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ON A GROUP OF MIXED-TYPE REVERSE-ORDER LAWS FOR GENERALIZED INVERSES OF A TRIPLE MATRIX PRODUCT WITH APPLICATIONS*

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Abstract. Necessary and sufficient conditions are established for a group of mixed-type reverse-order laws for generalized inverses of a triple matrix product to hold. Some applications of the reverse-order laws to generalized inverses of the sum of two matrices are also given.

Key words. Elementary block matrix operations, $\{i,\dots,j\}$-inverse of matrix, Matrix product, Moore-Penrose inverse, Range of matrix, Rank of matrix, Reverse-order law, Sum of matrices.

AMS subject classifications. 15A03, 15A09.

1. Introduction. Throughout this paper, $\mathbb{C}^{m\times n}$ denotes the set of all $m\times n$ complex matrices; the symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, the rank and the range (column space) of a matrix $A\in\mathbb{C}^{m\times n}$, respectively. The Moore-Penrose inverse of A, denoted by $A^\dagger$, is defined to be the unique solution X to the four matrix equations

$$\text{(i)}\ \ AXA=A,\quad \text{(ii)}\ \ XAX=X,\quad \text{(iii)}\ \ (AX)^*=AX,\quad \text{(iv)}\ \ (XA)^*=XA.$$

Further, let E_A and F_A stand for the two orthogonal projectors $E_A=I-AA^\dagger$ and $F_A=I-A^\dagger A$. A matrix $X\in\mathbb{C}^{n\times m}$ is called an $\{i,\dots,j\}$-inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the ith, $\dots$, jth equations of the four matrix equations above. The set of all $\{i,\dots,j\}$-inverses of A is denoted by $\{A^{(i,\dots,j)}\}$. In particular, a $\{1\}$-inverse of A is called g-inverse of A, $\{1,2\}$-inverse of A is called reflexive g-inverse of A, $\{1,3\}$-inverse of A is called least-squares g-inverse of A, and $\{1,4\}$-inverse of A is called minimum-norm g-inverse of A.

Let A, B and C be three matrices such that the product ABC exists. If each of the triple matrices is nonsingular, then the product ABC is nonsingular too, and the inverse of ABC satisfies the reverse-order law $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$. This law, however, cannot trivially be extended to generalized inverses of ABC when the product is a singular matrix. In other words, the reverse-order law

$$(ABC)^{(i,\dots,j)}=C^{(i,\dots,j)}B^{(i,\dots,j)}A^{(i,\dots,j)} \tag{1.1}$$

does not automatically hold for $\{i,\dots,j\}$-inverses of matrices. One of the fundamental research problems in the theory of generalized inverses of matrices is to give necessary

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and sufficient conditions for various reverse-order laws for $\{i,\ldots,j\}$-inverses of matrix products to hold. For the Moore-Penrose inverse of ABC, the reverse-order law $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$ was studied by some authors; see, e.g., [4, 6, 7].

In addition to the standard reverse-order law $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, the inverse of ABC can also be written as the mixed-type reverse-order law $(ABC)^{-1} = (BC)^{-1}B(AB)^{-1}$. Correspondingly, the mixed-type reverse-order law for $\{i,\ldots,j\}$-inverses of a general triple matrix product ABC can be written as

$$(ABC)^{(i,\ldots,j)} = (BC)^{(i,\ldots,j)}B(AB)^{(i,\ldots,j)}. \tag{1.2}$$

The special case $(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger}$ of (1.2) was investigated in [3, 6, 7]. Another motivation for considering (1.2) comes from the following expression for the sum of two matrices

$$A + B = [\, I,\, I\,]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I \\ I \end{bmatrix} \overset{\text{def}}{=} PNQ. \tag{1.3}$$

In this case, applying (1.2) to PNQ gives the following equality for $\{i,\ldots,j\}$-inverses of $A+B$:

$$(\, A + B\,)^{(i,\ldots,j)} = (NQ)^{(i,\ldots,j)}N(PN)^{(i,\ldots,j)} = \begin{bmatrix} A \\ B \end{bmatrix}^{(i,\ldots,j)}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}[\, A,\, B\,]^{(i,\ldots,j)}. \tag{1.4}$$

This equality establishes an essential relationship between $\{i,\ldots,j\}$-inverses of $A+B$ and $\{i,\ldots,j\}$-inverses of two block matrices $[\, A,\, B\,]$ and $\begin{bmatrix} A \\ B \end{bmatrix}$.

