The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 4, pp. 19-31, August 1998.
ISSN 1081-3810.

ELA

ON THE CHARACTERISTIC POLYNOMIAL OF MATRICES WITH PRESCRIBED COLUMNS AND THE STABILIZATION AND OBSERVABILITY OF LINEAR SYSTEMS*

SUSANA FURTADO† AND FERNANDO C. SILVA‡

Abstract. Let $A \in F^{n \times n}$, $B \in F^{n \times t}$, where F is an arbitrary field. In this paper, the possible characteristic polynomials of $[\,A \quad B\,]$, when some of its columns are prescribed and the other columns vary, are described. The characteristic polynomial of $[\,A \quad B\,]$ is defined as the largest determinantal divisor (or the product of the invariant factors) of $[\,xI_n - A \quad -B\,]$. This result generalizes a previous theorem by H. Wimmer which studies the same problem when $t = 0$. As a consequence, it is extended to arbitrary fields a result, already proved for infinite fields, that describes all the possible characteristic polynomials of a square matrix when an arbitrary submatrix is fixed and the other entries vary. Finally, applications to the stabilization and observability of linear systems by state feedback are studied.

AMS subject classifications. 15A18, 93B60

Key words. Matrix completion, eigenvalue, linear system.

1. Introduction. Throughout this paper, F denotes a field. If $f(x)$ is a polynomial, $d(f)$ denotes its degree.

Several results are known that study the existence of matrices (matrix completions) with a fixed submatrix and satisfying certain conditions. For example, the following theorem, due to Wimmer, describes the possible characteristic polynomials of a matrix when a certain number of rows are fixed and the others vary.

THEOREM 1.1. [20] *Let $A_{1,1} \in F^{p \times p}$, $A_{1,2} \in F^{p \times q}$ and $m = p + q$. Let $f \in F[x]$ be a monic polynomial of degree m. Let $\alpha_1 \mid \cdots \mid \alpha_p$ be the invariant factors of*

$$\begin{bmatrix} xI_p - A_{1,1} & -A_{1,2} \end{bmatrix}.$$

Then, there exist $A_{2,1} \in F^{q \times p}$ and $A_{2,2} \in F^{q \times q}$ such that

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

has characteristic polynomial f if and only if

$$\alpha_1 \cdots \alpha_p \mid f. \tag{1}$$

Later, Zaballa [22] characterized the possible invariant polynomials of a matrix when a certain number of rows are fixed and the others vary. As a square matrix is

*This work was partially supported by the Praxis Program (praxis/2/2.1/mat/73/94) and was done within the activities of the Centro de Álgebra da Universidade de Lisboa. Received by the editors on 26 October 1997. Final manuscript accepted on 7 August 1998. Handling editor: Daniel Hershkowitz.

†Faculdade de Economia, Universidade do Porto, Rua Dr. Roberto Frias, 4200 Porto, Portugal (sbf@fep.up.pt).

‡Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, Rua Ernesto de Vasconcelos, 1700 Lisboa, Portugal (fcsilva@fc.ul.pt).

similar to its transpose, these results also describe the possible characteristic polynomials and the possible invariant polynomials of a matrix when a certain number of columns are fixed and the others vary. For other results of this type, see, e.g., [1, 9, 10, 11, 12, 15, 17, 19, 22, 25].

Recall that the characteristic polynomial of a matrix $A \in F^{m\times m}$ is the product of the invariant factors of $xI_m - A$. Now call *characteristic polynomial* of a matrix $\begin{bmatrix} A & B \end{bmatrix}$, where $A \in F^{m\times m}$, $B \in F^{m\times n}$, to the product of the invariant factors of

$$\begin{bmatrix} xI_m - A & -B \end{bmatrix}.$$

Note that the condition (1) says that the characteristic polynomial of $\begin{bmatrix} A_{1,1} & A_{1,2} \end{bmatrix}$ divides f.

In a previous paper [7], we have described all possible characteristic polynomials of $\begin{bmatrix} A & B \end{bmatrix}$ when some of its rows are fixed and the others vary. The main purpose of this paper is to describe all the possible characteristic polynomials of $\begin{bmatrix} A & B \end{bmatrix}$ when some of its columns are fixed and the others vary. Note that, in order to solve this problem, we may assume that the columns fixed in A are the first ones and that the columns fixed in B are also the first ones. In fact, if $P \in F^{m\times m}$ and $Q \in F^{n\times n}$ are permutation matrices, then

$$\begin{bmatrix} A & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} P^{-1}AP & P^{-1}BQ \end{bmatrix}$$

have the same characteristic polynomial.

Analogously, call *characteristic polynomial* of a matrix $\begin{bmatrix} A^t & C^t \end{bmatrix}^t$, where $A \in F^{m\times m}$, $C \in F^{s\times m}$, to the product of the invariant factors of

$$\begin{bmatrix} xI_m - A^t & -C^t \end{bmatrix}^t.$$

Two matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \tag{2}$$

$A, A' \in F^{m\times m}$, $B, B' \in F^{m\times n}$, $C, C' \in F^{s\times m}$ and $D, D' \in F^{s\times n}$, are said to be *m-similar* if there exist matrices $P \in F^{m\times m}$, $Q \in F^{s\times s}$, $R \in F^{n\times n}$, $S \in F^{m\times s}$, $T \in F^{n\times m}$ such that P, Q and R are nonsingular and

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} P^{-1} & S \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & 0 \\ T & R \end{bmatrix}. \tag{3}$$

It is easy to see that the matrices (2) are m-similar if and only if the pencils

$$\begin{bmatrix} xI_m - A & -B \\ -C & -D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} xI_m - A' & -B' \\ -C' & -D' \end{bmatrix} \tag{4}$$

are strictly equivalent. Therefore the m-similarity classes are completely described by the Kronecker invariants for strict equivalence. Moreover, a canonical form for m-similarity results easily from the Kronecker canonical form for strict equivalence. An explicit canonical form for m-similarity was presented in [3, Lemma 2]. For details about strict equivalence, see, e.g., [8]. Note that the proof of the existence of the Kronecker canonical form presented in [8] fails in finite fields. However the Kronecker canonical form is valid in arbitrary fields; see [6].

