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A GENERALIZATION OF ROTATIONS AND HYPERBOLIC MATRICES AND ITS APPLICATIONS*

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Abstract. In this paper, A-factor circulant matrices with the structure of a circulant, but with the entries below the diagonal multiplied by the same factor A are introduced. Then the generalized rotation and hyperbolic matrices are defined, using an idea due to Ungar. Considering the exponential property of the generalized rotation and hyperbolic matrices, additive formulae for corresponding matrices are also obtained. Also introduced is the block Fourier matrix as a basis for generalizing the Euler formula. The special functions associated with the corresponding Lie group are the functions $F^A_{n,k}(x)$ $(k = 0, 1, \cdots, n-1)$. As an application, the fundamental solutions of the second order matrix differential equation $y''(x) = \Pi_A y(x)$ with initial conditions $y(0) = I$ and $y'(0) = 0$ are obtained using the generalized trigonometric functions $\cos_A(x)$ and $\sin_A(x)$.

Key words. Circulant matrix, A-Factor circulant matrices, Block Vandermonde and Fourier matrices, Rotation and hyperbolic matrices, Generalized Euler formula matrices, Periodic solutions.

AMS subject classifications. 39B30, 15A57.

1. Introduction.

The trigonometric functions can be generalized in many ways, some of them indispensable to the applications of mathematics. We mention, for example, the Bessel, elliptic and hypergeometric functions and their various generalizations. More recently the present authors have given a generalization of trigonometric functions using the generalization of the circle $x^2 + y^2 = 1$ and hyperbola $x^2 - y^2 = 1$ to higher order curves $x^4 \pm y^4 = 1$. This idea can be generalized even for more general curves $x^n \pm y^n = 1$. But they have not been able to find any useful addition formula for their hypergonometric functions [1]. Therefore it is of special interest to find a new class of functions that preserve the "elegance" and "simplicity" of the trigonometric function and specially their addition formulas. One way to do this is to use the linear algebra tools. The idea is to use the rotation and hyperbolic matrices $\mathcal{R}(x)$ and $\mathcal{H}(x)$, as follows:

$$\mathcal{R}(x) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}, \qquad \mathcal{H}(x) = \begin{bmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{bmatrix},$$

or the unique solutions of the following differential equations

$$\mathcal{R}'(x) = A_{\mathcal{R}}\mathcal{R}(x) \qquad \mathcal{H}'(x) = A_{\mathcal{H}}\mathcal{H}(x)$$

in which

$$A_{\mathcal{R}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad A_{\mathcal{H}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

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where $A_{\mathcal{R}}$ and $A_{\mathcal{H}}$ are circulant and anti-circulant matrices, respectively. This idea has been extended by Kittappa [2] and Ungar [3] and also Ungar and Mouldoon [4], using $n \times n$, α-factor circulant matrices

$$\Pi_\alpha = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and exponential map $\exp(\Pi_\alpha x)$. Clearly the rotation and hyperbolic matrices are the special cases of $\exp(\Pi_\alpha x)$ for $\alpha = \pm 1$ and $n = 2$. In this work, using first the idea of A-factor circulant matrices by Ruiz-Claeyssen, Davila and Tsukazan, we consider the basic A-factor circulant matrix, as follows:

$$\Pi_A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Next, using the exponential map $\mathcal{R}(x) = \exp(\Pi_A x)$, we can generalize the idea of Ungar and Mouldoon by introducing the generalized rotation and hyperbolic matrices. Using these new matrices, we generate the generalized sine and cosine functions $F^A_{n,k}(x), (k = 0, 1, \cdots, n-1)$ by the following matrix equation:

$$\mathcal{R}(x) = \begin{bmatrix} F^A_{n,0}(x) & F^A_{n,1}(x) & F^A_{n,2}(x) & \cdots & F^A_{n,n-1}(x) \\ AF^A_{n,n-1}(x) & F^A_{n,0}(x) & F^A_{n,1}(x) & \cdots & F^A_{n,n-2}(x) \\ AF^A_{n,n-2}(x) & F^A_{n,n-1}(x) & F^A_{n,0}(x) & \cdots & F^A_{n,n-3}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ AF^A_{n,1}(x) & AF^A_{n,2}(x) & AF^A_{n,3}(x) & \cdots & F^A_{n,0}(x) \end{bmatrix}.$$

The advantage of these new trigonometric functions is that they satisfy the following addition formula

$$F^A_{n,k}(x+y) = \sum_{r=0}^{n-1} \mu F^A_{n,k}(x) F^A_{n,t_r}(y),$$

and also the generalized Euler's formula

$$\exp(\sqrt[n]{A}x) = \sum_{k=0}^{n-1} \left(\sqrt[n]{A}\right)^k F^A_{n,k}(x).$$

Finally, as an application, we use $F^A_{n,k}(x)$ in order to find the fundamental solutions of the differential equation $y''(x) = \Pi_A y(x)$ with the initial conditions $y(0) = I$ and $y'(0) = 0$.

