Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

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SOME ALGEBRAIC AND STATISTICAL PROPERTIES OF WLSES UNDER A GENERAL GROWTH CURVE MODEL*

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Abstract. Growth curve models are used to analyze repeated measures data (longitudinal data), which are functions of time. General expressions of weighted least-squares estimators (WLSEs) of parameter matrices were given under a general growth curve model. Some algebraic and statistical properties of the estimators are also derived through the matrix rank method.

AMS subject classifications. 62F11, 62H12, 15A03, 15A09.

Key words. Growth curve model, Moore-Penrose inverse of matrix, Rank formulas for matrices, Parameter matrix, WLSE, Extremal rank, Uniqueness, Unbiasedness.

1. Introduction. Throughout this paper, $\mathbb{R}^{m\times n}$ stands for the collections of all $m\times n$ real matrices. The symbols $\mathbf{A}'$, $r(\mathbf{A})$, $\mathscr{R}(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A})$ stand for the transpose, the rank, the range (column space) and the trace of a matrix $\mathbf{A}$, respectively. The Kronecker product of any two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined to be $\mathbf{A}\otimes\mathbf{B} = (a_{ij}\mathbf{B})$. The vec operation of any matrix $\mathbf{A} = [\,\mathbf{a}_1, \ldots, \mathbf{a}_n\,]$ is defined to be $\mathrm{vec}(\mathbf{A}) = [\,\mathbf{a}_1', \ldots, \mathbf{a}_n'\,]'$. A well-known property of the vec operation of a triple matrix product is $\mathrm{vec}(\mathbf{AZB}) = (\mathbf{B}'\otimes\mathbf{A})\mathrm{vec}(\mathbf{Z})$.

A longitudinal data set is a set consisting of a given sample of individuals over time. They provide multiple observations on each individuals in the sample. Longitudinal data can be used to establish regression models with respect to various possible regressors. In statistical applications, a special type of linear longitudinal data model is the following well-known growth curve model

$$\mathbf{Y} = \mathbf{X}_1\Theta\mathbf{X}_2 + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\boldsymbol{\Sigma}_2\otimes\boldsymbol{\Sigma}_1). \tag{1.1}$$

The model can also be written in the triplet form

$$\mathscr{M} = \{\mathbf{Y},\, \mathbf{X}_1\Theta\mathbf{X}_2,\, \sigma^2(\boldsymbol{\Sigma}_2\otimes\boldsymbol{\Sigma}_1)\}, \tag{1.2}$$

where

$\mathbf{Y} = (y_{ij}) \in \mathbb{R}^{n\times m}$ is an observable random matrix (a longitudinal data set),
$\mathbf{X}_1 = (x_{1ij}) \in \mathbb{R}^{n\times p}$ and $\mathbf{X}_2 = (x_{2ij}) \in \mathbb{R}^{q\times m}$ are two known model matrices of arbitrary ranks,
$\Theta = (\theta_{ij}) \in \mathbb{R}^{p\times q}$ is a matrix of unknown parameters to be estimated,
$\boldsymbol{\Sigma}_1 = (\sigma_{1ij}) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{\Sigma}_2 = (\sigma_{2ij}) \in \mathbb{R}^{m\times m}$ are two known nonnegative definite matrices of arbitrary ranks, and

* Received by the editors 5 March 2007. Accepted for publication 27 July 2007. Handling Editor: Ravindra B. Bapat.
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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

σ^2 is a positive unknown scalar.

If one of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ is a singular matrix, (1.1) is also said to be a singular growth curve model.

Through the Kronecker product and the vec operation of matrices, the model in (1.1) can alternatively be written as

$$\mathrm{vec}(\mathbf{Y}) = (\mathbf{X}_2' \otimes \mathbf{X}_1)\mathrm{vec}(\Theta) + \mathrm{vec}(\boldsymbol{\varepsilon}), \ E[\mathrm{vec}(\boldsymbol{\varepsilon})] = \mathbf{0}, \ Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1). \tag{1.3}$$

Because (1.1) is a linear model, many results on linear models carry over to (1.1). Note, however, that both $\mathrm{vec}(\mathbf{Y})$ and $\mathrm{vec}(\Theta)$ in (1.3) are vectors, and many properties of the two matrices $\mathbf{Y}$ and Θ in (1.1), such as their ranks, range, singularity, symmetry, partitioned representations, can hardly be demonstrated in the expressions of $\mathrm{vec}(\mathbf{Y})$ and $\mathrm{vec}(\Theta)$. Conversely, not all estimators of $\mathrm{vec}(\Theta)$ and $(\mathbf{X}_2' \otimes \mathbf{X}_1)\mathrm{vec}(\Theta)$ under (1.3) can be written in the forms of matrices in (1.1). That is to say, some problems on the model (1.1) can be studied through (1.3), others can only be done with the original model (1.1).

The growth curve model in (1.1) is an extension of multivariate linear models. This model was originally proposed by Potthoff and Roy [11] in studying longitudinal data and was subsequently studied by many authors, such as, Frees [4], Hsiao [5], Khatri [7], Pan and Fang [9], Rao [14, 15], Seber [18], von Rosen [31, 32], and Woolson and Leeper [33], among many others. The purpose of the present paper is to give some general expressions of weighted least-squares estimators (WLSEs) of Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$ under the general assumption in (1.1), and then study the maximal and minimal possible ranks of the estimators, as well as the unbiasedness and the uniqueness of the estimators.

The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{R}^{m\times n}$, denoted by $\mathbf{A}^+$, is defined to be the unique solution $\mathbf{G}$ to the four matrix equations

$$\text{(i) } \mathbf{AGA} = \mathbf{A}, \quad \text{(ii) } \mathbf{GAG} = \mathbf{G}, \quad \text{(iii) } (\mathbf{AG})' = \mathbf{AG}, \quad \text{(iv) } (\mathbf{GA})' = \mathbf{GA}.$$

A matrix $\mathbf{G}$ is called a generalized inverse (g-inverse) of $\mathbf{A}$, denoted by $\mathbf{G} = \mathbf{A}^-$, if it satisfies (i). Further, let $\mathbf{P_A}$, $\mathbf{F_A}$ and $\mathbf{E_A}$ stand for the three orthogonal projectors $\mathbf{P_A} = \mathbf{AA}^+$, $\mathbf{E_A} = \mathbf{I}_m - \mathbf{AA}^+$ and $\mathbf{F_A} = \mathbf{I}_n - \mathbf{A}^+\mathbf{A}$.

In order to simplify various matrix expressions consisting of the Moore-Penrose inverses of matrices, we need some formulas for ranks of matrices. The following rank formulas are due to Marsaglia and Styan [8] .