When $s = 0$, m-similarity appeared in [2] with the name of feedback equivalence. (Also see [22].) Note that, in this case, the pencils (4), which do not have the second row of blocks, have neither row minimal indices nor infinite elementary divisors. When $s = n = 0$, m-similarity is the usual relation of similarity.

2. The Characteristic Polynomial of Matrices with Prescribed Columns.

The following result was obtained by Zaballa when $t = 0$ [22] and when $q = 0$ [23]. The general case was established in [4].

THEOREM 2.1. *Let $A_{1,1} \in F^{p\times p}$ and $A_{2,1} \in F^{q\times p}$. Let $\alpha_1 \mid \cdots \mid \alpha_p$ be the invariant factors and $k_1 \geq \cdots \geq k_q$ be the row minimal indices of*

$$\begin{bmatrix} xI_p - A_{1,1} \\ -A_{2,1} \end{bmatrix}. \tag{5}$$

Let $m = p+q$, $B_1 \in F^{m\times m}$ and $B_2 \in F^{m\times t}$. Let $\gamma_1 \mid \cdots \mid \gamma_m$ be the invariant factors and $s_1 \geq \cdots \geq s_\rho > s_{\rho+1} = \cdots = s_t (= 0)$ be the column minimal indices of

$$\begin{bmatrix} xI_m - B_1 & -B_2 \end{bmatrix}. \tag{6}$$

Then, there exist $A_{1,2} \in F^{p\times q}$, $A_{1,3} \in F^{p\times t}$, $A_{2,2} \in F^{q\times q}$ and $A_{2,3} \in F^{q\times t}$ such that

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \end{bmatrix} \tag{7}$$

and $\begin{bmatrix} B_1 & B_2 \end{bmatrix}$ are feedback equivalent if and only if

$(i_{2.1})$ $\gamma_i \mid \alpha_i \mid \gamma_{i+q+\rho}, \qquad i \in \{1,\ldots,p\}$,

$(ii_{2.1})$ $m \geq d(\zeta^q)$ *and*

$$(k_1+1,\ldots,k_q+1) \prec (m - d(\zeta^{q-1}), d(\zeta^{q-1}) - d(\zeta^{q-2}), \ldots, d(\zeta^1) - d(\zeta^0)),$$

where $\zeta^j = \zeta_1^j \cdots \zeta_{p+j}^j$ *and* $\zeta_i^j = \text{l.c.m.}\{\alpha_{i-j}, \gamma_i\}$, $i \in \{1,\ldots,p+j\}, j \in \{0,\ldots,q\}$;

$(iii_{2.1})$ $m \geq d(\eta^\rho)$ *and*

$$(s_1,\ldots,s_\rho) \prec (m - d(\eta^{\rho-1}), d(\eta^{\rho-1}) - d(\eta^{\rho-2}), \ldots, d(\eta^1) - d(\eta^0)),$$

where $\eta^j = \eta_1^j \cdots \eta_{m-\rho+j}^j$ *and* $\eta_i^j = \text{l.c.m.}\{\alpha_{i-q}, \gamma_{i+\rho-j}\}$, $i \in \{1,\ldots,m-\rho+j\}, j \in \{0,\ldots,\rho\}$.

[We make convention that $\gamma_i = 0$ whenever $i > m$, and $\alpha_i = 1$ whenever $i \leq 0$.]

The following lemma is not hard to prove; see [17, Lemma 8] for details.

LEMMA 2.2. *Let $t_1,\ldots,t_m,t'_1,\ldots,t'_m,t''$ be integers such that $t_1 \leq \cdots \leq t_m, t'_1 \leq \cdots \leq t'_m, t_1+\cdots+t_m \leq t'' \leq t'_1+\cdots+t'_m$. Then there exist integers $t''_1,\ldots,t''_m$ such that $t''_1 \leq \cdots \leq t''_m, t_i \leq t''_i \leq t'_i, i \in \{1,\ldots,m\}, t''_1+\cdots+t''_m = t''$.*

Corollary 2.3. *Let $A_{1,1} \in F^{p\times p}$ and $A_{2,1} \in F^{q\times p}$. Let $\alpha_1 \mid \cdots \mid \alpha_p$ be the invariant factors of (5). Let $m = p+q$ and let t be a positive integer. Let $f \in F[x]$ be a monic polynomial. Then, there exist $A_{1,2} \in F^{p\times q}$, $A_{1,3} \in F^{p\times t}$, $A_{2,2} \in F^{q\times q}$ and $A_{2,3} \in F^{q\times t}$ such that (7) has characteristic polynomial f if and only if $d(\text{l.c.m.}\{\alpha_1\cdots\alpha_p, f\}) \leq m$ and*

$$\alpha_1\cdots\alpha_{p-\rho} \mid f, \tag{8}$$

where $\rho = \min\{m - d(f), t\}$.

Proof. Necessity. Suppose that there exists a matrix of the form (7) with characteristic polynomial f. Let $\gamma_1 \mid \cdots \mid \gamma_m$ be the invariant factors of

$$\begin{bmatrix} xI_p - A_{1,1} & -A_{1,2} & -A_{1,3} \\ -A_{2,1} & xI_q - A_{2,2} & -A_{2,3} \end{bmatrix}. \tag{9}$$

According to Theorem 2.1, $m \geq d(\zeta^q)$. Clearly $\text{l.c.m.}\{\alpha_1\cdots\alpha_p, f\} \mid \zeta^q$ and, therefore, $d(\text{l.c.m.}\{\alpha_1\cdots\alpha_p, f\}) \leq m$. It also follows from Theorem 2.1, that $\alpha_1\cdots\alpha_{p-\rho'} \mid \gamma_{1+q+\rho'}\cdots\gamma_m$, where ρ' is the number of nonzero column minimal indices of (9). Note that $\rho \geq \rho'$. Therefore $\alpha_1\cdots\alpha_{p-\rho} \mid f$.