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2. A-Factor Circulant Matrices. Let $C_1, C_2, \cdots, C_m$ and A be square matrices, each of order n. We assume that A is nonsingular and that it commutes with each of the C_k's. By an A-factor block circulant matrix of type (m, n) we mean a $mn \times mn$ matrix of the from

$$\mathcal{C} = \operatorname{circ}_A(C_1, C_2, \cdots, C_m) = \begin{bmatrix} C_1 & C_2 & \cdots & C_{m-1} & C_m \\ AC_m & C_1 & \cdots & C_{m-2} & C_{m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ AC_3 & AC_4 & \cdots & C_1 & C_2 \\ AC_2 & AC_3 & \cdots & AC_m & C_1 \end{bmatrix}.$$

It follows that any A-factor circulant can be expressed as

$$\mathcal{C} = \sum_{k=0}^{m-1} C_{k+1} \Pi_A^k,$$

where Π_A denotes the basic A-factor circulant, as following

$$\Pi_A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix polynomial

$$p(z) = \sum_{k=0}^{m-1} C_{k+1} z^k,$$

will be referred as the polynomial representer of the factor circulant. Factor circulants of the type (m,1) will be called scalar factor circulants. In this case A reduces to a nonzero scalar denoted by α, when needed. When A is the identity matrix I, we drop the term "factor" in the above definition. This kind of matrices are just block circulants. It is clear that the set of all factor circulants is an algebra with identity because $\Pi_A^k = A^q \Pi_A^p$ for $k = qm + p$, $p = 1, 2, \cdots, m-1$ and $q = 1, 2, \cdots$.

Factor circulant matrices are the only matrices that commute with each Π_A. The relationships

$$\Pi_A^t = \Pi_{A^t}^T, \qquad \Pi_A^{-1} = \Pi_{A^{-1}}^T,$$

where t denotes usual transpose, imply that $\mathcal{C}$ is an A-factor circulant (resp. $(A^{-1})^t$-factor circulant). This property differs from the case of block circulants on which A is symmetric and idempotent, since A reduces to the identity matrix I.

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3. The Generalized Rotation and Hyperbolic Matrices. In this section, using the basic A-factor circulant, we introduce functional matrices $F^A_{n,k}(x)$, which are the generalizations of hyperbolic and rotation and also α-hyperbolic functions for $A = 1$, $A = -1$ and $A = \alpha$ respectively (see [4]).

Definition 3.1. For any $x \in \mathbb{C}$ and $k = 0, 1, \cdots, n-1$, we define $F^A_{n,k}(x)$ as follows:

$$F^A_{n,k}(x) = \sum_{t=0}^{\infty} A^t \frac{x^{tn+k}}{(tn+k)!}, \qquad F^A_{n,0}(0) = I_m.$$

The function $F^A_{n,k}(x)$ is called the A-factor function of order n and of kind k. It is clear to see that,

$$\frac{d^i}{dx^i} F^A_{n,k}(x) = A^{k-i} F^A_{n,r}(x) \qquad (k = 0, 1, \cdots, n-1), \tag{3.1}$$

in which r is the smallest residue of $k - i \pmod n$.

Furthermore, $F^A_{n,k}(x)$ is the solution of the initial matrix value problem:

$$\begin{cases} y^{(n)}(x) = Ay(x) \\ y^{(t)}(0) = \delta_{t,k} I_m \qquad (t = 0, 1, \cdots, n-1), \end{cases}$$

where $k = 0, 1, \cdots, n-1$. These solutions will be referred as the *fundamental solutions* and $F^A_{n,n-1}(x)$ as the *dynamic solution*.