Because $\{i,\ldots,j\}$-inverses of a matrix are not necessarily unique, there are, in fact, four relationships between both sides of (1.2):

$$\begin{aligned}
&\{\,(ABC)^{(i,\ldots,j)}\} \cap \{\,(BC)^{(i,\ldots,j)}B(AB)^{(i,\ldots,j)}\} \neq \emptyset,\\
&\{\,(ABC)^{(i,\ldots,j)}\} \subseteq \{\,(BC)^{(i,\ldots,j)}B(AB)^{(i,\ldots,j)}\},\\
&\{\,(ABC)^{(i,\ldots,j)}\} \supseteq \{\,(BC)^{(i,\ldots,j)}B(AB)^{(i,\ldots,j)}\},\\
&\{\,(ABC)^{(i,\ldots,j)}\} = \{\,(BC)^{(i,\ldots,j)}B(AB)^{(i,\ldots,j)}\}.
\end{aligned}$$

It is a huge task to reveal the relationships for all $\{i,\ldots,j\}$-inverses of matrices. In this paper, we consider the following several special cases of (1.2):

$$(ABC)^{(1)} = (BC)^{(1)}B(AB)^{(1)}, \tag{1.5}$$

$$(ABC)^{(1)} = (BC)^{(1,i)}B(AB)^{(1,i)}, \quad i = 3,\ 4, \tag{1.6}$$

$$(ABC)^{(1)} = (BC)^{\dagger}B(AB)^{\dagger}, \tag{1.7}$$

$$(ABC)^{(1,i)} = (BC)^{\dagger}B(AB)^{\dagger}, \quad i = 3,\ 4, \tag{1.8}$$

$$(ABC)^{(1,i)} = (BC)^{(1,i)}B(AB)^{(1,i)}, \quad i = 3,\ 4, \tag{1.9}$$

$$(ABC)^{\dagger} = (BC)^{(1,4)}B(AB)^{\dagger}, \tag{1.10}$$

$$(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{(1,3)}, \tag{1.11}$$

$$(ABC)^{\dagger} = (BC)^{(1,2,4)}B(AB)^{(1,2,3)}. \tag{1.12}$$

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We use ranks of matrices to derive a variety of necessary and sufficient conditions for the reverse-order laws to hold.

Recall that the rank of a matrix is defined to the dimension of the row (column) space of the matrix. Also recall that $A = 0$ if and only if $r(A) = 0$. From this simple fact, we see that two matrices A and B of the same size are equal if and only if $r(\,A - B\,) = 0$; two sets S_1 and S_2 consisting of matrices of the same size have a common matrix if and only if

$$\min_{A \in S_1,\, B \in S_2} r(\,A - B\,) = 0;$$

the set inclusion $S_1 \subseteq S_2$ holds if and only if

$$\max_{A \in S_1} \min_{B \in S_2} r(\,A - B\,) = 0.$$

If some formulas for the rank of $A - B$ can be derived, they can be used to characterize the equality $A = B$, as well as relationships between the two matrix sets. This method has widely been applied to characterize various reverse-order laws for $\{i, \ldots, j\}$-inverses of matrix products, see, e.g., [6, 7, 10, 11, 12].

In order to use the rank method to characterize (1.5)–(1.12), we need the following formulas for ranks of matrices.

LEMMA 1.1. [5] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ *and* $D \in \mathbb{C}^{l\times k}$. *Then*

$$r[\,A,\, B\,] = r(A) + r(B - AA^{(1)}B) = r(B) + r(A - BB^{(1)}A), \tag{1.13}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^{(1)}A) = r(C) + r(A - AC^{(1)}C), \tag{1.14}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[\,(I_m - BB^{(1)})A(I_n - C^{(1)}C)\,], \tag{1.15}$$

where the ranks are invariant with respect to the choices of $A^{(1)}$, $B^{(1)}$ *and* $C^{(1)}$. *If* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, *then*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(\,D - CA^{\dagger}B\,). \tag{1.16}$$

The following lemma provides a group of formulas for the minimal and maximal ranks of the Schur complement $D - CA^{(i,\ldots,j)}B$ with respect to $\{i, \ldots, j\}$-inverses of A.