Sufficiency. Suppose that $d(\text{l.c.m.}\{\alpha_1\cdots\alpha_p, f\}) \leq m$ and $\alpha_1\cdots\alpha_{p-\rho} \mid f$, where $\rho = \min\{m-d(f), t\}$. Suppose that

$$\begin{aligned} \alpha_i &= \pi_1^{l_{i,1}}\cdots\pi_\tau^{l_{i,\tau}}, \qquad i \in \{1,\dots,p\}, \\ f &= \pi_1^{l_1}\cdots\pi_\tau^{l_\tau}, \end{aligned}$$

were $\pi_1,\dots,\pi_\tau$ are monic, pairwise distinct, irreducible polynomials and $l_{i,j}$ and l_j are nonnegative integers. For each $j \in \{1,\dots,\tau\}$, let

$$\begin{aligned} l'_j &= \max\{l_{1,j}+\cdots+l_{p,j}, l_j\}, \\ l'_{i,j} &= l_{i,j}, \qquad i \in \{1,\dots,p-1\}, \\ l'_{p,j} &= l_{p,j} + l'_j - (l_{1,j}+\cdots+l_{p,j}). \end{aligned} \tag{10}$$

We have

$$l_{1,j}+\cdots+l_{p-\rho,j} \leq l_j \leq l'_j = l'_{1,j}+\cdots+l'_{p,j}.$$

The first inequality follows from (8). Take $l_{i,j} = l'_{i,j} = 0$ whenever $i \leq 0$. According to Lemma 2.2, there exist integers $l''_{1,j},\dots,l''_{m,j}$ such that

$$\begin{aligned} & l''_{1,j} \leq \cdots \leq l''_{m,j}, \\ & l_{i-q-\rho,j} \leq l''_{i,j} \leq l'_{i-q,j}, \qquad i \in \{1,\dots,m\}, \\ & l_j = l''_{1,j}+\cdots+l''_{m,j}. \end{aligned} \tag{11}$$

Let

$$\gamma_i = \pi_1^{l''_{i,1}}\cdots\pi_\tau^{l''_{i,\tau}}, \qquad i \in \{1,\dots,m\}.$$

From (10) and (11), it follows that

$$\begin{aligned} \alpha_{i-q-\rho} \quad &| \quad \gamma_i, \qquad i \in \{1, \ldots, m\}, \\ \gamma_i \quad &| \quad \alpha_{i-q} \mid \alpha_i, \qquad i \in \{1, \ldots, m-1\}, \end{aligned} \tag{12}$$

where $\alpha_i = 0$ whenever $i > p$. If $q > 0$, then the condition $(i_{2.1})$ is already proved. Now suppose that $q = 0$. Then $d(\alpha_1 \cdots \alpha_p) = p$. From $d(\text{l.c.m.}\{\alpha_1 \cdots \alpha_p, f\}) \leq m$ it follows that $f \mid \alpha_1 \cdots \alpha_p$. Therefore $l'_j = l_{1,j} + \cdots + l_{p,j}$ and $l'_{p,j} = l_{p,j}$, $j \in \{1, \ldots, \tau\}$. From (11) it follows that $\gamma_p \mid \alpha_p$. In any case, the condition $(i_{2.1})$ is proved.

Let $j \in \{1, \ldots, \tau\}$. Note that

$$\max\{l_{i-q,j}, l''_{i,j}\} \leq l'_{i-q,j}, \qquad i \in \{1, \ldots, m\}.$$

Therefore

$$\sum_{i=1}^{m} \max\{l_{i-q,j}, l''_{i,j}\} \leq \sum_{i=1}^{m} l'_{i-q,j} = l'_j.$$

Also note that

$$\zeta^q = \eta^\rho = \prod_{i=1}^{m} \text{l.c.m.}\{\alpha_{i-q}, \gamma_i\} = \pi_j^{\sum_{i=1}^{m} \max\{l_{i-q,j}, l''_{i,j}\}} \sigma_j,$$

where $\text{g.c.d.}\{\pi_j, \sigma_j\} = 1$, and $\text{l.c.m.}\{\alpha_1 \cdots \alpha_p, f\} = \pi_j^{l'_j} \sigma'_j$, where $\text{g.c.d.}\{\pi_j, \sigma'_j\} = 1$. Therefore $\zeta^q = \eta^\rho \mid \text{l.c.m.}\{\alpha_1 \cdots \alpha_p, f\}$ and $d(\zeta^q) = d(\eta^\rho) \leq m$.

From (12), it follows that $\zeta^j = \alpha_1 \cdots \alpha_p$, $j \in \{0, \ldots, q-1\}$. As $\gamma_1 \cdots \gamma_m = f$ and $d(f) \leq m - \rho$, we have $\gamma_1 = \cdots = \gamma_\rho = 1$. Therefore, from (12), it follows that $\eta^0 = \gamma_{\rho+1} \cdots \gamma_m = \gamma_1 \cdots \gamma_m = f$. Suppose that $m - d(f) = \rho g + h$, where g and h are integers and $0 \leq h < \rho$. Let

$$\begin{aligned} s_i &= g + 1, \qquad i \in \{1, \ldots, h\}, \\ s_i &= g, \qquad i \in \{h+1, \ldots, \rho\}. \end{aligned}$$

Note that $d(\gamma_1 \cdots \gamma_m) + s_1 + \cdots + s_\rho = m$. Looking at the normal form for feedback equivalence, it is easy to find $B_1 \in F^{m \times m}$ and $B_2 \in F^{m \times t}$ such that (6) has invariant factors $\gamma_1 \mid \cdots \mid \gamma_m$ and column minimal indices $s_1 \geq \cdots \geq s_\rho > s_{\rho+1} = \cdots = s_t (= 0)$.