The following relationships among the dynamic solutions and fundamental ones can be easily established by uniqueness arguments:

$$\begin{aligned} &\text{(i) } F^A_{n,k}(x) = d^{n-k-1}\big/dx^{n-k-1} F^A_{n,n-1}(x) \qquad (k = 0, 1, \cdots, n-1), \\ &\text{(ii) } \frac{d^i}{dx^i} F^A_{n,k}(x) = F^A_{n,k-1}(x) \qquad (1 \le i < k \le n-1), \\ &\text{(iii) } \frac{d^i}{dx^i} F^A_{n,k}(x) = A F^A_{n,n-(i-k)}(x) \qquad (1 \le k < i \le n-1). \end{aligned} \tag{3.2}$$

Example 3.2. Let us consider the particular case of $A = \alpha$. The function $F^\alpha_{n,k}(x)$ is called the α-hyperbolic function of order n and of kind k [4]. There is a single α-hyperbolic function of order 1; it is the exponential function $F^\alpha_{1,0}(x) = e^{\alpha x}$. There are two α-hyperbolic functions of order 2; $F^\alpha_{2,0}(x) = \cosh(\sqrt{\alpha}x)$ and $F^\alpha_{2,1}(x) = \frac{1}{\sqrt{\alpha}} \sinh(\sqrt{\alpha}x)$. In the case $\alpha = 1$, the three α-hyperbolic functions of order 3 are

$$F^1_{3,0}(x) = \frac{1}{3}\left[e^x + 2e^{\frac{-x}{2}} \cos(\frac{\sqrt{3}x}{2})\right],$$

$$F^1_{3,1}(x) = \frac{1}{3}\left[e^x - 2e^{\frac{-x}{2}} \cos(\frac{\sqrt{3}x}{2} + \frac{\pi}{3})\right],$$

$$F^1_{3,2}(x) = \frac{1}{3}\left[e^x - 2e^{\frac{-x}{2}} \cos(\frac{\sqrt{3}x}{2} - \frac{\pi}{3})\right],$$

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indicating that the ordinary differential equation $y'''(x) = y(x)$ has three linearly independent solutions $F^1_{3,r}(x), (r = 0, 1, 2)$. In the case $n = 4$, we get the elegant formulas

$$\begin{aligned}
F^1_{4,0}(x) &= \frac{1}{2}(\cosh(x) + \cos(x)),\\
F^1_{4,1}(x) &= \frac{1}{2}(\sinh(x) + \sin(x)),\\
F^1_{4,2}(x) &= \frac{1}{2}(\cosh(x) - \cos(x)),\\
F^1_{4,3}(x) &= \frac{1}{2}(\sinh(x) - \sin(x)).
\end{aligned}$$

Theorem 3.3. *Consider Π_A as defined in a previous section. For any $x \in \mathbb{C}$, $k = 0, 1, \cdots, n-1$, the functions $F^A_{n,k}(x)$ satisfy the following equality:*

$$\exp(x\Pi_A) = \mathrm{circ}_A[F^A_{n,0}(x), F^A_{n,1}(x), \cdots, F^A_{n,n-1}(x)]. \tag{3.3}$$

Proof. We have

$$\begin{aligned}
\exp(x\Pi_A) &= \sum_{k=0}^{\infty} \frac{1}{k!}(x\Pi_A)^k\\
&= \sum_{k=0}^{\infty} F^A_{n,k}(x)\Pi^k_A\\
&= \mathrm{circ}_A[F^A_{n,0}(x), F^A_{n,1}(x), \cdots, F^A_{n,n-1}(x)],
\end{aligned}$$

since, if we have $m = nq + r$ $(0 \le r \le n-1)$, then $\Pi^m_A = A^q\Pi_A$. ∎

Lemma 3.4. *For any square matrix M, we have*

$$\det(\exp(M)) = \exp(tr(M)).$$

Proof. See [6]. ∎

Put $\mathcal{R}(x) = \exp(x\Pi_A)$. It is clear that $\mathcal{R}(0) = I_{mn}$.

Theorem 3.5. $\det(\mathcal{R}(x)) = 1$.

Proof. Considering Theorem 3.3 and Lemma 3.4, we have

$$\det(\mathcal{R}(x)) = \det(\exp(x\Pi_A)) = \exp(\mathrm{tr}(x\Pi_A)) = \exp(0) = 1. \quad ∎$$

The above theorem shows the solutions of the differential equation (3.1) are independent.

