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THE MOORE-PENROSE INVERSE OF A FREE MATRIX*

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Abstract. A matrix is free, or generic, if its nonzero entries are algebraically independent. Necessary and sufficient combinatorial conditions are presented for a complex free matrix to have a free Moore-Penrose inverse. These conditions extend previously known results for square, nonsingular free matrices. The result used to prove this characterization relates the combinatorial structure of a free matrix to that of its Moore-Penrose inverse. Also, it is proved that the bipartite graph or, equivalently, the zero pattern of a free matrix uniquely determines that of its Moore-Penrose inverse, and this mapping is described explicitly. Finally, it is proved that a free matrix contains at most as many nonzero entries as does its Moore-Penrose inverse.

Key words. Free matrix, Moore-Penrose inverse, Bipartite graph, Directed graph.

AMS subject classifications. 05C50, 15A09.

1. The main results. A set of complex numbers S is *algebraically independent* over the rational numbers $\mathbb{Q}$ if $p(s_1, \ldots, s_n) \neq 0$ whenever $s_1, \ldots, s_n$ are distinct elements of S and $p(x_1, \ldots, x_n)$ is a nonzero polynomial with rational coefficients.

LEMMA 1.1. *Let S_1 and S_2 be finite sets of complex numbers so that each element of S_2 may be written as a rational form in elements of S_1, and conversely. If S_1 is algebraically independent, then S_2 is algebraically independent if and only if $|S_1| = |S_2|$.*

Proof. For each finite set of complex numbers S, let $dt_{\mathbb{Q}}\, S$ denote the maximal cardinality of an algebraically independent subset of S and note that $dt_{\mathbb{Q}} S = dt_{\mathbb{Q}} \mathbb{Q}(S)$. Each element of S_2 may be written as a rational form in entries of S_1, so $S_2 \subseteq \mathbb{Q}(S_1)$. Similarly, $S_1 \subseteq \mathbb{Q}(S_2)$; thus, $\mathbb{Q}(S_1) = \mathbb{Q}(S_2)$. If S_1 is algebraically independent, then $|S_1| = dt_{\mathbb{Q}}\, S_1 = dt_{\mathbb{Q}}\, \mathbb{Q}(S_1) = dt_{\mathbb{Q}}\, \mathbb{Q}(S_2) = dt_{\mathbb{Q}}\, S_2$, and the lemma follows. ◻

A matrix with complex entries is *free*, or *generic*, if the multiset of nonzero entries is algebraically independent. These nonzero entries may be viewed as indeterminants over the rational numbers; see [5, Chap. 6]. Free matrices have been used to represent objects from transversal theory, extremal poset theory, electrical network theory, and other combinatorial areas; see [3, Chap. 9] for a partial overview. The advantage of such representations is that they allow methods from linear algebra to be applied to combinatorial problems, most often via the connection given in Theorem 1.2 below. A *nonzero partial diagonal* of a matrix A is a collection of nonzero entries no two of which lie in the same row or column, and the *term rank* of A is the maximal size of a nonzero partial diagonal in A.

THEOREM 1.2. [4, 6] *The term rank of a free matrix equals its rank.*

Note that the rank and term rank are identical for each submatrix of a free matrix.

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For any complex $m \times n$ matrix $A = [a_{ij}]$, the *Moore-Penrose inverse* $A^\dagger$ is the unique matrix that satisfies the following four properties [7, 8]:

$$A^\dagger A A^\dagger = A^\dagger \qquad A A^\dagger A = A \qquad (A^\dagger A)^T = A^\dagger A \qquad (AA^\dagger)^T = AA^\dagger\,.$$

If A is a square, nonsingular matrix, then $A^\dagger = A^{-1}$. Hence, Moore-Penrose inversion generalizes standard matrix inversion. For more information on the Moore-Penrose inverse, see [1] and the extensive bibliography therein.

This article describes combinatorial properties of the Moore-Penrose inverses of free matrices. The first two main results, Theorems 1.3 and 1.4 below, describe how the combinatorial structure of a free matrix A relates to that of the Moore-Penrose inverse $A^\dagger$. Stronger but more technical results are proved in Sections 2 and 3.