From the previous remarks, it is not hard to deduce that the conditions $(i_{2.1})$–$(iii_{2.1})$ are satisfied. According to Theorem 2.1, there exist $A_{1,2} \in F^{p \times q}$, $A_{1,3} \in F^{p \times t}$, $A_{2,2} \in F^{q \times q}$ and $A_{2,3} \in F^{q \times t}$ such that (7) and $\begin{bmatrix} B_1 & B_2 \end{bmatrix}$ are feedback equivalent. Therefore (7) has characteristic polynomial $\gamma_1 \cdots \gamma_m = f$. $\square$

The following lemma is easy to prove.

LEMMA 2.4. *Let $A_{1,1}, A'_{1,1} \in F^{p \times p}$, $A_{1,3}, A'_{1,3} \in F^{p \times t}$, $A_{2,1}, A'_{2,1} \in F^{q \times p}$, $A_{2,3}, A'_{2,3} \in F^{q \times t}$. Let $m = p + q$, and let $f \in F[x]$ be a monic polynomial. Suppose that*

$$\begin{bmatrix} A_{1,1} & A_{1,3} \\ A_{2,1} & A_{2,3} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} A'_{1,1} & A'_{1,3} \\ A'_{2,1} & A'_{2,3} \end{bmatrix}$$

are p-similar. Then, there exist $A_{1,2} \in F^{p\times q}$, $A_{1,4} \in F^{p\times u}$, $A_{2,2} \in F^{q\times q}$, $A_{2,4} \in F^{q\times u}$ *such that*

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \end{bmatrix}$$

has characteristic polynomial f if and only if there exist $A'_{1,2} \in F^{p\times q}$, $A'_{1,4} \in F^{p\times u}$, $A'_{2,2} \in F^{q\times q}$, $A'_{2,4} \in F^{q\times u}$ *such that*

$$\begin{bmatrix} A'_{1,1} & A'_{1,2} & A'_{1,3} & A'_{1,4} \\ A'_{2,1} & A'_{2,2} & A'_{2,3} & A'_{2,4} \end{bmatrix}$$

has characteristic polynomial f.

THEOREM 2.5. *Let* $A_{1,1} \in F^{p\times p}$, $A_{1,3} \in F^{p\times t}$, $A_{2,1} \in F^{q\times p}$ *and* $A_{2,3} \in F^{q\times t}$. *Let* $\beta_1 \mid \cdots \mid \beta_{p+\epsilon}$ *be the invariant factors,* ϵ *the number of infinite elementary divisors,* l *the sum of the degrees of the infinite elementary divisors,* w *the sum of the column minimal indices of*

$$\begin{bmatrix} xI_p - A_{1,1} & -A_{1,3} \\ -A_{2,1} & -A_{2,3} \end{bmatrix}. \tag{13}$$

Let $m = p+q$, *and let* $f \in F[x]$ *be a monic polynomial. Then, there exist* $A_{1,2} \in F^{p\times q}$, $A_{2,2} \in F^{q\times q}$, $A_{1,4} \in F^{p\times u}$ *and* $A_{2,4} \in F^{q\times u}$ *such that*

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \end{bmatrix} \tag{14}$$

has characteristic polynomial f if and only if one of the following conditions is satisfied.

($I_{2.5}$) $\max\{\epsilon, u\} > 0, d(\text{l.c.m.}\{\beta_1 \cdots \beta_{p+\epsilon}, f\}) \leq m - w - l$ *and*

$$\beta_1 \cdots \beta_{p+\epsilon-\rho} \mid f,$$

where $\rho = \min\{m - w - l - d(f), \epsilon + u\}$.

($II_{2.5}$) $\epsilon = u = 0, d(f) = m - w$ *and*

$$\beta_1 \cdots \beta_p \mid f.$$

Proof. Note that $\beta_1 = \cdots = \beta_{w+l} = 1$. With permutations of the rows and columns of the normal form for p-similarity of

$$\begin{bmatrix} A_{1,1} & A_{1,3} \\ A_{2,1} & A_{2,3} \end{bmatrix} \tag{15}$$

one can get a p-similar matrix of the form

$$\left[\begin{array}{cc|c} D_{1,1} & 0 & 0 \\ 0 & E_{1,1} & E_{1,3} \\ \hline D_{2,1} & 0 & 0 \\ 0 & E_{2,1} & E_{2,3} \end{array}\right], \tag{16}$$

where $D_{1,1} \in F^{(p-w-l+\epsilon)\times(p-w-l+\epsilon)}$, $D_{2,1} \in F^{(q-\epsilon)\times(p-w-l+\epsilon)}$, $E_{1,1} \in F^{(w+l-\epsilon)\times(w+l-\epsilon)}$, the matrix

$$\begin{bmatrix} xI_{p-w-l+\epsilon} - D_{1,1} \\ -D_{2,1} \end{bmatrix}$$

has invariant factors $\beta_{w+l+1}, \ldots, \beta_{p+\epsilon}$, and the matrix

$$\begin{bmatrix} xI_{w+l-\epsilon} - E_{1,1} & -E_{1,3} \\ -E_{2,1} & -E_{2,3} \end{bmatrix} \tag{17}$$

has $w+l$ invariant factors equal to 1. Bearing in mind Lemma 2.4, one may assume, without loss of generality, that (15) has the form (16).

Necessity. Suppose that there exists a matrix of the form (14) with characteristic polynomial f. Bearing in mind that (17) has $w+l$ invariant factors equal to 1, and that, therefore, its Smith normal form is $\begin{bmatrix} I_{w+l} & 0 \end{bmatrix}$, it is not hard to deduce that

$$\begin{bmatrix} xI_p - A_{1,1} & -A_{1,2} & -A_{1,3} & -A_{1,4} \\ -A_{2,1} & xI_q - A_{2,2} & -A_{2,3} & -A_{2,4} \end{bmatrix} \tag{18}$$

is equivalent to a matrix of the form

$$\begin{bmatrix} xI_{p-w-l+\epsilon} - D_{1,1} & -D_{1,2} & -D_{1,3} & 0 & 0 \\ -D_{2,1} & xI_{q-\epsilon} - D_{2,2} & -D_{2,3} & 0 & 0 \\ 0 & 0 & 0 & I_{w+l} & 0 \end{bmatrix}, \tag{19}$$

where $D_{1,3} \in F^{(p-w-l+\epsilon)\times(\epsilon+u)}$. Consequently,

$$\begin{bmatrix} D_{1,1} & D_{1,2} & D_{1,3} \\ D_{2,1} & D_{2,2} & D_{2,3} \end{bmatrix} \tag{20}$$

has characteristic polynomial f.