Example 3.6. For $n = 2$ and $A = \alpha$, the identity $\det(\mathcal{R}(x)) = 1$ is

$$(F^\alpha_{2,0}(x))^2 - \alpha(F^\alpha_{2,1}(x))^2 = 1,$$

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for $\alpha = -1$ (or $\alpha = 1$), and we have, $\sin^2(x)+\cos^2(x) = 1$ (or $\cosh^2(x)-\sinh^2(x) = 1$). For $n = 3$, and $A = \alpha$ dropping the superscript, we have

$$(F^{\alpha}_{3,0}(x))^3 + \alpha(F^{\alpha}_{3,1}(x))^3 + \alpha^2(F^{\alpha}_{3,2}(x))^3 - 3\alpha F^{\alpha}_{3,0}(x)F^{\alpha}_{3,1}(x)F^{\alpha}_{3,2}(x) = 1.$$

Theorem 3.7. *For $x, y \in \mathbb{C}$, we have*

$$\mathcal{R}(x + y) = \mathcal{R}(x)\mathcal{R}(y), \tag{3.4}$$

$$\mathcal{R}^{-1}(x) = \mathcal{R}(-x). \tag{3.5}$$

Proof. Since the matrices $x\Pi_A$ and $y\Pi_A$ commute with each other, we have

$$\mathcal{R}(x + y) = \exp[(x + y)\Pi_A] = \exp(x\Pi_A)\exp(y\Pi_A) = \mathcal{R}(x)\mathcal{R}(y). \square$$

By using Theorem 3.7, for any $n \in \mathbb{Z}$ and $x \in \mathbb{C}$, we obtain

$$\mathcal{R}^n(x) = \mathcal{R}(nx).$$

Applying the formula (3.4), there is an additive formula for functions $F^A_{n,k}(x)$ as obtained in the following corollary:

Corollary 3.8.

$$F^A_{n,k}(x + y) = \sum_{r=0}^{n-1} \mu F^A_{n,k}(x)F^A_{n,t_r}(y), \tag{3.6}$$

where t_r is the smallest nonnegative remainder of $(k-r)$ with modulo m, and $\mu = I_m$ if $r \leq k$ and $\mu = A$ if $r > k$.

Proof. Comparing both sides of the matrix equality (3.4), the equality is clearly proved. $\square$

Corollary 3.9. *In the equality (3.6), if we put $y = -x$, then we conclude that*

$$I_m = \sum_{r=0}^{n-1} \mu F^A_{n,k}(x)F^A_{n,t_r}(-x). \tag{3.7}$$

4. The Block Fourier Matrix and the Generalized Euler Formula.

Definition. Let $H_1, H_2, \cdots, H_m$ be square matrices each of order n. The block matrix

$$\mathcal{V}_n(H_1, H_2, \cdots, H_m) = \begin{bmatrix} I & I & \cdots & I \\ H_1 & H_2 & \cdots & H_m \\ H_1^2 & H_2^2 & \cdots & H_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ H_1^{m-1} & H_2^{m-1} & \cdots & H_m^{m-1} \end{bmatrix},$$

will be referred to as the *block Vandermonde matrix* of the H_k's. When $n = 1$, the definition reduces to the usual one.

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DEFINITION 4.1. Let $\omega = \exp(2\pi i/m)$ denote the basic m-th root of unity. We define the *block Fourier matrix* $\mathcal{F}_{mn}$ as

$$\mathcal{F}_{mn} = \frac{\mathcal{V}_n(I, \overline{\omega} I, \overline{\omega}^2 I, \cdots, \overline{\omega}^{(m-1)} I)}{\sqrt{m}},$$

where I denote the identity matrix of order n.

It follows that the block conjugate transpose of the Fourier matrix is given by

$$\mathcal{F}^{\star}_{mn} = \frac{\mathcal{V}_n(I, \omega I, \omega^2 I, \cdots, \omega^{(m-1)} I)}{\sqrt{m}},$$

and we have

$$\mathcal{F}^{\star}_{mn}\mathcal{F}_{mn} = I_{mn}. \tag{4.1}$$

In fact, let $E = [E_{kj}]$ be the block product matrix $\mathcal{F}^{\star}_{mn}\mathcal{F}_{mn}$. Then

$$mE_{kj} = \sum_{s=0}^{m-1} \omega^{sk}\omega^{-sj} I = \sum_{s=0}^{m-1} \omega^{s(k-j)} I = m\delta_{kj} I$$

implies the validity of (4.1).