Lemma 1.1. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$ *and* $\mathbf{D} \in \mathbb{R}^{l\times k}$. *Then*

$$r[\,\mathbf{A}, \mathbf{B}\,] = r(\mathbf{A}) + r(\mathbf{E_A B}) = r(\mathbf{B}) + r(\mathbf{E_B A}), \tag{1.4}$$

$$r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r(\mathbf{CF_A}) = r(\mathbf{C}) + r(\mathbf{AF_C}), \tag{1.5}$$

$$r\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = r(\mathbf{B}) + r(\mathbf{C}) + r(\mathbf{E_B A F_C}). \tag{1.6}$$

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A publication of the International Linear Algebra Society
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If $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$ *and* $\mathscr{R}(\mathbf{C}') \subseteq \mathscr{R}(\mathbf{A}')$, *then*

$$r\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = r(\mathbf{A}) + r(\,\mathbf{D} - \mathbf{C}\mathbf{A}^{+}\mathbf{B}\,). \tag{1.7}$$

Lemma 1.2. *Suppose* $\mathscr{R}(\mathbf{B}_1') \subseteq \mathscr{R}(\mathbf{C}_1')$, $\mathscr{R}(\mathbf{B}_2) \subseteq \mathscr{R}(\mathbf{C}_1)$, $\mathscr{R}(\mathbf{B}_2') \subseteq \mathscr{R}(\mathbf{C}_2')$ *and* $\mathscr{R}(\mathbf{B}_3) \subseteq \mathscr{R}(\mathbf{C}_2)$. *Then*

$$r(\,\mathbf{A} - \mathbf{B}_1\mathbf{C}_1^{+}\mathbf{B}_2\mathbf{C}_2^{+}\mathbf{B}_3\,) = r\begin{bmatrix} \mathbf{0} & \mathbf{C}_2 & \mathbf{B}_3 \\ \mathbf{C}_1 & \mathbf{B}_2 & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{0} & -\mathbf{A} \end{bmatrix} - r(\mathbf{C}_1) - r(\mathbf{C}_2). \tag{1.8}$$

Proof. In terms of the Moore-Penrose inverses of matrices, the given conditions are equivalent to

$$\mathbf{B}_1\mathbf{C}_1^{+}\mathbf{C}_1 = \mathbf{B}_1, \quad \mathbf{C}_1\mathbf{C}_1^{+}\mathbf{B}_2 = \mathbf{B}_2, \quad \mathbf{B}_2\mathbf{C}_2^{+}\mathbf{C}_2 = \mathbf{B}_2, \quad \mathbf{C}_2\mathbf{C}_2^{+}\mathbf{B}_3 = \mathbf{B}_3. \tag{1.9}$$

Also recall that elementary block matrix operations (EBMOs) do not change the rank of a matrix. Applying (1.9) and EBMOs to the block matrix in (1.8) gives

$$\begin{aligned} r\begin{bmatrix} \mathbf{0} & \mathbf{C}_2 & \mathbf{B}_3 \\ \mathbf{C}_1 & \mathbf{B}_2 & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{0} & -\mathbf{A} \end{bmatrix} &= r\begin{bmatrix} \mathbf{0} & \mathbf{C}_2 & \mathbf{B}_3 \\ \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_1\mathbf{C}_1^{+}\mathbf{B}_2 & -\mathbf{A} \end{bmatrix} \\ &= r\begin{bmatrix} \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{A} + \mathbf{B}_1\mathbf{C}_1^{+}\mathbf{B}_2\mathbf{C}_2^{+}\mathbf{B}_3 \end{bmatrix} \\ &= r(\,\mathbf{A} - \mathbf{B}_1\mathbf{C}_1^{+}\mathbf{B}_2\mathbf{C}_2^{+}\mathbf{B}_3\,) + r(\mathbf{C}_1) + r(\mathbf{C}_2), \end{aligned}$$

establishing (1.8). ☐

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ and $\mathbf{C} \in \mathbb{R}^{l\times n}$ be given, and suppose $\mathbf{Z} \in \mathbb{R}^{k\times l}$ is a variable matrix. It was shown in [19, 23] that the matrix pencil $\mathbf{A} - \mathbf{BZC}$ satisfies the following two rank identities

$$r(\,\mathbf{A} - \mathbf{BZC}\,) = r[\,\mathbf{A},\, \mathbf{B}\,] + r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} - r\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} + r[\,\mathbf{E}_{\mathbf{T}_1}(\,\mathbf{Z} + \mathbf{T}\mathbf{M}^{-}\mathbf{S}\,)\mathbf{F}_{\mathbf{S}_1}] \tag{1.10}$$

and

$$r(\,\mathbf{A} - \mathbf{BZC}\,) = r[\,\mathbf{A},\, \mathbf{B}\,] + r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} - r\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} + r(\,\mathbf{E}_{\mathbf{Q}}\mathbf{A}\mathbf{F}_{\mathbf{P}} - \mathbf{E}_{\mathbf{Q}}\mathbf{BZC}\mathbf{F}_{\mathbf{P}}), \tag{1.11}$$

where $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}$, $\mathbf{T} = [\,\mathbf{0},\, \mathbf{I}_k\,]$, $\mathbf{S} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_l \end{bmatrix}$, $\mathbf{T}_1 = \mathbf{T}\mathbf{F}_{\mathbf{M}}$, $\mathbf{S}_1 = \mathbf{E}_{\mathbf{M}}\mathbf{S}$, $\mathbf{P} = \mathbf{E}_{\mathbf{B}}\mathbf{A}$ and $\mathbf{Q} = \mathbf{A}\mathbf{F}_{\mathbf{C}}$. It is easy to verify that the two matrix equations

$$\mathbf{E}_{\mathbf{T}_1}(\,\mathbf{Z} + \mathbf{T}\mathbf{M}^{-}\mathbf{S}\,)\mathbf{F}_{\mathbf{S}_1} = \mathbf{0} \;\text{ and }\; \mathbf{E}_{\mathbf{Q}}\mathbf{BZC}\mathbf{F}_{\mathbf{P}} = \mathbf{E}_{\mathbf{Q}}\mathbf{A}\mathbf{F}_{\mathbf{P}} \tag{1.12}$$

are solvable for $\mathbf{Z}$. The following result is derived from (1.10) and (1.11).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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LEMMA 1.3. *The maximal and minimal ranks of* $\mathbf{A} - \mathbf{BZC}$ *are given by*

$$\max_{\mathbf{Z}\in\mathbb{R}^{k\times l}} r(\,\mathbf{A} - \mathbf{BZC}\,) = \min\left\{ r[\,\mathbf{A},\,\mathbf{B}\,], \quad r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}\right\} \tag{1.13}$$

and

$$\min_{\mathbf{Z}\in\mathbb{R}^{k\times l}} r(\,\mathbf{A} - \mathbf{BZC}\,) = r[\,\mathbf{A},\,\mathbf{B}\,] + r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} - r\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}. \tag{1.14}$$

The matrices satisfying (1.13) and (1.14) can be derived from (1.12). The following two rank formulas are given in [20]. Note that the difference $\mathbf{Y} - \mathbf{X}_1\Theta\mathbf{X}_2$ corresponding to (1.1) is a matrix pencil with respect to the parameter matrix Θ, so that the maximal and minimal ranks of $\mathbf{Y} - \mathbf{X}_1\Theta\mathbf{X}_2$ with respect to Θ can also be derived from Lemma 1.3.