For each integer $r \geq 1$, let $A[1:r]$ denote the leading principal (i.e. upper left) $r \times r$ submatrix of A. Furthermore, let $B(A)$ be the bipartite graph with vertices $U \cup V$ and edges $\big\{\{u_i, v_j\} \;:\; u_i \in U,\ v_j \in V,\ a_{ij} \neq 0\big\}$ where $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ are disjoint sets. If $m = n$, then let $D(A)$ be the digraph with vertices $W = \{w_1, \ldots, w_n\}$ and arcs $E = \{(w_i, w_j) \;:\; a_{ij} \neq 0\}$. Thus, for instance, $D(I_n)$ denotes the digraph with arcs $\{(w, w) \,:\, w \in W\}$. For any digraph D, let $\overline{D}$ be the transitive closure of D. For (di)graphs D_1 and D_2, let the notation "$D_1 \subseteq D_2$" indicate that D_1 is a sub-(di)graph of D_2.

Theorem 1.3. *If A is a free matrix, then $B(A)$ uniquely determines $B(A^\dagger)$.*

The above result follows immediately from Theorem 2.3 in Section 2.

Theorem 1.4. *Let A be a free matrix of rank r for which $D(I_r) \subseteq D(A[1:r])$. Then $A = [a_{ij}]$ and $A^\dagger = [\alpha_{ij}]$ can be written as*

$$A = \begin{bmatrix} C & F \\ G & 0 \end{bmatrix} \qquad \text{and} \qquad A^\dagger = \begin{bmatrix} S & X \\ Y & Z \end{bmatrix}$$

with $C = A[1:r]$, $D(I_r) \subseteq D(C) \subseteq \overline{D(C)} \subseteq D(S)$, $B(F) \subseteq B(Y^T)$, and $B(G) \subseteq B(X^T)$.

A proof of Theorem 1.4 is given in Section 3.

Corollary 1.5. *The Moore-Penrose inverse $A^\dagger$ of a free matrix A has at least as many nonzero entries as A.*

In the nonsingular case, Theorem 1.4 has the following simple form.

Theorem 1.6. *For any nonsingular $n \times n$ free matrix A such that $D(I_n) \subseteq D(A)$, the digraph $D(A^{-1})$ is equal to the transitive closure $\overline{D(A)}$.*

Proof. Since A^{-1} is a polynomial in A, we have $D(A^{-1}) \subseteq \overline{D(A) \cup D(I_n)} = \overline{D(A)}$. Conversely, Theorem 1.4 asserts that $\overline{D(A)} \subseteq D(A^{-1})$. Hence, $D(A^{-1}) = \overline{D(A)}$. □

Remark 1.7. No generality is lost by requiring that $D(I_r) \subseteq D(A[1:r])$ in Theorem 1.4. Indeed for each free matrix A of rank r, choose any permutation matrices P and Q for which $A' := PAQ$ satisfies this condition, and apply the theorem to A'. Similarly, no generality is lost by requiring that $D(I_n) \subseteq D(A)$ in Theorem 1.6.

The final main result is expressed as Theorem 1.8 below. It provides necessary and sufficient conditions for the Moore-Penrose inverse $A^\dagger$ of a free matrix A to be free, and thereby generalizes [2, Theorem 3.1].

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Theorem 1.8. *Let A be a free matrix of rank r and let P and Q be permutation matrices for which $A' := PAQ$ satisfies $D(I_r) \subseteq D(A'[1:r])$. Write*

$$A' = \begin{bmatrix} C & F \\ G & 0 \end{bmatrix} \qquad \text{and} \qquad A'^{\dagger} = \begin{bmatrix} S & X \\ Y & Z \end{bmatrix}$$

where $C = A'[1:r]$ and S are $r \times r$ matrices. The following statements are equivalent:

1. *$A^{\dagger}$ is free;*
2. *A and $A^{\dagger}$ have an equal number of nonzero (or zero) entries;*
3. *$D(C) = D(S)$, $B(F) = B(Y^T)$, $B(G) = B(X^T)$, and $Z = 0$.*

Proof. Conditions *1.* and *2.* are equivalent by Lemma 1.1. Since $A^{\dagger}$ is free if and only if $A'^{\dagger}$ is free, $A^{\dagger}$ is free if and only if A' and $A'^{\dagger}$ have equal numbers of zero entries. By Theorem 1.4, this is true if and only if condition *3.* is satisfied. □

Corollary 1.9. *If A is a free matrix, then the bipartite graph $B(A)$ determines whether the Moore-Penrose inverse $A^{\dagger}$ is also free.*

Proof. By Theorem 1.3, $B(A)$ determines $B(A^{\dagger})$ and thus whether A and $A^{\dagger}$ contain equal numbers of zero entries. Now apply Theorem 1.8. □

Remark 1.10. Let A be a free matrix represented as in Theorems 1.4 and 1.8. If $A^{\dagger}$ is free, then, by these theorems, $D(C) = \overline{D(C)} = D(S)$.