Suppose that $\max\{\epsilon, u\} > 0$. According to Corollary 2.3, $d(\text{l.c.m.}\{\beta_1 \cdots \beta_{p+\epsilon}, f\}$ $= d(\text{l.c.m.}\{\beta_{l+w+1} \cdots \beta_{p+\epsilon}, f\} \leq m - w - l$ and

$$\beta_1 \cdots \beta_{p+\epsilon-\rho} = \beta_{l+w+1} \cdots \beta_{p+\epsilon-\rho} \mid f,$$

where $\rho = \min\{m - l - w - d(f), \epsilon + u\}$.

Now suppose that $\epsilon = u = 0$. Then $l = 0$ and, as (20) is a square matrix, its characteristic polynomial, f, has degree $m - w$. According to Theorem 1.1,

$$\beta_1 \cdots \beta_p = \beta_{w+1} \cdots \beta_p \mid f.$$

Sufficiency. The arguments are similar to the ones used to prove necessity. Suppose that $\max\{\epsilon, u\} > 0$. According to Corollary 2.3, there exist $D_{1,2} \in F^{(p-w-l+\epsilon)\times(q-\epsilon)}$, $D_{1,3} \in F^{(p-w-l+\epsilon)\times(\epsilon+u)}$, $D_{2,2} \in F^{(q-\epsilon)\times(q-\epsilon)}$, and $D_{2,3} \in F^{(q-\epsilon)\times(\epsilon+u)}$, such that (20) has characteristic polynomial f. It is not hard to see that the matrices (18) and (19) are equivalent for certain blocks $A_{1,2}, A_{2,2}, A_{1,4}$ and

$A_{2,4}$. For these blocks (14) has characteristic polynomial f. The case $\epsilon = u = 0$ is analogous. □

The next theorem was established in [5] for infinite fields. Now it can be deduced, for arbitrary fields, as a simple consequence of Theorems 1.1 and 2.5. This theorem describes the possible characteristic polynomials of a matrix with a prescribed arbitrary submatrix. Note that, with permutations of rows and columns that correspond to similarity transformations, the general problem can be reduced to the case where the prescribed submatrix lies in the position considered in the next statement; see [5], for details.

THEOREM 2.6. *Let* $A_{1,1} \in F^{p\times p}$, $A_{1,3} \in F^{p\times t}$, $A_{2,1} \in F^{q\times p}$ *and* $A_{2,3} \in F^{q\times t}$. *Let* $\beta_1 \mid \cdots \mid \beta_{p+\epsilon}$ *be the invariant factors,* ϵ *the number of infinite elementary divisors,* l *the sum of the degrees of the infinite elementary divisors,* w *the sum of the column minimal indices of* (13). *Let* u *be a nonnegative integer and* $m = p + q + t + u$, *and let* $f \in F[x]$ *be a monic polynomial of degree* m.

If $\epsilon = u = 0$, *then there exist* $A_{1,2} \in F^{p\times q}$, $A_{2,2} \in F^{q\times q}$, $A_{1,4} \in F^{p\times u}$, $A_{2,4} \in F^{q\times u}$, $A_{3,1} \in F^{t\times p}$, $A_{3,2} \in F^{t\times q}$, $A_{3,3} \in F^{t\times t}$, $A_{3,4} \in F^{t\times u}$, $A_{4,1} \in F^{u\times p}$, $A_{4,2} \in F^{u\times q}$, $A_{4,3} \in F^{u\times t}$, $A_{4,4} \in F^{u\times u}$, *such that*

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix} \tag{21}$$

has characteristic polynomial f *if and only if there exists a polynomial* g *of degree* $p + q - w$ *such that*

$$\beta_1 \cdots \beta_p \mid g \mid f.$$

If $\max\{\epsilon, u\} > 0$, *then such a matrix completion exists if and only if*

$$\beta_1 \cdots \beta_{p-u} \mid f.$$

Proof. Necessity. Let g be the characteristic polynomial of (14). According to Theorem 1.1, $g \mid f$.

If $\epsilon = u = 0$, then according to Theorem 2.5, g has degree $p+q-w$ and $\beta_1 \cdots \beta_p \mid g$.

Now suppose that $\max\{\epsilon, u\} > 0$. According to Theorem 2.5,
$d(\text{l.c.m.}\{\beta_1 \cdots \beta_{p+\epsilon}, g\}) \leq p + q - w - l$ and $\beta_1 \cdots \beta_{p+\epsilon-\rho} \mid g$, where $\rho = \min\{p + q - w - l - d(g), \epsilon + u\}$. Therefore $\beta_1 \cdots \beta_{p-u} \mid g \mid f$.

Sufficiency. Suppose that $\epsilon = u = 0$. According to Theorem 2.5, there exists a matrix of the form (14) with characteristic polynomial g. According to Theorem 1.1, there exists a matrix of the form (21) with characteristic polynomial f.

From now on suppose that $\max\{\epsilon, u\} > 0$. Let $g = \beta_1 \cdots \beta_{p-u}$ and $\rho = \min\{p + q - w - l - d(g), \epsilon + u\}$.