LEMMA 1.4. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B}_i \in \mathbb{R}^{m\times k_i}$, $\mathbf{C}_i \in \mathbb{R}^{l_i\times n}$ *be given with* $\mathscr{R}(\mathbf{B}_1) \subseteq \mathscr{R}(\mathbf{B}_2)$ *and* $\mathscr{R}(\mathbf{C}_2') \subseteq \mathscr{R}(\mathbf{C}_1')$. *Then*

$$\max_{\mathbf{Z}_1,\,\mathbf{Z}_2} r(\,\mathbf{A} - \mathbf{B}_1\mathbf{Z}_1\mathbf{C}_1 - \mathbf{B}_2\mathbf{Z}_2\mathbf{C}_2\,) = \min\left\{ r[\,\mathbf{A},\,\mathbf{B}_2\,],\; r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}_1\end{bmatrix},\; r\begin{bmatrix}\mathbf{A} & \mathbf{B}_1\\ \mathbf{C}_2 & \mathbf{0}\end{bmatrix}\right\} \tag{1.15}$$

and

$$\begin{aligned}&\min_{\mathbf{Z}_1,\,\mathbf{Z}_2} r(\,\mathbf{A} - \mathbf{B}_1\mathbf{Z}_1\mathbf{C}_1 - \mathbf{B}_2\mathbf{Z}_2\mathbf{C}_2\,)\\ &= r[\,\mathbf{A},\,\mathbf{B}_2\,] + r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}_1\end{bmatrix} + r\begin{bmatrix}\mathbf{A} & \mathbf{B}_1\\ \mathbf{C}_2 & \mathbf{0}\end{bmatrix} - r\begin{bmatrix}\mathbf{A} & \mathbf{B}_1\\ \mathbf{C}_1 & \mathbf{0}\end{bmatrix} - r\begin{bmatrix}\mathbf{A} & \mathbf{B}_2\\ \mathbf{C}_2 & \mathbf{0}\end{bmatrix}.\end{aligned} \tag{1.16}$$

In particular,

$$\max_{\mathbf{Z}_1,\,\mathbf{Z}_2} r(\,\mathbf{A} - \mathbf{B}\mathbf{Z}_1 - \mathbf{Z}_2\mathbf{C}\,) = \min\left\{ m,\; n,\; r\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\right\} \tag{1.17}$$

and

$$\min_{\mathbf{Z}_1,\,\mathbf{Z}_2} r(\,\mathbf{A} - \mathbf{B}\mathbf{Z}_1 - \mathbf{Z}_2\mathbf{C}\,) = r\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{bmatrix} - r(\mathbf{B}) - r(\mathbf{C}). \tag{1.18}$$

The following result is well known, see, e.g., [10, 16].

LEMMA 1.5. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then the matrix equation* $\mathbf{BZC} = \mathbf{A}$ *is solvable for* $\mathbf{Z}$ *if and only if* $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B})$ *and* $\mathscr{R}(\mathbf{A}') \subseteq \mathscr{R}(\mathbf{C}')$, *or equivalently,* $\mathbf{BB}^+\mathbf{AC}^+\mathbf{C} = \mathbf{A}$. *In this case, the general solution of the equation can be written as* $\mathbf{Z} = \mathbf{B}^+\mathbf{AC}^+ + \mathbf{F}_{\mathbf{B}}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{C}}$, *where* $\mathbf{U}_1$ *and* $\mathbf{U}_2$ *are arbitrary.*

The rank formulas in Lemmas 1.1–1.4 together with EBMOs can be used to simplify various expressions involving the Moore-Penrose inverses of matrices and arbitrary matrices. We call the method of solving problems through rank formulas of matrices and EBMOs *the matrix rank method*. In regression analysis, this method can be used to investigate

(a) consistency of regression models,
(b) explicit and implicit restrictions to parameters in regression models,
(c) superfluous observations,
(d) estimability of parametric functions,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

(e) reduced-rank estimators,
(f) unbiasedness of estimators,
(g) uniqueness of estimators,
(h) equalities of estimators,
(i) additive and block decomposability of estimators,
(j) unrelatedness of estimators,
(k) independence of estimators,
(l) orthogonality of estimators,
(m) proportionality of estimators,
(n) parallelness of estimators,
(o) relations between original and misspecified models,

etc., see several recent papers [12, 13, 21, 22, 24, 25, 26, 27, 28, 29] on these topics by the rank method.

2. Expressions of WLSEs under the growth curve model. Let $\mathbf{Z} \in \mathbb{R}^{n\times m}$, let $\mathbf{V}_1 \in \mathbb{R}^{n\times n}$ and $\mathbf{V}_2 \in \mathbb{R}^{m\times m}$ be two nonnegative definite matrices of arbitrary ranks, and define

$$f(\mathbf{Z}) = \mathrm{tr}(\mathbf{Z}'\mathbf{V}_1\mathbf{Z}\mathbf{V}_2) = \mathrm{vec}'(\mathbf{Z})(\mathbf{V}_2 \otimes \mathbf{V}_1)\mathrm{vec}(\mathbf{Z}). \tag{2.1}$$

The WLSE of Θ under (1.1) with respect to the loss function in (2.1), denoted by $\mathrm{WLSE}(\Theta)$, is defined to be

$$\hat{\Theta} = \underset{\Theta}{\mathrm{argmin}}\, f(\,\mathbf{Y} - \mathbf{X}_1\Theta\mathbf{X}_2\,), \tag{2.2}$$

the WLSE of $\mathbf{X}_1\Theta\mathbf{X}_2$ under (1.1) is defined to be

$$\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) = \mathbf{X}_1\mathrm{WLSE}(\Theta)\mathbf{X}_2, \tag{2.3}$$

the WLSE of the matrix $\mathbf{K}_1\Theta\mathbf{K}_2$ under (1.1) is defined to be

$$\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) = \mathbf{K}_1\mathrm{WLSE}(\Theta)\mathbf{K}_2, \tag{2.4}$$

where $\mathbf{K}_1 \in \mathbb{R}^{k_1\times p}$ and $\mathbf{K}_2 \in \mathbb{R}^{q\times k_2}$ are two given matrices.

The normal equation corresponding to (2.2) is given by

$$(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)\Theta(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2') = (\mathbf{X}_1'\mathbf{V}_1)\mathbf{Y}(\mathbf{V}_2\mathbf{X}_2'). \tag{2.5}$$

This equation is always consistent. Solving the equation by Lemma 1.5 gives the following well-known result on the WLSEs of Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$ under (1.1).

THEOREM 2.1. *The general expressions of the WLSEs of* Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ *and* $\mathbf{K}_1\Theta\mathbf{K}_2$ *under* (1.1) *are given by*

$$\begin{aligned} \mathrm{WLSE}(\Theta) &= (\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+ \\ &\quad + \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}, & (2.6) \\ \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2 \\ &\quad + \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{X}_2 + \mathbf{X}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2, & (2.7) \\ \mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) &= \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{K}_2 \\ &\quad + \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{K}_2 + \mathbf{K}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2, & (2.8) \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

where $\mathbf{U}_1,\ \mathbf{U}_2 \in \mathbb{R}^{p\times q}$ *are arbitrary. If the two arbitrary matrices have no relation with* $\mathbf{Y}$, *then*

$$\begin{aligned} E[\mathrm{WLSE}(\Theta)] &= (\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{V}_1\mathbf{X}_1\Theta\mathbf{X}_2\mathbf{V}_2(\mathbf{X}_2\mathbf{V}_2)^{+} \\ &\quad + \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}, \qquad (2.9) \\ E[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] &= \mathbf{X}_1(\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{V}_1\mathbf{X}_1\Theta\mathbf{X}_2\mathbf{V}_2(\mathbf{X}_2\mathbf{V}_2)^{+}\mathbf{X}_2 \\ &\quad + \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{X}_2 + \mathbf{X}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2, \qquad (2.10) \\ E[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)] &= \mathbf{K}_1(\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{V}_1\mathbf{X}_1\Theta\mathbf{X}_2\mathbf{V}_2(\mathbf{X}_2\mathbf{V}_2)^{+}\mathbf{K}_2 \\ &\quad + \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{K}_2 + \mathbf{K}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2. \qquad (2.11) \end{aligned}$$