LEMMA 1.2. [8, 9] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ *and* $D \in \mathbb{C}^{l\times k}$. *Then*

$$\begin{aligned} \min_{A^{(1)}} r(\,D - CA^{(1)}B\,) &= r(A) + r[\,C,\, D\,] + r\begin{bmatrix} B \\ D \end{bmatrix} + r\begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &\quad - r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}, \end{aligned} \tag{1.17}$$

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$$\max_{A^{(1)}} r(\, D - CA^{(1)}B \,) = \min\left\{ r[\, C,\, D\,], \quad r\begin{bmatrix} B \\ D \end{bmatrix}, \quad r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}, \tag{1.18}$$

$$\min_{A^{(1,3)}} r(\, D - CA^{(1,3)}B \,) = r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}, \tag{1.19}$$

$$\max_{A^{(1,3)}} r(\, D - CA^{(1,3)}B \,) = \min\left\{ r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} - r(A), \quad r\begin{bmatrix} B \\ D \end{bmatrix} \right\}, \tag{1.20}$$

$$\min_{A^{(1,4)}} r(\, D - CA^{(1,4)}B \,) = r[\, C,\, D\,] + r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix}, \tag{1.21}$$

$$\max_{A^{(1,4)}} r(\, D - CA^{(1,4)}B \,) = \min\left\{ r[\, C,\, D\,],\ r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r(A) \right\}, \tag{1.22}$$

$$r(\, D - CA^{\dagger}B \,) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{1.23}$$

In particular, if $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times m}$ *and* $D \in \mathbb{C}^{l\times k}$, *then*

$$r(\, D - CAA^{\dagger}B \,) = r\begin{bmatrix} A^*A & A^*B \\ CA & D \end{bmatrix} - r(A). \tag{1.24}$$

The following results are derived from (1.19) and (1.21).

Lemma 1.3. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $G \in \mathbb{C}^{n\times m}$. *Then*:

(a) $G \in \{A^{(1,3)}\}$ *if and only if* $A^*AG = A^*$.

(b) $G \in \{A^{(1,4)}\}$ *if and only if* $GAA^* = A^*$.

Lemma 1.4. [13] *Let* $P,\, Q \in \mathbb{C}^{m\times m}$, *and suppose* $P^2 = P$ *and* $Q^2 = Q$. *Then*

$$r(\, P - Q \,) = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[\, P,\, Q\,] - r(P) - r(Q). \tag{1.25}$$

Lemma 1.5. [1] *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$. *Then*

$$r(\, AB - ABB^{\dagger}A^{\dagger}AB \,) = r[\, A^*,\, B\,] + r(AB) - r(A) - r(B). \tag{1.26}$$

In particular, $B^{\dagger}A^{\dagger} \in \{(AB)^{(1)}\}$ *if and only if* $r[\, A^*,\, B\,] = r(A) + r(B) - r(AB)$.

Lemma 1.6. [8] *Let* $A \in \mathbb{C}^{m\times n}$, $B_i \in \mathbb{C}^{m\times k_i}$ *and* $C_i \in \mathbb{C}^{l_i\times n}$ *be given,* $i = 1, 2$, *and let* $X_i \in \mathbb{C}^{k_i\times l_i}$ *be variable matrices,* $i = 1, 2$. *Then*

$$\min_{X_1,\, X_2} r(\, A - B_1X_1C_1 - B_2X_2C_2 \,) = r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r[\, A,\, B_1,\, B_2\,] + \max\{s_1, \quad s_2\}, \tag{1.27}$$

where

$$s_1 = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}.$$

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LEMMA 1.7. [7] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ *and* $C \in \mathbb{C}^{l\times n}$. *Then the product* $B^\dagger AC^\dagger$ *can be written as*

$$B^\dagger AC^\dagger = -[\,0,\, B^*\,]\begin{bmatrix} B^*AC^* & B^*BB^* \\ C^*CC^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ C^* \end{bmatrix} \overset{\text{def}}{=} -PJ^\dagger Q, \tag{1.28}$$

where the block matrices P, J *and* Q *satisfy*

$$r(J) = r(B) + r(C), \quad \mathscr{R}(Q) \subseteq \mathscr{R}(J) \ \text{ and } \ \mathscr{R}(P^*) \subseteq \mathscr{R}(J^*). \tag{1.29}$$

The following simple results are widely used in the context to simplify various operations on ranks and ranges of matrices:

$$\mathscr{R}(A) = \mathscr{R}(AA^*) = \mathscr{R}(AA^\dagger), \quad \mathscr{R}(A^*) = \mathscr{R}(A^*A) = \mathscr{R}(A^\dagger A), \tag{1.30}$$

$$\mathscr{R}(ABB^\dagger) = \mathscr{R}(ABB^*) = \mathscr{R}(AB), \quad \mathscr{R}(AC^\dagger C) = \mathscr{R}(AC^*C) = \mathscr{R}(AC^*), \tag{1.31}$$

$$r(ABB^\dagger) = r(ABB^*) = r(AB), \quad r(AC^\dagger C) = r(AC^*C) = r(AC^*), \tag{1.32}$$