Example 1.11. The above results are illustrated by Figures 1.1 and 1.2, each of which shows a free matrix A and its inverse $A^{\dagger}$. The entries a_{ij} and α_{ij} represent the nonzero entries of A and $A^{\dagger}$, respectively; for instance, $\alpha_{11} = a_{11}/(a_{11}^2 + a_{41}^2)$ in Figure 1.1. The matrices are represented as in Theorems 1.4 and 1.8, and the (di)graphs associated to them in these theorems are also shown. Note that in each figure $D(I_r) \subseteq D(C) \subseteq \overline{D(C)} \subseteq D(S)$, $B(F) \subseteq B(Y^T)$, and $B(G) \subseteq B(X^T)$, as asserted by Theorem 1.4. Note also that $A^{\dagger}$ contains at least as many nonzero entries as A, as asserted by Corollary 1.5. Finally, note that the Moore-Penrose inverse $A^{\dagger}$ in Figure 1.1 is not free but that $A^{\dagger}$ in Figure 1.2 is indeed free. By Theorem 1.8, this may be seen by comparing the numbers of zero entries in A and $A^{\dagger}$ or by checking whether the equalities $D(C) = D(S)$, $B(F) = B(Y^T)$, and $B(G) = B(X^T)$ hold.

2. The structure of the Moore-Penrose inverse of a free matrix. Let $Q_{k,l}$ be the family of ordered k-subsets of $(1, \ldots, l)$. Consider $\gamma = (\gamma_1, \ldots, \gamma_k) \in Q_{k,l}$. For each $i \in \gamma$, let $\gamma - i$ denote the ordered set $\gamma \backslash \{i\}$; also for each $i \notin \gamma$, let $(i;\gamma)$ denote the ordered set $(i, \gamma_1, \ldots, \gamma_k)$. If A is an $m \times n$ matrix and γ, δ are ordered subsets of $(1, \ldots, m)$ and $(1, \ldots, n)$, respectively, then let $A[\gamma|\delta]$ denote the $|\gamma| \times |\delta|$ matrix whose (i,j)th entry equals $a_{\gamma_i \delta_j}$. Note that $A[1:k] := A[1, \ldots, k|1, \ldots, k]$. Note also that $A[\gamma|\delta]$ is the submatrix of A with rows γ and columns δ, whereas $A[i;\gamma|j;\delta]$ has rows and columns ordered $(i;\gamma)$ and $(j;\delta)$, respectively.

Theorem 2.1. [7] *Let A be a complex $m \times n$ matrix A with rank $r \geq 2$ and let $A^{\dagger} = [\alpha_{ij}]$ denote the Moore-Penrose inverse of A. Then*

$$\alpha_{ij} = \frac{\displaystyle\sum_{\gamma \in Q_{r-1,m}, j \notin \gamma} \; \sum_{\delta \in Q_{r-1,n}, i \notin \delta} \det A[\gamma|\delta] \overline{\det A[j;\gamma|i;\delta]}}{\displaystyle\sum_{\rho \in Q_{r,m}} \; \sum_{\tau \in Q_{r,n}} \det A[\rho|\tau] \overline{\det A[\rho|\tau]}}.$$

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$$A = \begin{bmatrix} C & F \\ G & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & a_{33} & 0 \\ \hline 0 & a_{42} & 0 & 0 \end{array}\right] \qquad A^\dagger = \begin{bmatrix} S & X \\ Y & Z \end{bmatrix} = \left[\begin{array}{ccc|c} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \hline \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{array}\right]$$