Replace $\mathbf{U}_1$ and $\mathbf{U}_2$ in (2.6), (2.7) and (2.8) with

$$\begin{aligned} \mathbf{U}_1 &= \mathbf{L}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}, \\ \mathbf{U}_2 &= [\,(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1 + \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1\,]\mathbf{Y}\mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}, \end{aligned}$$

respectively, where $\mathbf{L}_1 \in \mathbb{R}^{p\times n}$ and $\mathbf{L}_2 \in \mathbb{R}^{m\times q}$ are arbitrary. Then (2.6), (2.7) and (2.8) can be written in the following homogeneous forms

$$\begin{aligned} \mathrm{WLSE}(\Theta) &= [\,(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1 + \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1\,]\mathbf{Y} \\ &\quad \times[\,\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+} + \mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\,], \qquad (2.12) \\ \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) &= [\,\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1 + \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1\,]\mathbf{Y} \\ &\quad \times[\,\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2 + \mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2\,], \qquad (2.13) \\ \mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) &= [\,\mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1 + \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1\,]\mathbf{Y} \\ &\quad \times[\,\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{K}_2 + \mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2\,] \qquad (2.14) \end{aligned}$$

with

$$\begin{aligned} E[\mathrm{WLSE}(\Theta)] &= [\,(\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{V}_1\mathbf{X}_1 + \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1\mathbf{X}_1\,]\Theta \\ &\quad \times[\,\mathbf{X}_2\mathbf{V}_2(\mathbf{X}_2\mathbf{V}_2)^{+} + \mathbf{X}_2\mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\,], \qquad (2.15) \\ E[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] &= [\,\mathbf{X}_1(\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{V}_1\mathbf{X}_1 + \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1\mathbf{X}_1\,]\Theta \\ &\quad \times[\,\mathbf{X}_2\mathbf{V}_2(\mathbf{X}_2\mathbf{V}_2)^{+}\mathbf{X}_2 + \mathbf{X}_2\mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2\,], \qquad (2.16) \\ E[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)] &= [\,\mathbf{K}_1(\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{V}_1\mathbf{X}_1 + \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1\mathbf{X}_1\,]\Theta \\ &\quad \times[\,\mathbf{X}_2\mathbf{V}_2(\mathbf{X}_2\mathbf{V}_2)^{+}\mathbf{K}_2 + \mathbf{X}_2\mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2\,]. \qquad (2.17) \end{aligned}$$

Eqs. (2.9)–(2.11) and (2.15)–(2.17) indicate that $\mathrm{WLSE}(\Theta)$, $\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)$ and $\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)$ are not necessarily unbiased for Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$, respectively. In Section 4, we shall give necessary and sufficient conditions such that $\mathrm{WLSE}(\Theta)$, $\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)$ and $\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)$ in (2.6)–(2.8) and (2.12)–(2.14) are unbiased.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

If $\mathbf{V}_1 = \mathbf{I}_n$ and $\mathbf{V}_2 = \mathbf{I}_m$, then (2.6), (2.7) and (2.8) reduce to the following ordinary least-squares estimators (OLSEs) of Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$ under (1.1)

$$\mathrm{OLSE}(\Theta) = \mathbf{X}_1^+\mathbf{Y}\mathbf{X}_2^+ + \mathbf{F}_{\mathbf{X}_1}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{X}_2}, \tag{2.18}$$

$$\mathrm{OLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) = \mathbf{P}_{\mathbf{X}_1}\mathbf{Y}\mathbf{P}_{\mathbf{X}_2}, \tag{2.19}$$

$$\mathrm{OLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) = \mathbf{K}_1\mathbf{X}_1^+\mathbf{Y}\mathbf{X}_2^+\mathbf{K}_2 + \mathbf{K}_1\mathbf{F}_{\mathbf{X}_1}\mathbf{U}_1\mathbf{K}_2 + \mathbf{K}_1\mathbf{U}_2\mathbf{E}_{\mathbf{X}_2}\mathbf{K}_2. \tag{2.20}$$

3. Ranks of WLSEs under the growth curve model. Observe that (2.6), (2.7) and (2.8) are three matrix pencils with two arbitrary matrices $\mathbf{U}_1$ and $\mathbf{U}_2$. One of the most fundamental results on WLSE(Θ), WLSE($\mathbf{X}_1\Theta\mathbf{X}_2$) and WLSE($\mathbf{K}_1\Theta\mathbf{K}_2$) in (2.6), (2.7) and (2.8) are their maximal and minimal possible ranks with respect to the choice of $\mathbf{U}_1$ and $\mathbf{U}_2$.

THEOREM 3.1. *Let* WLSE(Θ), WLSE($\mathbf{X}_1\Theta\mathbf{X}_2$) *and* WLSE($\mathbf{K}_1\Theta\mathbf{K}_2$) *be as given in* (2.6), (2.7) *and* (2.8). *Then*:

(a) *The maximal and minimal ranks of* WLSE(Θ) *are given by*

$$\max r[\mathrm{WLSE}(\Theta)] = \min\{p + q + r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2') - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2),\ p,\ q\} \tag{3.1}$$

and

$$\min r[\mathrm{WLSE}(\Theta)] = r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'). \tag{3.2}$$

In particular, the rank of WLSE(Θ) *is invariant if and only if one of the following holds*:

(i) $r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2') = p$,
(ii) $r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2') = q$,
(iii) $r(\mathbf{V}_1\mathbf{X}_1) = p$ *and* $r(\mathbf{X}_2\mathbf{V}_2) = q$.

(b) *The maximal and minimal ranks of* WLSE($\mathbf{X}_1\Theta\mathbf{X}_2$) *are given by*

$$\begin{aligned}&\max r[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)]\\ &= \min\{\, r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2') + r(\mathbf{X}_1) + r(\mathbf{X}_2) - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)\\ &\qquad r(\mathbf{X}_1),\ r(\mathbf{X}_2)\},\end{aligned} \tag{3.3}$$

and

$$\min r[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] = r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'). \tag{3.4}$$

In particular, the rank of WLSE($\mathbf{X}_1\Theta\mathbf{X}_2$) *is invariant if and only if one of the following holds*:

(i) $\mathscr{R}(\mathbf{X}_1'\mathbf{V}_1) = \mathscr{R}(\mathbf{X}_1')$ *and* $\mathscr{R}(\mathbf{X}_2\mathbf{V}_1) = \mathscr{R}(\mathbf{X}_2)$,
(ii) $\mathscr{R}(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2') = \mathscr{R}(\mathbf{X}_1')$,
(iii) $\mathscr{R}(\mathbf{X}_2\mathbf{V}_2\mathbf{Y}'\mathbf{V}_1\mathbf{X}_1) = \mathscr{R}(\mathbf{X}_2)$.

(c) *The maximal and minimal ranks of* WLSE($\mathbf{K}_1\Theta\mathbf{K}_2$) *are given by*

$$\begin{aligned}&\max r[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)]\\ &= \min\left\{ r\begin{bmatrix} \mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' & \mathbf{0} & \mathbf{K}_2 \\ \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \end{bmatrix} - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2),\right.\\ &\qquad \left. r(\mathbf{K}_1),\ r(\mathbf{K}_2)\right\}\end{aligned} \tag{3.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

and

$$\begin{aligned}&\min r[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)]\\&= r\begin{bmatrix}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1 & \mathbf{0}\\ \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' & \mathbf{0} & \mathbf{K}_2\\ \mathbf{0} & \mathbf{K}_1 & \mathbf{0}\end{bmatrix} - r\begin{bmatrix}\mathbf{K}_1\\ \mathbf{V}_1\mathbf{X}_1\end{bmatrix} - r[\,\mathbf{K}_2,\ \mathbf{X}_2\mathbf{V}_2\,].\end{aligned} \tag{3.6}$$