Our definition can be shown to lead to the generalized Euler formula

$$\exp(\sqrt[n]{A}x) = \sum_{k=0}^{n-1} \left(\sqrt[n]{A}\right)^k F^A_{n,k}(x), \tag{4.2}$$

where $\sqrt[n]{A}$ is an arbitrarily specified nth root of A.

EXAMPLE 4.2. Obviously, this reduces to $e^{ix} = \cos(x) + i\sin(x)$ in the case $n = 2$ and $A = -1$. Also, for $n = 3$ and $A = 1$, we have

$$e^{\sqrt[3]{1}x} = F^1_{3,0}(x) + \sqrt[3]{1}\, F^1_{3,1}(x) + (\sqrt[3]{1})^2 F^1_{3,2}(x).$$

Since there are n, n-th roots of A, we see that (4.2) is actually a system of n liner equations. We will use the Fourier matrix to show that the system (4.2) can be solved for the $F^A_{n,k}(x), k = 0, \cdots, n-1$.

COROLLARY 4.3. *Suppose $\omega_n = \exp[2\pi i/n]$ is a primitive nth root of unity. Then, we have*

$$F^A_{n,r}(x) = \left(\sqrt[n]{A}\right)^{-r} \sum_{k=0}^{n-1} \omega_n^{-rk} \exp\left[\omega_n^k \sqrt[n]{A}x\right]. \tag{4.3}$$

Proof. Since the n-th roots of A are of the form

$$\sqrt[n]{A}, \omega_n \sqrt[n]{A}, \cdots, \omega_n^{n-1} \sqrt[n]{A}$$

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(4.2) is actually a set of n equations, which in matrix from may be written:

$$\begin{pmatrix} \exp(\sqrt[n]{A}x) \\ \exp(\omega_n \sqrt[n]{A}x) \\ \vdots \\ \exp(\omega_n^{n-1} \sqrt[n]{A}x) \end{pmatrix} = \mathcal{F}_{mn} \begin{pmatrix} F_{n,0}^{A}(x) \\ \sqrt[n]{A}F_{n,1}^{A}(x) \\ \vdots \\ \left(\sqrt[n]{A}\right)^{n-1} F_{n,n-1}^{A}(x) \end{pmatrix}.$$

Using $\mathcal{F}_{mn}^{-1} = \mathcal{F}_{mn}^{\star}$, we easily invert (4.2) to get (4.3). □

5. Lie Group and Lie Algebra Properties. To develop the Lie group properties, further work on the functions $F_{n,k}^{\alpha}$ appears necessary. Lie group of transformations, including the rotation group and the Lorentz group, are fundamental in advanced theoretical physics. It is possible that the following generalizations could have such applications.

Considering the matrix Π_α, it can be easily verified that $\{z\Pi_\alpha : z \in \mathbb{C}\}$ is a Lie algebra with usual Lie product. Also, consider the relation $z_1 \sim z_2 \Leftrightarrow \exp(z_1\Pi_\alpha) = \exp(z_2\Pi_\alpha)$. Let S be the quotient set under the above relation. Then, it follows that $\{z\Pi_\alpha : z \in S\}$ is a linear experimentation of an abstract Lie group with the associated special function F (see [7]).

THEOREM 5.1. *The infinitesimal operator of one parameter Lie group of transformations, corresponding to the Lie group above, is*

$$X(\mathcal{R}) = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + \cdots + x_n\frac{\partial}{\partial x_{n-1}} + \alpha x_1\frac{\partial}{\partial x_n}. \tag{5.1}$$

REMARK 5.2. A proof of the above theorem has been presented in [2], in the special cases $\alpha = \pm 1$.

Proof. [Proof Theorem 5.1] Consider the following transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$X' = \mathcal{R}X.$$

For a small change $\triangle z$ in z from $z = 0$, let x changes to $\overline{x} - x = \mathcal{R}(\triangle z)x - \mathcal{R}(0)x$. Now, since $\mathcal{R}(0) = I$, we have

$$dx = \left.\frac{d\mathcal{R}(z)}{dz}\right|_{z=0} xdz = \Pi_\alpha xdz.$$

For a function $f(x)$, we have

$$df = \mathrm{grad}(f)dx = \mathrm{grad}(f)\Pi_\alpha xdz.$$

So, the infinitesimal operator is $\mathrm{grad}(.)\Pi_\alpha xdz$, which is equivalent (5.1). Using Π_A, we obtain another generalization of the infinitesimal operator as $\mathrm{grad}(.)\Pi_A x$, where x and $\mathrm{grad}(.)$ are vectors of order mn. □

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6. An Application in Differential Equations. Consider the fundamental solutions of the second order matrix differential equation

$$y''(x) = \Pi_A y(x), \tag{6.1}$$

with the initial conditions $y(0) = I$ and $y'(0) = 0$, where Π_A is the A-factor circulants matrix of order $j \times j$.