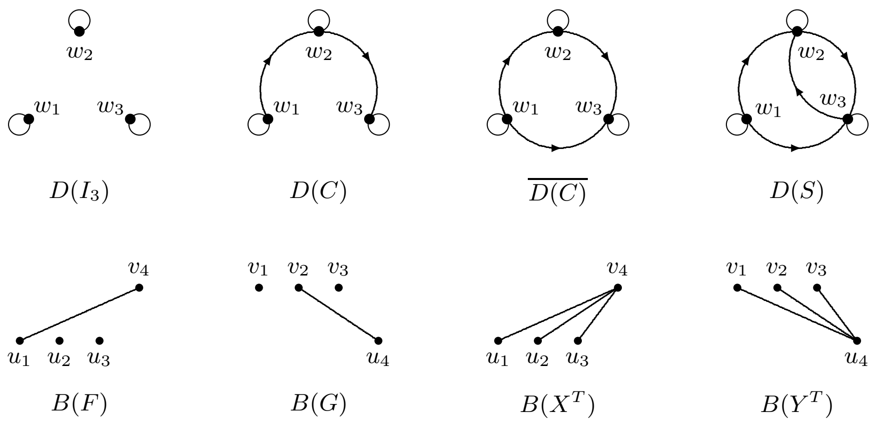

FIG. 1.1. *The combinatorial structure of a free matrix and its Moore-Penrose inverse*

$$A = \begin{bmatrix} C & F \\ G & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad A^\dagger = \begin{bmatrix} S & X \\ Y & Z \end{bmatrix} = \left[\begin{array}{cccc|c} \alpha_{11} & \alpha_{12} & \alpha_{23} & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{44} & 0 \\ \hline 0 & 0 & 0 & \alpha_{54} & 0 \end{array}\right]$$

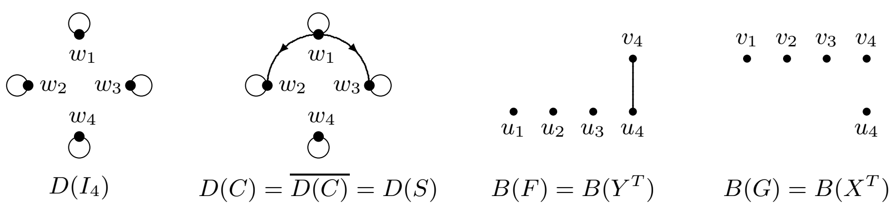

FIG. 1.2. *A free matrix whose Moore-Penrose inverse is also free*

Throughout the remainder of this section, let $A = [a_{ij}]$ denote a free $m \times n$ matrix of rank r and let $A^\dagger = [\alpha_{ij}]$ denote its Moore-Penrose inverse.

REMARK 2.2. If $r = 0$, then $A^\dagger = A^T = 0$, and $B(A)$ and $B(A^\dagger)$ have no edges. If $r = 1$, then, by Theorem 1.2, rows and columns of A may be swapped so that A or A^T has the form $[\mathbf{v}\, 0]$ for a nonzero vector $\mathbf{v}$ and a possibly empty zero matrix, so

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$A^\dagger = A^*/||\mathbf{v}||^2$. Hence, $\{u_i, v_j\}$ is an edge of $B(A^\dagger)$ if and only if $\{u_j, v_i\}$ is an edge of $B(A)$.

Theorem 2.3 below describes how the bipartite graph of a free matrix A uniquely determines the bipartite graph of the Moore-Penrose inverse $A^\dagger$. A *matching* in a graph is a collection of edges no two of which are incident.

THEOREM 2.3. *$\{u_i, v_j\}$ is an edge of $B(A^\dagger)$ if and only if $B(A)$ contains a path p from v_i to u_j of length $2s+1$ with $s \geq 0$ and a matching with $r-s-1$ edges, none of which is incident to p.*