(d) *The maximal and minimal ranks of the residual matrix* $\mathbf{Y} - \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)$ *are given by*

$$\begin{aligned}&\max r[\,\mathbf{Y} - \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)\,]\\&= \min\left\{ r\begin{bmatrix}\mathbf{Y} & \mathbf{0} & \mathbf{X}_1\\ \mathbf{0} & -\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1\\ \mathbf{X}_2 & \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' & \mathbf{0}\end{bmatrix} - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2),\right.\\&\qquad\qquad \left. r[\,\mathbf{Y},\ \mathbf{X}_1\,],\ \ r\begin{bmatrix}\mathbf{Y}\\ \mathbf{X}_2\end{bmatrix}\right\}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\min r[\,\mathbf{Y} - \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)\,]\\&= r[\,\mathbf{Y},\ \mathbf{X}_1\,] + r\begin{bmatrix}\mathbf{Y}\\ \mathbf{X}_2\end{bmatrix} + r\begin{bmatrix}\mathbf{Y} & \mathbf{0} & \mathbf{X}_1\\ \mathbf{0} & -\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1\\ \mathbf{X}_2 & \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' & \mathbf{0}\end{bmatrix}\\&\quad - r\begin{bmatrix}\mathbf{Y} & \mathbf{X}_1 & \mathbf{0}\\ \mathbf{X}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2\end{bmatrix} - r\begin{bmatrix}\mathbf{Y} & \mathbf{X}_1\\ \mathbf{X}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{V}_1\mathbf{X}_1\end{bmatrix}.\end{aligned} \tag{3.8}$$

Proof. Applying (1.17) and (1.18) to (2.6) gives

$$\begin{aligned}&\max r[\mathrm{WLSE}(\Theta)]\\&= \max_{\mathbf{U}_1,\mathbf{U}_2} r[\,(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+ + \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\,]\\&= \min\left\{p,\ q,\ r\begin{bmatrix}(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+ & \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)} & \mathbf{0}\end{bmatrix}\right\}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\min r[\mathrm{WLSE}(\Theta)]\\&= \min_{\mathbf{U}_1,\mathbf{U}_2} r[\,(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+ + \mathbf{F}_{\mathbf{V}_1\mathbf{X}_1}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{X}_2\mathbf{V}_2}\,]\\&= r\begin{bmatrix}(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+ & \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)} & \mathbf{0}\end{bmatrix}\\&\quad - r(\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}) - r(\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}).\end{aligned} \tag{3.10}$$

It can be derived from (1.4), (1.5) and (1.6) that

$$r(\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}) = p - r(\mathbf{V}_1\mathbf{X}_1),\quad r(\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}) = q - r(\mathbf{X}_2\mathbf{V}_2),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

$$
\begin{aligned}
& r\begin{bmatrix} (\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+ & \mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)} & \mathbf{0} \end{bmatrix} \\
& = r[(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+] + r(\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}) + r(\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}) \\
& = r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2') + p + q - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2),
\end{aligned}
$$

so that (3.1) and (3.2) follow. Applying (1.15) and (1.16) to (2.7) gives

$$
\begin{aligned}
& \max r[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] \\
& = \max_{\mathbf{U}_1,\mathbf{U}_2} r[\,\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2 \\
& \qquad + \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{X}_2 + \mathbf{X}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2\,] \\
& = \min\left\{ r\begin{bmatrix} \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2 & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix}, \right. \\
& \qquad \left. r(\mathbf{X}_1),\ r(\mathbf{X}_2) \right\} \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
& \min r[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] \\
& = \min_{\mathbf{U}_1,\mathbf{U}_2} r[\,\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2 \\
& \qquad + \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{X}_2 + \mathbf{X}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2\,] \\
& = r\begin{bmatrix} \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2 & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix} \\
& \qquad - r(\mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}) - r(\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2). \qquad (3.12)
\end{aligned}
$$

Applying (1.4), (1.5) and (1.6) and simplifying by EBMOs give

$$
r(\mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}) = r(\mathbf{X}_1) - r(\mathbf{V}_1\mathbf{X}_1), \quad r(\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2) = r(\mathbf{X}_2) - r(\mathbf{X}_2\mathbf{V}_2),
$$

$$
\begin{aligned}
& r\begin{bmatrix} \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2 & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix} \\
& = r\begin{bmatrix} \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2 & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2 \\ \mathbf{0} & \mathbf{V}_1\mathbf{X}_1 & \mathbf{0} \end{bmatrix} \\
& \qquad - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2) \\
& = r\begin{bmatrix} \mathbf{0} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2\mathbf{V}_2 \end{bmatrix} \\
& \qquad - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2) \\
& = r(\mathbf{X}_1) + r(\mathbf{X}_2) + r[\mathbf{V}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2\mathbf{V}_2] \\
& \qquad - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2) \\
& = r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2') + r(\mathbf{X}_1) + r(\mathbf{X}_2) - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2),
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

so that (3.3) and (3.4) follow. Applying (1.15) and (1.16) to (2.8) gives

$$\begin{aligned}
&\max r[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)]\\
&= \max_{\mathbf{U}_1,\mathbf{U}_2} r[\,\mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{K}_2\\
&\qquad\quad + \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{K}_2 + \mathbf{K}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2\,]\\
&= \min\left\{ r\begin{bmatrix} \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{K}_2 & \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2 & \mathbf{0} \end{bmatrix}, \right.\\
&\qquad\quad \left. r(\mathbf{K}_1),\ r(\mathbf{K}_2) \right\} \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
&\min r[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)]\\
&= \min_{\mathbf{U}_1,\mathbf{U}_2} r[\,\mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{K}_2\\
&\quad + \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{K}_2 + \mathbf{K}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2\,]\\
&= r\begin{bmatrix} \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{K}_2 & \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2 & \mathbf{0} \end{bmatrix}\\
&\quad - r(\mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}) - r(\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2). \qquad (3.14)
\end{aligned}$$

Applying (1.4), (1.5) and (1.6) and simplifying by EBMOs give

$$r(\mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}) = r\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{V}_1\mathbf{X}_1 \end{bmatrix} - r(\mathbf{V}_1\mathbf{X}_1), \quad r(\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2) = r[\,\mathbf{K}_2,\, \mathbf{X}_2\mathbf{V}_2\,] - r(\mathbf{X}_2\mathbf{V}_2),$$

$$\begin{aligned}
&r\begin{bmatrix} \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{K}_2 & \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2 & \mathbf{0} \end{bmatrix}\\
&= r\begin{bmatrix} \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{K}_2 & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{K}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2 \\ \mathbf{0} & \mathbf{V}_1\mathbf{X}_1 & \mathbf{0} \end{bmatrix}\\
&\quad - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)\\
&= r\begin{bmatrix} \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{K}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2 \\ \mathbf{0} & \mathbf{V}_1\mathbf{X}_1 & \mathbf{V}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^+\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^+\mathbf{X}_2\mathbf{V}_2 \end{bmatrix}\\
&\quad - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)\\
&= r\begin{bmatrix} \mathbf{0} & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{K}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' \\ \mathbf{0} & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1 & \mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' \end{bmatrix} - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2),
\end{aligned}$$

so that (3.5) and (3.6) follow. Applying (1.15) and (1.16) to $\mathbf{Y} - \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)$ gives