Firstly suppose that $\rho = p + q - w - l - d(g) < \epsilon + u$. From the normal form for p-similarity [3, Lemma 2] of (15) [or the normal form for strict equivalence of (13)], it follows that

$$p = d(\beta_1 \cdots \beta_{p+\epsilon}) + k + w + l - \epsilon,$$

where k is the sum of the row minimal indices of (13), and that $q \geq \epsilon$. If $d(\beta_{p+\epsilon-\rho}) > 0$, then $d(\beta_{p-u+1} \cdots \beta_{p+\epsilon}) > \rho = p+q-w-l-d(g)$ and $d(\beta_1 \cdots \beta_{p+\epsilon}) > p+q-w-l \geq p-w-k-l+\epsilon$, which is impossible. Therefore, $1 = \beta_1 \cdots \beta_{p+\epsilon-\rho} = \beta_1 \cdots \beta_{p-u} = g$.

For any of the two possible values of ρ, according to Theorem 2.5, there exists a matrix of the form (14) with characteristic polynomial g. According to Theorem 1.1, there exists a matrix of the form (21) with characteristic polynomial f. $\square$

Lemma 2.7. *Let $A, A' \in F^{m\times m}$, $B, B' \in F^{m\times t}$, $C, C' \in F^{s\times m}$ and $D, D' \in F^{s\times t}$. Let $f \in F[x]$ be a monic polynomial. Suppose that*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$$

are m-similar. Then, there exists $X \in F^{t\times m}$ such that

$$\begin{bmatrix} A+BX \\ C+DX \end{bmatrix} \tag{22}$$

has characteristic polynomial f if and only if there exists $X' \in F^{t\times m}$ such that

$$\begin{bmatrix} A'+B'X' \\ C'+D'X' \end{bmatrix} \tag{23}$$

has characteristic polynomial f.

Proof. Suppose that (3) is satisfied. Suppose that there exists $X \in F^{t\times m}$ such that (22) has characteristic polynomial f. Let $X' = R^{-1}(XP - T)$. Then

$$\begin{bmatrix} A'+B'X' \\ C'+D'X' \end{bmatrix} = \begin{bmatrix} P^{-1} & S \\ 0 & Q \end{bmatrix} \begin{bmatrix} A+BX \\ C+DX \end{bmatrix} P,$$

which shows that (22) and (23) are feedback equivalent and, therefore, have the same characteristic polynomial. $\square$

Theorem 2.8. *Let $A \in F^{m\times m}$, $B \in F^{m\times t}$, $C \in F^{s\times m}$ and $D \in F^{s\times t}$. Let $\beta_1 \mid \cdots \mid \beta_{m+\epsilon}$ be the invariant factors, ϵ the number of infinite elementary divisors, l the sum of the degrees of the infinite elementary divisors, k the sum of the row minimal indices of*

$$\begin{bmatrix} xI_m - A & -B \\ -C & -D \end{bmatrix}. \tag{24}$$

Let $f \in F[x]$ be a monic polynomial. Then, there exists $X \in F^{t\times m}$ such that

$$\begin{bmatrix} A+BX \\ C+DX \end{bmatrix} \tag{25}$$

has characteristic polynomial f if and only if one of the following conditions is satisfied.

$(I_{2.8})$ $\epsilon > 0, d(\text{l.c.m.}\{\beta_1 \cdots \beta_{m+\epsilon}, f\}) \leq m-k-l+\epsilon$ *and*

$$\beta_1 \cdots \beta_{m+\epsilon-\rho} \mid f,$$

where $\rho = \min\{m-k-l+\epsilon-d(f), \epsilon\}$.

ELA

$(II_{2.8})$ $\epsilon = 0, d(f) = m - k$ *and*

$$\beta_1 \cdots \beta_m \mid f.$$

Proof. By performing permutations of rows and columns in the normal form for m-similarity [3, Lemma 2] of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{26}$$

one get an m-similar matrix of the form

$$\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] = \left[\begin{array}{cc|ccc} A_{1,1} & A_{1,2} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ \hline C_{3,1} & C_{3,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_u & 0 \end{array}\right], \tag{27}$$

where $A_{1,1} \in F^{p\times p}$, $A_{1,2} \in F^{p\times q}$, $m = p + q$, and

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ C_{3,1} & C_{3,2} \end{bmatrix}$$

has a normal form for p-similarity. Note that u is the number of infinite elementary divisors of (24) of degree equal to 1. Let $(1 =)\ l_1 = \cdots = l_u < l_{u+1} \leq \cdots \leq l_\epsilon$ be the degrees of the infinite elementary divisors of (24). It is not hard to deduce that $\beta_{q+u+1}, \ldots, \beta_{m+\epsilon}$ are the invariant factors and $l_{u+1} - 1, \ldots, l_\epsilon - 1$ are the degrees of the infinite elementary divisors of

$$\begin{bmatrix} xI_p - A_{1,1} & -A_{1,2} \\ -C_{3,1} & -C_{3,2} \end{bmatrix}.$$

Moreover, $\beta_1 = \ldots = \beta_{q+u} = 1$. Bearing in mind Lemma 2.7, one may assume, without loss of generality, that (26) has already the form (27).

Necessity. Suppose that there exists $X \in F^{t\times m}$ such that (25) has characteristic polynomial f. Note that (25) has the form

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ C_{3,1} & C_{3,2} \\ C_{4,1} & C_{4,2} \end{bmatrix}, \tag{28}$$

where $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$, $C_{4,1} \in F^{u\times p}$, $C_{4,2} \in F^{u\times q}$. According to Theorem 2.5, one of the conditions $(I_{2.8})$, $(II_{2.8})$ is satisfied.

Sufficiency. Suppose that one of the conditions $(I_{2.8})$, $(II_{2.8})$ is satisfied. According to Theorem 2.5, there exist $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$, $C_{4,1} \in F^{u\times p}$, $C_{4,2} \in F^{u\times q}$ such that (28) has characteristic polynomial f. Note that (28) has the form (25) for some $X \in F^{t\times m}$. □

3. Stabilization and Observability of Linear Systems by Linear Feedback. Let $\mathbb{F}$ be the field of complex numbers, $\mathbb{C}$, or the field of real numbers, $\mathbb{R}$. Consider a linear system $\mathcal{S}$

$$\left\{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ v(t) &= Cx(t) + Du(t), \end{aligned}\right.$$

where $A \in \mathbb{F}^{m\times m}$, $B \in \mathbb{F}^{m\times n}$, $C \in \mathbb{F}^{s\times m}$, $D \in \mathbb{F}^{s\times n}$, $x(t)$ is the state, $u(t)$ is the input and $v(t)$ is the output.