Proof. If $r = 0$ or $r = 1$, then the proof follows trivially from Remark 2.2, so suppose that $r \geq 2$. Assume that $\alpha_{ij} \neq 0$. By Theorem 2.1, there are sequences $\gamma \in Q_{r-1,m}$ and $\delta \in Q_{r-1,n}$ so that $j \notin \gamma$, $i \notin \delta$, and $\det A[\gamma|\delta]\overline{\det A[j;\gamma|i;\delta]} \neq 0$. Then $B(A[\gamma|\delta])$ and $B(A[j;\gamma|i;\delta])$ each contains a matching with $r-1$ and r edges, respectively, say E_1 and E_2. Let G denote the bipartite graph with edges $E_1 \cup E_2$ and delete from G each isolated edge containing neither v_i nor u_j. In G, vertices v_i and u_j have degree 1 and all other vertices have degree 2, so G is the disjoint union of some path p from v_i to u_j and some cycles. Hence, $B(A[j;\gamma|i;\delta])$ contains p and at least one (perhaps empty) matching E that covers all vertices not in p. Thus, there is an integer $s \geq 0$ so that p has length $2s+1$ and so that E contains $r-s-1$ edges. Both p and E are contained in $B(A)$ and satisfy the conditions stated in the theorem.

Now assume that $B(A)$ contains a path p from v_i to u_j

$$v_i - u_{j_1} - v_{i_1} - u_{j_2} - v_{i_2} - \cdots - u_{j_s} - v_{i_s} - u_j$$

of length $2s+1$ with $s \geq 0$ and at least one matching with $r-s-1$ edges, say $\big\{ \{u_{j'_t}, v_{i'_t}\} \,:\, t = 1, \ldots, r-s-1 \big\}$, none of which is incident to p. Order the sets

$$\begin{array}{rccc} & \{j_1, \ldots, j_s\} & \cup & \{j'_t \,:\, t = 1, \ldots, r-s-1\} \\ \text{and} & \{i_1, \ldots, i_s\} & \cup & \{i'_t \,:\, t = 1, \ldots, r-s-1\} \end{array}$$

to form increasing sequences, γ' and δ', respectively. Then

$$\begin{array}{rccc} & \{a_{j_1 i_1}, \ldots, a_{j_s i_s}\} & \cup & \{a_{j'_t i'_t} \,:\, t = 1, \ldots, r-s-1\} \\ \text{and} & \{a_{j_1 i}, a_{j_2 i_1}, \ldots, a_{j_s i_{s-1}}, a_{j i_s}\} & \cup & \{a_{j'_t i'_t} \,:\, t = 1, \ldots, r-s-1\} \end{array}$$

are nonzero partial diagonals of $A[\gamma'|\delta']$ and $A[j;\gamma'|i;\delta']$, respectively. Therefore,

$$S = \sum_{\gamma \in Q_{r-1,m}, j \notin \gamma} \; \sum_{\delta \in Q_{r-1,n}, i \notin \delta} \det A[\gamma|\delta]\overline{\det A[j;\gamma|i;\delta]}$$

is a sum of signed monomials of degree $2r-1$, one of which is

$$a_{j_1 i_1} \cdots a_{j_s i_s} a_{j'_1 i'_1} \cdots a_{j'_{r-s-1} i'_{r-s-1}} \overline{a_{j_1 i} a_{j_2 i_1} \cdots a_{j_s i_{s-1}} a_{j i_s} a_{j'_1 i'_1} \cdots a_{j'_{r-s-1} i'_{r-s-1}}}\,.$$

Since A is free, these monomials do not vanish, so $S \neq 0$. By Theorem 2.1, $\alpha_{ij} \neq 0$. ☐

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PROPOSITION 2.4. *Let $\{u_j, v_i\}$ be an edge of $B(A)$. Then $\{u_i, v_j\}$ is an edge of $B(A^\dagger)$ if and only if $\{u_j, v_i\}$ is contained in some matching of $B(A)$ with r edges. If $\{u_j, v_i\}$ is contained in every such matching, then $\alpha_{ij} = \frac{1}{a_{ji}}$. If $\{u_i, v_j\}$ is contained in no such matching, and E is such a matching, then E contains edges $\{u_i, v_{i_E}\}$ and $\{u_{j_E}, v_j\}$ for which $\{u_{i_E}, v_{j_E}\}$ is an edge of $B(A^\dagger)$.*

Proof. If $\{u_j, v_i\}$ is contained in some matching E in $B(A)$ with r edges, then $u_j - v_i$ is a path that together with the matching $E - \{u_j, v_i\}$ satisfies the conditions in Theorem 2.3, so $\{u_i, v_j\}$ is an edge of $B(A^\dagger)$.