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A publication of the International Linear Algebra Society
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$$
\begin{aligned}
&\max r[\,\mathbf{Y} - \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)\,]\\
&= \max_{\mathbf{U}_1,\mathbf{U}_2} r(\,\mathbf{Y} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2\\
&\qquad - \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{X}_2 - \mathbf{X}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2\,)\\
&= \min\left\{ r\begin{bmatrix} \mathbf{Y} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2 & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix},\right.\\
&\qquad\qquad \left. r[\,\mathbf{Y},\, \mathbf{X}_1\,],\ r\begin{bmatrix} \mathbf{Y} \\ \mathbf{X}_2 \end{bmatrix} \right\}
\end{aligned} \tag{3.15}
$$

and

$$
\begin{aligned}
&\min r[\,\mathbf{Y} - \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)\,]\\
&= \min_{\mathbf{U}_1,\mathbf{U}_2} r[\,\mathbf{Y} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2\\
&\qquad - \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{U}_1\mathbf{X}_2 - \mathbf{X}_1\mathbf{U}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2\,]\\
&= r[\,\mathbf{Y},\, \mathbf{X}_1\,] + r\begin{bmatrix} \mathbf{Y} \\ \mathbf{X}_2 \end{bmatrix} - r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{X}_2 & \mathbf{0} \end{bmatrix} - r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix}\\
&\qquad + r\begin{bmatrix} \mathbf{Y} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2 & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix}.
\end{aligned} \tag{3.16}
$$

Applying (1.4), (1.5) and (1.6) and simplifying by EBMOs give

$$
r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{X}_2 & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 \\ \mathbf{X}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1\mathbf{X}_1 \end{bmatrix} - r(\mathbf{V}_1\mathbf{X}_1),
$$

$$
r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2 \end{bmatrix} - r(\mathbf{X}_2\mathbf{V}_2),
$$

$$
\begin{aligned}
&r\begin{bmatrix} \mathbf{Y} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2 & \mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} \\ \mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 & \mathbf{0} \end{bmatrix}\\
&= r\begin{bmatrix} \mathbf{Y} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2 & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2 \\ \mathbf{0} & \mathbf{V}_1\mathbf{X}_1 & \mathbf{0} \end{bmatrix}\\
&\quad - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)\\
&= r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 & -\mathbf{Y}\mathbf{V}_2 \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{0} \\ -\mathbf{V}_1\mathbf{Y} & \mathbf{0} & \mathbf{V}_1\mathbf{Y}\mathbf{V}_2 - \mathbf{V}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2\mathbf{V}_2 \end{bmatrix}\\
&\quad - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)\\
&= r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2 \\ \mathbf{0} & \mathbf{V}_1\mathbf{X}_1 & -\mathbf{V}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2\mathbf{V}_2 \end{bmatrix}\\
&\quad -r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

$$= r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' \\ \mathbf{0} & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1 & -\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' \end{bmatrix}$$
$$-r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)$$
$$= r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' \\ \mathbf{0} & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1 & -\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' \end{bmatrix} - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2),$$

so that (3.7) and (3.8) follow. ◻

Partition the parameter matrix Θ in (1.1) as

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, \tag{3.17}$$

where $\Theta_{11} \in \mathbb{R}^{p_1\times q_1}$. Then Θ_{11} can be written as $\Theta_{11} = \mathbf{K}_1\Theta\mathbf{K}_2$, where

$$\mathbf{K}_1 = [\,\mathbf{I}_{p_1},\, \mathbf{0}\,] \ \ \textit{and} \ \ \mathbf{K}_2 = [\,\mathbf{I}_{q_1},\, \mathbf{0}\,]'. \tag{3.18}$$

Applying Theorem 3.1(c) to Θ_{11} yields the following result.

Corollary 3.2. *The maximal and minimal ranks of* WLSE(Θ_{11}) *are given by*

$$\max r[\mathrm{WLSE}(\Theta_{11})]$$
$$= \min\left\{p_1,\ q_1,\ r\begin{bmatrix} \mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_{12} \\ \mathbf{X}_{22}\mathbf{V}_2\mathbf{X}_2' & \mathbf{0} \end{bmatrix} + p_1 + q_1 - r(\mathbf{V}_1\mathbf{X}_1) - r(\mathbf{X}_2\mathbf{V}_2)\right\} \tag{3.19}$$

and

$$\min r[\mathrm{WLSE}(\Theta_{11})] = r\begin{bmatrix} \mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_{12} \\ \mathbf{X}_{22}\mathbf{V}_2\mathbf{X}_2' & \mathbf{0} \end{bmatrix} - r(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_{12}) - r(\mathbf{X}_{21}\mathbf{V}_2\mathbf{X}_2'), \tag{3.20}$$

where

$$\mathbf{X}_1 = [\,\mathbf{X}_{11},\, \mathbf{X}_{12}\,] \ \ \textit{and} \ \ \mathbf{X}_2 = [\,\mathbf{X}_{21}',\, \mathbf{X}_{22}'\,]'. \tag{3.21}$$

Eqs. (3.1)–(3.8), (3.19) and (3.20) only give the maximal and minimal ranks of the WLSEs of Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$, $\mathbf{K}_1\Theta\mathbf{K}_2$ and Θ_{11} under (1.1). A more challenging task is to give the WLSEs of Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$ that achieve the extremal ranks. In addition, it is of interest to construct WLSE(Θ), WLSE$(\mathbf{X}_1\Theta\mathbf{X}_2)$, WLSE$(\mathbf{K}_1\Theta\mathbf{K}_2)$ and Θ_{11} with a given rank between the two maximal and minimal ranks. These problems are referred to as reduced-rank regression in the literature. Some previous work on reduced-rank regression for multivariate linear models can be found, e.g., in [1, 2, 3, 6, 17, 30].

4. Unbiasedness of WLSEs under the growth curve model. One of the most important and desirable properties of an estimator under (1.1) is its unbiasedness for corresponding parametric functions.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

LEMMA 4.1. *Let* $\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)$ *be given as in* (2.14), *and denote*

$$\mathbf{P}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1} = \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1 + \mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)}\mathbf{L}_1 \tag{4.1}$$

and

$$\mathbf{Q}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2} = \mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{K}_2 + \mathbf{L}_2\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2. \tag{4.2}$$

Then the estimator is unbiased for $\mathbf{K}_1\Theta\mathbf{K}_2$ *under* (1.1) *if and only if*

$$\mathbf{P}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}\mathbf{X}_1\Theta\mathbf{X}_2\mathbf{Q}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2} = \mathbf{K}_1\Theta\mathbf{K}_2. \tag{4.3}$$

In particular, if

$$\mathbf{P}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}\mathbf{X}_1 = \mathbf{K}_1 \quad \text{and} \quad \mathbf{X}_2\mathbf{Q}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2} = \mathbf{K}_2, \tag{4.4}$$

then the estimator $\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)$ *in* (2.14) *is unbiased for* $\mathbf{K}_1\Theta\mathbf{K}_2$ *under* (1.1).

Solving the two equations in (4.4) by Lemma 1.5, we obtain the following result.

