le n$,*

$$rank\, X(1:v-1, u+1:n) = \begin{cases} v-1 & \text{if} \quad u \le n-1,\ v=2 \\ v-u & \text{if} \quad u \le n-1,\ v > u+1 \\ 2 & \text{if} \quad u \le n-2,\ 2 < v \le u+1 \\ n-u & \text{if} \quad u = n-1,\ v > 2 \end{cases}.$$

Proof. We remark that the existence and uniqueness of the reflexive generalized inverse that satisfies the hypotheses of the theorem are well-known, see, for example, ([3, p. 71]). In fact, X can be obtained by inverting a bordering matrix (see [3, p. 198]).

Consider the bordered matrix

$$B = \begin{bmatrix} A & w \\ z^* & 0 \end{bmatrix}.$$

Then B^{-1} has the form

$$B^{-1} = \begin{bmatrix} X & \cdot \\ \cdot & 0 \end{bmatrix}.$$

We claim that for $1 \le u \le n, 2 \le v \le n$,

$$rank\, B(1:u \cup \{n+1\}, v:n+1) = \begin{cases} u & \text{if} \quad u \le n-1,\ v=2 \\ 2 & \text{if} \quad u \le n-1,\ v > u+1 \\ u-v+4 & \text{if} \quad u \le n-2,\ 2 < v \le u+1 \\ n-v+2 & \text{if} \quad u \ge n-1,\ v > 2 \end{cases}.$$

We illustrate the proof of the claim. Suppose $u \le n-2$ and $2 < v \le u+1$. Then $B(1:u \cup \{n+1\}, v:n+1)$ has the form

$$\begin{array}{c} \\ 1 \\ \vdots \\ v-1 \\ \vdots \\ u \\ n+1 \end{array} \begin{array}{c} \begin{array}{cccccc} v & \cdots & u+1 & \cdots & n & n+1 \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & w_1 \\ \vdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ \times & \cdots & \cdots & \cdots & \vdots & w_{v-1} \\ \vdots & \ddots & & & & \vdots \\ \cdots & \cdots & \times & \cdots & & w_u \\ z_v & \cdots & z_{u+1} & \cdots & z_n & 0 \end{pmatrix} \end{array}.$$

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The submatrix $B(v-1:u\cup\{1,n+1\},v:u+2\cup\{n+1\})$ of order $v-u+4$ of $B(1:u\cup\{n+1\},v:n+1)$ can be seen to be nonsingular, in view of $w_1\neq 0$ and $z_u\neq 0$, and therefore $B(1:u\cup\{n+1\},v:n+1)$ has rank $u-v+4$. The remaining cases are treated similarly and therefore the claim is proved.

¿From the rank of $B(1:u\cup\{n+1\},v:n+1)$ we may compute its row nullity, which is the same as the row nullity of its complementary submatrix in B^{-1}, by Theorem 1.1. Note that a complementary submatrix of $B(1:u\cup\{n+1\},v:n+1)$ in B^{-1} is in fact a submatrix of X. Therefore, we get the following information,

$$\eta(X(1:v-1,u+1:n))=\begin{cases} 1 & \text{if} \quad u\leq n-1,\ v=2\\ u-1 & \text{if} \quad u\leq n-1,\ v>u+1\\ v-3 & \text{if} \quad u\leq n-2,\ 2<v\leq u+1\\ u+v-n-1 & \text{if} \quad u\geq n-1,\ v>2\end{cases}.$$

¿From the row nullity of $X(1:v-1,u+1:n)$ we see that its rank must be as asserted in the Theorem and the proof is complete. □

We now provide an application of Theorem 3.1 to predicting the rank of minors above the diagonal in the Moore-Penrose inverse of a tridiagonal matrix, which generalizes (as well as proves) the observation made in the example at the beginning of this section.

Corollary 3.2. *Let T be an $n\times n$ tridiagonal matrix with $\operatorname{rank} T=n-1$. Suppose T admits right and left null vectors with no zero coordinate. Then for $1\leq s\leq n-1$ and $2\leq t\leq n$,*

$$\operatorname{rank} T^{+}(1:s,t:n)=\begin{cases} 1 & \text{if} \quad s=1\\ 1 & \text{if} \quad t=n\\ 2 & \text{if} \quad 2\leq s\leq n-1,\ s<t\leq n-1\end{cases},$$

that is, any upper-right submatrix of T^+ which is above the main diagonal and has at least 2 rows and columns has rank 2.

Proof. Let w and z be vectors with no zero coordinate such that $Tw=0$ and $T^*z=0$. Then the unique reflexive generalized inverse X of T which satisfies $Xw=0$ and $z^*X=0$ is T^+ (see [3, p.196]). Therefore, the result follows from Theorem 3.1. □

We remark that the hypothesis of Corollary 3.2 is satisfied if T has rank $n-1$ and every $(n-1)\times(n-1)$ principal submatrix of T is nonsingular. As an example, if T is the $n\times n$ tridiagonal matrix with $t_{ij}=-1$ for $|i-j|=1$ and with zero row sums, then the hypothesis of Corollary 3.2 is satisfied.

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