Conversely, suppose that $\{u_i, v_j\}$ is an edge of $B(A^\dagger)$. By Theorem 2.3, $B(A)$ contains a path p from v_i to u_j of length $2s+1$,

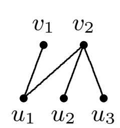
$$v_i - u_{j_1} - v_{i_1} - u_{j_2} - v_{i_2} - \cdots - u_{j_s} - v_{i_s} - u_j \ ,$$

as well as a matching E with $r-s-1$ edges, none of which are incident to p. Then $E \cup \big\{\{u_j, v_i\}, \{u_{j_1}, v_{i_1}\}, \ldots, \{u_{j_s}, v_{i_s}\}\big\}$ is a matching in $B(A)$ with r edges, including $\{u_j, v_i\}$. If $\{u_j, v_i\}$ is in every matching in $B(A)$ with r edges, then $\det A[j;\gamma|i;\delta] = a_{ji} \det A[\gamma|\delta]$ for all $\gamma \in Q_{r-1,m}$ and $\delta \in Q_{r-1,n}$ with $j \notin \gamma$, $i \notin \delta$. By Theorem 2.1, $\alpha_{ij} = \frac{\overline{a_{ji}}}{a_{ji}\overline{a_{ji}}} = \frac{1}{a_{ji}}$. If E has no edges incident to u_i or v_j, then $E \cup \{u_i, v_j\}$ is a matching in $B(A)$ with $r+1$ edges, so, by Theorem 1.2, the rank of A is at least $r+1$, a contradiction. Thus, E contains at least one such edge, say $\{u_i, v_{i_E}\}$. If E does not also contain an edge $\{u_{j_E}, v_j\}$, then $(E - \{u_i, v_{i_E}\}) \cup \{u_i, v_j\}$ is a matching of $B(A)$ with r edges, one of which is $\{u_i, v_j\}$, a contradiction. Thus, E contains edges $\{u_i, v_{i_E}\}$ and $\{u_{j_E}, v_j\}$ for some vertices v_{i_E} and u_{j_E}. Then $v_{i_E} - u_i - v_j - u_{j_E}$ is a path from u_{j_E} to v_{i_E} of length $2s+1$ with $s = 1$ and $E - \{u_i, v_{i_E}\} - \{u_{j_E}, v_j\}$ is a matching in $B(A)$ with $r-s-1 = r-2$ edges, none of which contain u_{j_E}, u_i, v_j, or v_{i_E}. By Theorem 2.3, $\{u_{i_E}, v_{j_E}\}$ is an edge of $B(A^\dagger)$. Then $\{u_{j_E}, v_{i_E}\}$ is not an edge of $B(A)$ since this would imply that $\big(E - \{u_{j_E}, v_j\} - \{u_i, v_{i_E}\}\big) \cup \{u_i, v_j\} \cup \{u_{j_E}, v_{i_E}\}$ is a matching in $B(A)$ with r edges, one of which is $\{u_i, v_j\}$. □

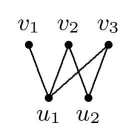
EXAMPLE 2.5. To illustrate the results in this section, let A be as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & a_{32} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{a_{11}} & \frac{a_{12}a_{22}}{a_{11}c} & \frac{a_{12}a_{32}}{a_{11}c} \\ 0 & \frac{a_{22}}{c} & \frac{a_{32}}{c} \end{bmatrix}$$

A $\qquad$ $A^\dagger$



$B(A)$



$B(A^\dagger)$

Here, $c = a_{22}^2 + a_{32}^2$ and the rank of A is $r = 2$. The graph $B(A)$ contains the path $v_1 - u_1 - v_2 - u_3$ of length $2s+1$ with $s = 1$. Then $r - s - 1 = 0$ and Theorem 2.3 implies that $\{u_1, v_3\}$ is an edge of $B(A^\dagger)$, as is indeed the case. The graph $B(A)$ contains precisely two matchings with $r = 2$ edges, namely

$$E_1 = \big\{\{u_1, v_1\}, \{u_2, v_2\}\big\} \quad \text{and} \quad E_2 = \big\{\{u_1, v_1\}, \{u_3, v_2\}\big\}.$$

Thus, the edge $\{u_1, v_1\}$ is contained in all matchings in $B(A)$ with $r = 2$ edges and the edge $\{u_1, v_2\}$ is contained in none of these. By Proposition 2.4, $\alpha_{11} = \frac{1}{a_{11}}$, $\alpha_{21} = 0$, and the edges $\{u_1, v_2\}$ and $\{u_1, v_3\}$ are edges of $B(A^\dagger)$.