THEOREM 4.2. *There exist* $\mathbf{P}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}$ *and* $\mathbf{Q}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2}$ *such that the two equalities in* (4.4) *hold if and only if*

$$\mathscr{R}(\mathbf{K}_1') \subseteq \mathscr{R}(\mathbf{X}_1') \quad \text{and} \quad \mathscr{R}(\mathbf{K}_2) \subseteq \mathscr{R}(\mathbf{X}_2).$$

In this case, the general expressions of $\mathbf{P}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}$ *and* $\mathbf{Q}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2}$ *satisfying the two equalities in* (4.4) *can be written as*

$$\widehat{\mathbf{P}}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1} = \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1 + \mathbf{K}_1\mathbf{F}_{\mathbf{V}_1\mathbf{X}_1}\mathbf{X}_1^{+} + \mathbf{K}_1\mathbf{F}_{\mathbf{V}_1\mathbf{X}_1}\mathbf{W}_1\mathbf{E}_{\mathbf{X}_1} \tag{4.5}$$

and

$$\widehat{\mathbf{Q}}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2} = \mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{K}_2 + \mathbf{X}_2^{+}\mathbf{E}_{\mathbf{X}_2\mathbf{V}_2}\mathbf{K}_2 + \mathbf{F}_{\mathbf{X}_2}\mathbf{W}_2\mathbf{E}_{\mathbf{X}_2\mathbf{V}_2}\mathbf{K}_2, \tag{4.6}$$

or equivalently,

$$\widehat{\mathbf{P}}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1} = \mathbf{K}_1\mathbf{X}_1^{-} + \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1(\,\mathbf{I}_n - \mathbf{X}_1\mathbf{X}_1^{-}\,) \tag{4.7}$$

and

$$\widehat{\mathbf{Q}}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2} = \mathbf{X}_2^{-}\mathbf{K}_2 + (\,\mathbf{I}_m - \mathbf{X}_2^{-}\mathbf{X}_2\,)\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{K}_2, \tag{4.8}$$

where $\mathbf{W}_1 \in \mathbb{R}^{p\times n}$ *and* $\mathbf{W}_2 \in \mathbb{R}^{m\times q}$ *are arbitrary, and* $\mathbf{X}_1^{-}$ *and* $\mathbf{X}_2^{-}$ *are any g-inverses of* $\mathbf{X}_1$ *and* $\mathbf{X}_2$. *Correspondingly,*

$$\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) = \widehat{\mathbf{P}}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}\mathbf{Y}\widehat{\mathbf{Q}}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2}$$

is unbiased for $\mathbf{K}_1\Theta\mathbf{K}_2$ *under* $\mathscr{M}$.

COROLLARY 4.3. *The linear estimator*

$$\begin{aligned}\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) = {} & [\,\mathbf{X}_1\mathbf{X}_1^{-} + \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1(\,\mathbf{I}_n - \mathbf{X}_1\mathbf{X}_1^{-}\,)\,]\mathbf{Y} \\ & \times[\,\mathbf{X}_2^{-}\mathbf{X}_2 + (\,\mathbf{I}_m - \mathbf{X}_2^{-}\mathbf{X}_2\,)\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2\,]\end{aligned}$$

is unbiased for $\mathbf{X}_1\Theta\mathbf{X}_2$ *under* (1.1) *for any two nonnegative definite matrices* $\mathbf{V}_1$ *and* $\mathbf{V}_2$.

COROLLARY 4.4. *Let* $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be as given in* (3.18) *and* (3.21). *Then there are* $\mathbf{P}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}$ *and* $\mathbf{Q}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2}$ *as in* (4.1) *and* (4.2) *such that the two equalities in* (4.4) *hold if and only if*

$$r(\mathbf{X}_{11}) = p_1, \quad r(\mathbf{X}_{21}) = q_1, \quad \mathscr{R}(\mathbf{X}_{11}) \cap \mathscr{R}(\mathbf{X}_{12}) = \{\mathbf{0}\}, \quad \mathscr{R}(\mathbf{X}_{21}') \cap \mathscr{R}(\mathbf{X}_{22}') = \{\mathbf{0}\}.$$

In this case, $\mathrm{WLSE}(\Theta_{11}) = \widehat{\mathbf{P}}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}\mathbf{Y}\widehat{\mathbf{Q}}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2}$ *is unbiased for* Θ_{11} *under* $\mathscr{M}$, *where* $\widehat{\mathbf{P}}_{\mathbf{K}_1:\mathbf{X}_1:\mathbf{V}_1}$ *and* $\widehat{\mathbf{Q}}_{\mathbf{K}_2:\mathbf{X}_2:\mathbf{V}_2}$ *are as given in* (4.7) *and* (4.8).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

5. Uniqueness of WLSEs under the growth curve model. Concerning the uniqueness of WLSE(Θ), WLSE$(\mathbf{X}_1\Theta\mathbf{X}_2)$, WLSE$(\mathbf{K}_1\Theta\mathbf{K}_2)$ and WLSE(Θ_{11}), we have the following results.

THEOREM 5.1. *Let* WLSE(Θ), WLSE$(\mathbf{X}_1\Theta\mathbf{X}_2)$ *and* WLSE$(\mathbf{K}_1\Theta\mathbf{K}_2)$ *be as given in* (2.6), (2.7) *and* (2.8). *Then,*

(a) WLSE(Θ) *is unique if and only if* $r(\mathbf{V}_1\mathbf{X}_1) = p$ *and* $r(\mathbf{X}_2\mathbf{V}_2) = q$. *In this case, the unique* WLSE(Θ) *can be written as*

$$\mathrm{WLSE}(\Theta) = (\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{-1} \tag{5.1}$$

with

$$\begin{aligned} r[\mathrm{WLSE}(\Theta)] &= r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'), \\ E[\mathrm{WLSE}(\Theta)] &= \Theta, \\ Cov[\mathrm{WLSE}(\Theta)] &= \sigma^2[(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{V}_2\mathbf{\Sigma}_2] \otimes [(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{V}_1\mathbf{\Sigma}_1]. \end{aligned}$$

(b) WLSE$(\mathbf{X}_1\Theta\mathbf{X}_2)$ *is unique if and only if* $\mathscr{R}(\mathbf{X}_1'\mathbf{V}_1) = \mathscr{R}(\mathbf{X}_1')$ *and* $\mathscr{R}(\mathbf{X}_2\mathbf{V}_2) = \mathscr{R}(\mathbf{X}_2)$. *In this case, the unique* WLSE$(\mathbf{X}_1\Theta\mathbf{X}_2)$ *can be written as*

$$\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) = \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2 \tag{5.2}$$

with

$$\begin{aligned} r[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] &= r(\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'), \\ r[\,\mathbf{Y} - \mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)\,] &= r\begin{bmatrix} \mathbf{Y} & \mathbf{0} & \mathbf{X}_1 \\ \mathbf{0} & -\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2' & \mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1 \\ \mathbf{X}_2 & \mathbf{X}_2\mathbf{V}_2\mathbf{X}_2' & \mathbf{0} \end{bmatrix} \\ &\quad - r(\mathbf{X}_1) - r(\mathbf{X}_2) \end{aligned}$$

and

$$\begin{aligned} E[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] &= \mathbf{X}_1\Theta\mathbf{X}_2, \\ Cov[\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2)] &= \sigma^2[\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2'\mathbf{V}_2\mathbf{\Sigma}_2 \otimes \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{\Sigma}_1]. \end{aligned}$$

(c) WLSE$(\mathbf{K}_1\Theta\mathbf{K}_2)$ *is unique if and only if* $\mathscr{R}(\mathbf{K}_1') \subseteq \mathscr{R}(\mathbf{X}_1'\mathbf{V}_1)$ *and* $\mathscr{R}(\mathbf{K}_2) \subseteq \mathscr{R}(\mathbf{X}_2\mathbf{V}_2)$. *In this case, the unique* WLSE$(\mathbf{K}_1\Theta\mathbf{K}_2)$ *can be written as*

$$\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) = \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{K}_2 \tag{5.3}$$

with

$$\begin{aligned} E[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)] &= \mathbf{K}_1\Theta\mathbf{K}_2, \\ Cov[\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2)] &= \sigma^2[\mathbf{K}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2'\mathbf{V}_2\mathbf{\Sigma}_2 \otimes \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{\Sigma}_1]. \end{aligned}$$