Several results are known that relate matrix completion problems to linear systems; see, e.g., [16, 18, 20, 24].

Recall that $\mathcal{S}$ is *stable* if and only if all the solutions of $\dot{x}(t) = Ax(t)$ converge to 0 as $t \to +\infty$ if and only if the real parts of the eigenvalues of A are negative; and that $\mathcal{S}$ is *stabilizable* if and only if there exists a state feedback $u(t) = Xx(t)$ such that $\dot{x}(t) = (A + BX)x(t)$ becomes stable if and only if the roots of the characteristic polynomial of $\begin{bmatrix} A & B \end{bmatrix}$ have their real parts negative. In this context, arises the problem of describing the possible eigenvalues of $A + BX$, when X varies, whose solution is known for a long time, and it is a particular case of Theorem 2.8.

Also recall that $\mathcal{S}$ is *completely observable* if and only if the characteristic polynomial of $\begin{bmatrix} A^t & C^t \end{bmatrix}^t$ is equal to 1 if and only if $\operatorname{rank}\mathcal{O}(A, C) = m$, where

$$\mathcal{O}(A, C) = \begin{bmatrix} C^t & (CA)^t & \cdots & (CA^{m-1})^t \end{bmatrix}^t$$

is the observability matrix of $\mathcal{S}$; and that $\mathcal{S}$ is *detectable* if and only if the roots of the characteristic polynomial of $\begin{bmatrix} A^t & C^t \end{bmatrix}^t$ have their real parts negative. For the meaning of these concepts in systems theory and other details, see, e.g., [13].

Now suppose that the entries of A and C are polynomials in variables $x_1, \ldots, x_h$ and let $\mathcal{A}$ be the set of all h-uples $(a_1, \ldots, a_h) \in \mathbb{F}^h$ such that

$$\operatorname{rank}\mathcal{O}(A(a_1, \ldots, a_h), C(a_1, \ldots, a_h)) < m.$$

Then $\mathcal{A}$ is an algebraic set. Therefore, if $\emptyset \neq \mathcal{A} \neq \mathbb{F}^h$, then, in any neighbourhood of any element of $\mathcal{A}$, there are elements of $\mathbb{F}^h$ that do not belong to $\mathcal{A}$.

The following results are simple consequences of Theorem 2.8. We use the notation of Theorem 2.8.

Corollary 3.1. *There exists a state feedback $u(t) = Xx(t)$ such that*

$$\left\{\begin{aligned} \dot{x}(t) &= (A + BX)x(t), \\ v(t) &= (C + DX)x(t) \end{aligned}\right. \tag{29}$$

becomes completely observable if and only if one of the following conditions is satisfied.

($I_{3.1}$) *The system is already completely observable with $u(t) \equiv 0$.*

($II_{3.1}$) *$\epsilon > 0$ and $\beta_1 \cdots \beta_m = 1$.*

Proof. Note that, from the normal form for m-similarity [3, Lemma 2] of (26) [or the normal form for strict equivalence of (24)], it follows that

$$m = d(\beta_1 \cdots \beta_{m+\epsilon}) + k + w + l - \epsilon, \tag{30}$$

where w is the sum of the column minimal indices of (28).

Necessity. The characteristic polynomial of (25) is $f = 1$. According to Theorem 2.8, $\beta_1 \cdots \beta_m = 1$.

Now suppose that $\epsilon = 0$. Then $l = 0$ and $m = d(\beta_1 \cdots \beta_m) + k + w$. According to Theorem 2.8, $m = k$ and, therefore, $w = 0$. It follows that $B = 0$ and $D = 0$. Consequently the characteristic polynomial of $\begin{bmatrix} A^t & C^t \end{bmatrix}^t$ is $\beta_1 \cdots \beta_m = 1$, that is, the system is already completely observable with $u(t) \equiv 0$.

Sufficiency. As the case $(I_{3.1})$ is trivial, suppose that $(II_{3.1})$ is satisfied. Take $f = 1 \in \mathbb{F}[x]$. Let $\rho = \min\{m - k - l + \epsilon, \epsilon\}$. Suppose that $\rho = m - k - l + \epsilon$. If $\beta_{k+l} = \beta_{m+\epsilon-\rho} \neq 1$, then $d(\beta_1 \cdots \beta_{m+\epsilon}) > \rho = m + \epsilon - k - l$, which is impossible. Then $(I_{2.8})$ is satisfied. If $\rho = \epsilon$, then $(I_{2.8})$ is also satisfied. According to Theorem 2.8, there exists $X \in F^{t \times m}$ such that (25) has characteristic polynomial f, that is, (29) is completely observable. □

Corollary 3.2. *There exists a state feedback $u(t) = Xx(t)$ such that (29) becomes detectable if and only if the roots of $\beta_1 \cdots \beta_m$ have their real parts negative.*

Proof. Necessity. Let f be the characteristic polynomial of (25). The roots of f have their real parts negative. According to Theorem 2.8, the roots of $\beta_1 \cdots \beta_m$ have their real parts negative.

Sufficiency. Note that (30) is satisfied.

If $\epsilon = 0$, then $d(\beta_1 \cdots \beta_m) \leq m - k$. Let $f = \beta_1 \cdots \beta_m g$ be a monic polynomial of degree $m - k$ such that the real parts of the roots of g are negative. According to Theorem 2.8, there exists $X \in F^{s \times m}$ such that (25) has characteristic polynomial f. Therefore (29) is detectable.

Now suppose that $\epsilon > 0$. From (30) it follows that $d(\beta_1 \cdots \beta_m) \leq m - k - l + \epsilon$. Let $\rho = \min\{m - k - l + \epsilon - d(\beta_1 \cdots \beta_m), \epsilon\}$.