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3. Proof of Theorem 1.4. Theorem 1.4 follows trivially from the next result.

THEOREM 3.1. *Let A be a free matrix of rank r for which $D(I_r) \subseteq D(A[1:r])$. Then $A = [a_{ij}]$ and $A^\dagger = [\alpha_{ij}]$ can be written as*

$$A = \begin{bmatrix} C & F \\ G & 0 \end{bmatrix} \qquad \text{and} \qquad A^\dagger = \begin{bmatrix} S & X \\ Y & Z \end{bmatrix}$$

where $C = A[1:r]$. Furthermore, let i, j, k be positive integers with $\max\{i,j\} \leq r < k$, and suppose that there is at least one path in $D(C)$ from w_i to w_j. Then

1. *$\alpha_{ij} \neq 0$;*
2. *if $a_{ik} \neq 0$, then $\alpha_{kj} \neq 0$;*
3. *if $a_{kj} \neq 0$, then $\alpha_{ik} \neq 0$;*
4. *if $a_{jk} \neq 0$ or $a_{ki} \neq 0$, then paths from w_i to w_j in $D(C)$ are also paths in $D(S^T)$.*

Proof. Write A and $A^\dagger$ as

$$A = \begin{bmatrix} C & F \\ G & H \end{bmatrix} \qquad \text{and} \qquad A^\dagger = \begin{bmatrix} S & X \\ Y & Z \end{bmatrix}$$

where $C = A[1:r]$ and S are $r \times r$ matrices. The $r = 0$ and $r = 1$ cases follow easily from Remark 2.2 so suppose that $r \geq 2$. Since C contains a diagonal of r nonzero entries, Theorem 1.2 implies that $H = 0$. Let

$$w_i \to w_{i_1} \to w_{i_2} \to \cdots \to w_{j_{s-1}} \to w_j$$

be a (perhaps trivial) path p from w_i to w_j in $D(C)$. To prove statement *1.*, first note that $D(I_r) \subseteq D(C)$. It follows that the graph $B(A)$ contains the path

$$v_i - u_i - v_{i_1} - u_{i_1} - v_{i_2} - u_{i_2} - \cdots - v_{i_{s-1}} - u_{i_{s-1}} - v_j - u_j$$

from v_i to u_j and the matching

$$\big\{\{u_t, v_t\} \,:\, t \in \{1,\ldots,r\} - \{i, i_1, i_2, \ldots, i_{s-1}, j\}\big\}.$$

By Theorem 2.3, $B(A^\dagger)$ contains the edge $\{u_i, v_j\}$, i.e., $\alpha_{ij} \neq 0$.

Similarly, if $a_{ik} \neq 0$, then $B(A)$ contains the path

$$v_k - u_i - v_{i_1} - u_{i_1} - v_{i_2} - u_{i_2} - \cdots - v_{i_{s-1}} - u_{i_{s-1}} - v_j - u_j$$

from v_k to u_j and the same matching as before. By Theorem 2.3, $\alpha_{kj} \neq 0$, which proves statement *2.* Statement *3.* follows from statement *2.* by transposing A.

To prove statement *4.*, suppose that $a_{jk} \neq 0$. Then

$$\begin{aligned} &\big\{\{u_i, v_{i_1}\}, \{u_{i_1}, v_{i_2}\}, \ldots, \{u_{i_{s-1}}, v_j\}, \{u_j, v_k\}\big\} \\ \cup \quad &\big\{\{u_t, v_t\} \,:\, t \in \{1,\ldots,r\} - \{i, i_1, i_2, \ldots, i_{s-1}, j\}\big\} \end{aligned}$$

is a matching E in $B(A)$ with r edges. By Proposition 2.4, $\{u_{j'}, v_{i'}\}$ is an edge of $B(A^\dagger)$ for each edge $\{u_{i'}, v_{j'}\}$ of E. Therefore, $(w_{j'}, w_{i'})$ is an arc of $D(S)$ for each arc $(w_{i'}, w_{j'})$ of p, so p is a path in $D(S^T)$. The $a_{ki} \neq 0$ case is proved similarly. ∎

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