(d) WLSE(Θ_{11}) *is unique if and only if*

$$r(\mathbf{V}_1\mathbf{X}_{11}) = p_1, \qquad r(\mathbf{X}_{21}\mathbf{V}_2) = q_1,$$

$$\mathscr{R}(\mathbf{V}_1\mathbf{X}_{11}) \cap \mathscr{R}(\mathbf{V}_1\mathbf{X}_{12}) = \{\mathbf{0}\}, \qquad \mathscr{R}(\mathbf{V}_2\mathbf{X}_{21}') \cap \mathscr{R}(\mathbf{V}_2\mathbf{X}_{22}') = \{\mathbf{0}\}.$$

In this case, WLSE(Θ_{11}) *is unbiased for* Θ_{11}.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

Proof. It can be seen from (2.6) that WLSE(Θ) is unique $\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} = \mathbf{0}$ and $\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)} = \mathbf{0}$, both of them are equivalent to $r(\mathbf{V}_1\mathbf{X}_1) = p$ and $r(\mathbf{X}_2\mathbf{V}_2) = q$. It can be seen from (2.7) that WLSE($\mathbf{X}_1\Theta\mathbf{X}_2$) is unique $\mathbf{X}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} = \mathbf{0}$ and $\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{X}_2 = \mathbf{0}$, both of which are equivalent to $r(\mathbf{V}_1\mathbf{X}_1) = r(\mathbf{X}_1)$ and $r(\mathbf{X}_2\mathbf{V}_2) = r(\mathbf{X}_2)$, or equivalently $\mathscr{R}(\mathbf{X}_1'\mathbf{V}_1) = \mathscr{R}(\mathbf{X}_1')$ and $\mathscr{R}(\mathbf{X}_2\mathbf{V}_2) = \mathscr{R}(\mathbf{X}_2)$. It can be seen from (2.8) that WLSE($\mathbf{K}_1\Theta\mathbf{K}_2$) is unique $\mathbf{K}_1\mathbf{F}_{(\mathbf{V}_1\mathbf{X}_1)} = \mathbf{0}$ and $\mathbf{E}_{(\mathbf{X}_2\mathbf{V}_2)}\mathbf{K}_2 = \mathbf{0}$, both of which are equivalent to $r\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{V}_1\mathbf{X}_1 \end{bmatrix} = r(\mathbf{V}_1\mathbf{X}_1)$ and $r[\,\mathbf{K}_2,\, \mathbf{X}_2\mathbf{V}_2\,] = r(\mathbf{X}_2\mathbf{V}_2)$, or equivalently, $\mathscr{R}(\mathbf{K}_1') \subseteq \mathscr{R}(\mathbf{X}_1'\mathbf{V}_1)$ and $\mathscr{R}(\mathbf{K}_2) \subseteq \mathscr{R}(\mathbf{V}_2\mathbf{X}_2)$. Result (d) is a direct consequence of (c). $\square$

Two special cases of WLSE($\mathbf{X}_1\Theta\mathbf{X}_2$) and WLSE($\mathbf{K}_1\Theta\mathbf{K}_2$) in (2.7) and (2.8) are

$$\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) = \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{X}_2 \tag{5.4}$$

and

$$\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) = \mathbf{K}_1(\mathbf{X}_1'\mathbf{V}_1\mathbf{X}_1)^{+}\mathbf{X}_1'\mathbf{V}_1\mathbf{Y}\mathbf{V}_2\mathbf{X}_2'(\mathbf{X}_2\mathbf{V}_2\mathbf{X}_2')^{+}\mathbf{K}_2, \tag{5.5}$$

where $\mathbf{V}_1 = (\mathbf{X}_1\mathbf{T}_1\mathbf{X}_1' + \mathbf{\Sigma}_1)^{+}$ and $\mathbf{V}_2 = (\mathbf{X}_2'\mathbf{T}_2\mathbf{X}_2 + \mathbf{\Sigma}_2)^{+}$, the two symmetric matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are taken such that $r(\mathbf{X}_1\mathbf{T}_1\mathbf{X}_1' + \mathbf{\Sigma}_1) = r[\,\mathbf{X}_1,\, \mathbf{\Sigma}_1\,]$ and $r(\mathbf{X}_2'\mathbf{T}_2\mathbf{X}_2 + \mathbf{\Sigma}_2) = r[\,\mathbf{X}_2',\, \mathbf{\Sigma}_2\,]$. In this case, (5.4) is the best linear unbiased estimator (BLUE) of $\mathbf{X}_1\Theta\mathbf{X}_2$ under (1.1), and (5.5) is the BLUE of $\mathbf{K}_1\Theta\mathbf{K}_2$ under (1.1). In particular, if both $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are positive definite, and $r(\mathbf{X}_1) = p$ and $r(\mathbf{X}_2) = q$, then

$$\mathrm{WLSE}(\mathbf{X}_1\Theta\mathbf{X}_2) = \mathbf{X}_1(\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}\mathbf{\Sigma}_2^{-1}\mathbf{X}_2'(\mathbf{X}_2\mathbf{\Sigma}_2^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2 \tag{5.6}$$

and

$$\mathrm{WLSE}(\mathbf{K}_1\Theta\mathbf{K}_2) = \mathbf{K}_1(\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}\mathbf{\Sigma}_2^{-1}\mathbf{X}_2'(\mathbf{X}_2\mathbf{\Sigma}_2^{-1}\mathbf{X}_2')^{-1}\mathbf{K}_2 \tag{5.7}$$

are the unique BLUEs of $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$ under (1.1), respectively.

Corollary 5.2. *The ranks of the BLUE of* $\mathbf{X}_1\Theta\mathbf{X}_2$ *in* (5.6) *and the residual matrix* $\mathbf{Y} - \mathrm{BLUE}(\mathbf{X}_1\Theta\mathbf{X}_2)$ *can be written as*

$$r[\mathrm{BLUE}(\mathbf{X}_1\Theta\mathbf{X}_2)] = r(\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}\mathbf{\Sigma}_2^{-1}\mathbf{X}_2')$$

and

$$r[\,\mathbf{Y} - \mathrm{BLUE}(\mathbf{X}_1\Theta\mathbf{X}_2)\,] = r\begin{bmatrix} \mathbf{Y} & \mathbf{0} & \mathbf{X}_1 \\ \mathbf{0} & -\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}\mathbf{\Sigma}_2^{-1}\mathbf{X}_2' & \mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{X}_1 \\ \mathbf{X}_2 & \mathbf{X}_2\mathbf{\Sigma}_2^{-1}\mathbf{X}_2' & \mathbf{0} \end{bmatrix} - p - q.$$

6. Conclusions. We have derived some algebraic and statistical properties of the WLSEs of Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$ under the general growth curve model in (1.1) through the matrix rank method. These basic properties can be used to further investigate various problems associated with these estimators under (1.1). It is expected that other interesting results of estimators of Θ, $\mathbf{X}_1\Theta\mathbf{X}_2$ and $\mathbf{K}_1\Theta\mathbf{K}_2$ can be derived in a similar manner. Furthermore, the matrix rank method can be used to investigate the restricted growth curve model

$$\mathbf{Y} = \mathbf{X}_1\Theta\mathbf{X}_2 + \boldsymbol{\varepsilon}, \quad \mathbf{A}_1\Theta\mathbf{A}_2 = \mathbf{B}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 187-203, August 2007

and the extended growth curve model

$$\mathbf{Y} = \mathbf{X}_{11}\Theta_1\mathbf{X}_{21} + \cdots + \mathbf{X}_{1k}\Theta_k\mathbf{X}_{2k} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Sigma}_1).$$

The method can also be used to derive algebraic and statistical properties of estimators under other types of regression model for longitudinal and panel data.

Acknowledgment. The authors are grateful to an anonymous referee for his/her helpful comments and constructive suggestions on an earlier version of this paper.

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A publication of the International Linear Algebra Society
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