Suppose that $\rho = m - k - l + \epsilon - d(\beta_1 \cdots \beta_m) < \epsilon$. If $d(\beta_{m+\epsilon-\rho}) > 0$, then $d(\beta_{m+1} \cdots \beta_{m+\epsilon}) > \rho = m - k - l + \epsilon - d(\beta_1 \cdots \beta_m)$, which is impossible. Therefore, $1 = \beta_1 \cdots \beta_{m+\epsilon-\rho} = \beta_1 \cdots \beta_m$.

For any value of ρ, according to Theorem 2.8, there exists $X \in F^{n \times m}$ such that (25) has characteristic polynomial $\beta_1 \cdots \beta_m$ and (29) is detectable. □

Corollary 3.3. *There exists a state feedback $u(t) = Xx(t)$ such that (29) becomes simultaneously stable and completely observable if and only if the roots of the characteristic polynomial of $\begin{bmatrix} A & B \end{bmatrix}$ have their real parts negative and one of the conditions $(I_{3.1})$, $(II_{3.1})$ is satisfied.*

Proof. The necessity follows immediately from the previous remarks and results.

Sufficiency. As the roots of the characteristic polynomial of $\begin{bmatrix} A & B \end{bmatrix}$ have their real parts negative, there exists $X_0 \in \mathbb{F}^{n \times m}$ such that $A + BX_0$ is stable. By continuity, there exists a neighbourhood $\mathcal{V}$ of X_0 such that $A + BX$ is stable for every $X \in \mathcal{V}$. Now let $\mathcal{A}$ be the set of all the matrices $X \in \mathbb{F}^{n \times m}$ such that $\operatorname{rank}(\mathcal{O}(A + BX, C + DX)) < m$. As one of the conditions $(I_{3.1})$, $(II_{3.1})$ is satisfied, $\mathcal{A} \neq \mathbb{F}^{n \times m}$.

If $X_0 \notin \mathcal{A}$, the proof is complete with $X = X_0$. If $X_0 \in \mathcal{A}$, there exists a matrix $X \in \mathcal{V} \setminus \mathcal{A}$ and the proof is also complete. □

Acknowledgement. We would like to thank one of the referees for detecting an error in Corollary 2.3 and for his comments.

REFERENCES

[1] I. Baragaña and I. Zaballa. Column completion of a pair of matrices, *Linear and Multilinear Algebra*, 27:243–273, 1990.

[2] P. Brunovsky. A classification of linear controllable systems, *Kybernetica*, 3:173–188, 1970.

[3] I. Cabral. Matrices with prescribed submatrices and number of invariant polynomials, *Linear Algebra Appl.*, 219:207–224, 1995.

[4] I. Cabral and F. C. Silva. Unified theorems on completions of matrix pencils, *Linear Algebra Appl.*, 159:43–54, 1991.

[5] I. Cabral and F. C. Silva. Similarity invariants of completions of submatrices, *Linear Algebra Appl.*, 169:151–161, 1992.

[6] M. Jean Dieudonné. Sur la réduction canonique des couples de matrices, *Bull. Soc. Math. France*, 74:130–146, 1946.

[7] S. Furtado and F. C. Silva. On the characteristic polynomial of matrices with prescribed rows, preprint, 1997.

[8] F. R. Gantmacher. *The Theory of Matrices*, vol. 2, Chelsea Publishing Co., New York, 1960.

[9] I. Gohberg, M. A. Kaashoek and F. van Schagen. Eigenvalues of completions of submatrices, *Linear and Multilinear Algebra*, 25:55–70, 1989.

[10] I. Gohberg, M. A. Kaashoek and F. van Schagen. *Partially Specified Matrices and Operators: Classification, Completion, Applications*, Birkhäuser-Verlag, Basel, 1995.

[11] G. N. Oliveira. Matrices with prescribed characteristic polynomial and a prescribed submatrix III, *Monastsh. Math.*, 75:441–446, 1971.

[12] G. N. Oliveira. Matrices with prescribed characteristic polynomial and several prescribed submatrices, *Linear and Multilinear Algebra*, 2:357–364, 1975.

[13] T. Kailath. *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.

[14] R. E. Kalman. Mathematical description of linear dynamical systems, *SIAM J. Control Optim.*, 1:152–192, 1963.

[15] E. M. Sá. Imbedding conditions for λ-matrices, *Linear Algebra Appl.*, 24:33–50, 1979.

[16] J. Rosenthal and X. A. Wang. Inverse eigenvalue problems for multivariable linear systems, in *Systems and Control in the Twenty-First Century*, C. I. Byrnes, B. N. Datta, D. S. Gilliam and C. F. Martin Eds., Birkhäuser, Boston, 1997, pp. 289–311.

[17] F. C. Silva. Matrices with prescribed similarity class and a prescribed nonprincipal submatrix, *Portugal. Math.*, 47:103–113, 1990.

[18] F. C. Silva. On feedback equivalence and completion problems, *Linear Algebra Appl.*, 265:231–245, 1997.

[19] R. C. Thompson. Interlacing inequalities for invariant factors, *Linear Algebra Appl.*, 24:1–31, 1979.

[20] H. K. Wimmer. Existenzsätze in der theorie der matrizen und lineare kontrolltheorie, *Monasth. Math.*, 78:256–263, 1974.

[21] W. M. Wonham. *Linear Multivariable Control*, 3rd ed., Springer-Verlag, New York, 1985.

[22] I. Zaballa. Matrices with prescribed rows and invariant factors, *Linear Algebra Appl.*, 87:113–146, 1987.

[23] I. Zaballa. Interlacing inequalities and control theory, *Linear Algebra Appl.*, 101:9–31, 1988.

[24] I. Zaballa. Interlacing and majorization in invariant factors assignment problems, *Linear Algebra Appl.*, 121:409–421, 1989.

[25] I. Zaballa. Matrices with prescribed invariant factors and off-diagonal submatrices, *Linear and Multilinear Algebra*, 25:39-54, 1989.