In fact the solutions of the equation $z^{(2j)}(x) = Az(x)$, are related to solutions of (6.1) by the change of variables $y_i = z^{2(i-1)}$ for $i = 1, 2, \cdots, j$, where $u = \mathrm{col}(y_1, y_2, \cdots, y_j)$.
We know, $F^A_{2j,k}(x)$ (for $k = 0, 1, \cdots, 2j-1$), is the fundamental solution of the above $2j$th order equation. This implies

$$\cos_A(x) := \begin{bmatrix} F^A_{2j,0}(x) & F^A_{2j,2}(x) & \cdots & F^A_{2j,2j-2}(x) \\ \frac{d^2}{dx^2}F^A_{2j,0}(x) & \frac{d^2}{dx^2}F^A_{2j,2}(x) & \cdots & \frac{d^2}{dx^2}F^A_{2j,2j-2}(x) \\ \frac{d^4}{dx^4}F^A_{2j,0}(x) & \frac{d^4}{dx^4}F^A_{2j,2}(x) & \cdots & \frac{d^4}{dx^2}F^A_{2j,2j-2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{2j-2}}{dx^{2j-2}}F^A_{2j,0}(x) & \frac{d^{2j-2}}{dx^{2j-2}}F^A_{2j,2}(x) & \cdots & \frac{d^{2j-2}}{dx^{2j-2}}F^A_{2j,2j-2}(x) \end{bmatrix},$$

and a similar expression holds for other solution that we show with $\sin_A(x)$ with the odd labeled $F^A_{2j,k}(x)$'s. From (3.2), we conclude that

THEOREM 6.1. *Let Π_A be the basic A-factor block circulant of order $j \times j$. Then*

$$\cos_A(x) = circ_A[F^A_{2j,0}(x), F^A_{2j,2}(x), \cdots, F^A_{2j,2j-2}(x)],$$
$$\sin_A(x) = circ_A[F^A_{2j,1}(x), F^A_{2j,3}(x), \cdots, F^A_{2j,2j-1}(x)],$$

where the $F^A_{2j,k}(x)$'s are the fundamental solutions of $y^{(2j)}(x) = Ay(x)$, with the initial conditions $y^{(k)}(0) = \delta_{ik}I$ for $k = 0, 1, \cdots, 2j-1$ and $i = 0, 1, \cdots, 2j-1$.

Note that this representation of fundamental solutions is simpler than the one due to Claeyssen et al. [9]. Indeed, we do not need to find the series representations of $\cos(\sqrt{-\Pi_A}t)$ and $\sin(\sqrt{-\Pi_A}t)/\sqrt{-\Pi_A}$.

It should be observed that when Π_A is of order $2j \times 2j$, that is, the companion matrix, then $\exp(\Pi_A x)$ will involve the even as well as the odd derivatives of the fundamental solutions $F^A_{2j,k}(x)$. This will not be needed when transforming an even order undamped equation into (6.1) which is of dimension j.

Using the generalized trigonometric functions $\cos_K(x)$ and $\sin_K(x)$, where $K = \sqrt[2j]{-A}$, we have an explicit formula for unique ω-periodic solution of the matrix differential equation $y^{(2j)} = Ay + f(x)$, (cf. [8, 9]):

THEOREM 6.2. *Let $-A$ be a square matrix of order n with no eigenvalues of the $(2k\pi/\omega)^{2j}$, k is an integer. Then for any continuous ω-periodic function $f(x)$ there is a unique ω-periodic solution of the matrix equation $y^{(2j)} = Ay + f(x)$, and it is given by*

$$y_f(x) = \frac{1}{2j}\int_x^{x+\omega}\sum_{k=1}^{j}(\sin_K(-\alpha_k\omega/2))^{-1}\cos_K(\alpha_k(x-s+\omega/2))f(s)ds,$$

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with $K = \sqrt[2j]{-A}$ *and the roots* α_k *of the equation* $\alpha^{2j} = (-1)^{j+1}$.

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