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \Leftrightarrow r[\,A,\, B\,] = r(B) \Leftrightarrow BB^\dagger A = A, \tag{1.33}$$

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \text{ and } r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B), \tag{1.34}$$

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) \subseteq \mathscr{R}(PB), \tag{1.35}$$

$$\mathscr{R}(A_1) = \mathscr{R}(A_2) \text{ and } \mathscr{R}(B_1) = \mathscr{R}(B_2) \Rightarrow r[\,A_1,\, B_1\,] = r[\,A_2,\, B_2\,]. \tag{1.36}$$

2. The reverse-order law $(ABC)^{(1)} = (BC)^{(1)}B(AB)^{(1)}$.

In this section, we investigate the reverse-order law in (1.5).

THEOREM 2.1. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, *and let* $M = ABC$. *Then*:

(a) *For any* $(AB)^{(1)}$, *there exists a* $(BC)^{(1)}$ *so that* $(BC)^{(1)}B(AB)^{(1)} \in \{M^{(1)}\}$.

(b) *For any* $(BC)^{(1)}$, *there exists a* $(AB)^{(1)}$ *so that* $(BC)^{(1)}B(AB)^{(1)} \in \{M^{(1)}\}$.

(c) *The set inclusion* $\{(BC)^{(1)}B(AB)^{(1)}\} \subseteq \{M^{(1)}\}$ *holds if and only if* $M = 0$ *or* $r(M) = r(AB) + r(BC) - r(B)$.

Proof. It can be seen from the definition of $\{1\}$-inverse that a matrix X is a $\{1\}$-inverse of A if and only if $r(\,A - AXA\,) = 0$. Also recall that elementary matrix operations do not change the rank of the matrix. Applying (1.17) to the difference $M - M(BC)^{(1)}B(AB)^{(1)}M$ and then simplifying by elementary block matrix operations, we obtain

$$\begin{aligned}
&\min_{(AB)^{(1)}} r[\,M - M(BC)^{(1)}B(AB)^{(1)}M\,] \\
&= r(AB) - r[\,AB,\, M\,] - r\begin{bmatrix} AB \\ M(BC)^{(1)}B \end{bmatrix} + r\begin{bmatrix} AB & M \\ M(BC)^{(1)}B & M \end{bmatrix} \\
&= r(AB) - r[\,AB,\, 0\,] - r\begin{bmatrix} AB \\ M(BC)^{(1)}B \end{bmatrix} + r\begin{bmatrix} AB & 0 \\ M(BC)^{(1)}B & 0 \end{bmatrix} = 0.
\end{aligned}$$

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This rank formula implies that for any $(BC)^{(1)}$, there exists a $(AB)^{(1)}$ such that $M(BC)^{(1)}B(AB)^{(1)}M = M$, so that the result in (b) is true. Similarly, we can show that

$$\min_{(BC)^{(1)}} r[\, M - M(BC)^{(1)}B(AB)^{(1)}M \,] = 0$$

holds for any $(AB)^{(1)}$, so that (a) follows.

Also from the definition of $\{1\}$-inverse, the set inclusion $\{(BC)^{(1)}B(AB)^{(1)}\} \subseteq \{M^{(1)}\}$ holds if and only if

$$\max_{(AB)^{(1)},\,(BC)^{(1)}} r[\, M - M(BC)^{(1)}B(AB)^{(1)}M \,] = 0.$$

Applying (1.18) to the difference $M - M(BC)^{(1)}B(AB)^{(1)}M$ and simplifying by elementary block matrix operations, we obtain

$$\begin{aligned} &\max_{(AB)^{(1)}} r[\, M - M(BC)^{(1)}B(AB)^{(1)}M \,] \\ &= \min\left\{ r(M), \quad r\begin{bmatrix} AB & M \\ M(BC)^{(1)}B & M \end{bmatrix} - r(AB) \right\} \\ &= \min\left\{ r(M), \quad r\begin{bmatrix} AB \\ M(BC)^{(1)}B \end{bmatrix} - r(AB) \right\}. \end{aligned} \tag{2.1}$$

Further, applying (1.18) to the column block matrix in (2.1) and simplifying by elementary block matrix operations and $r(M) \leq r(BC)$ give

$$\begin{aligned} &\max_{(BC)^{(1)}} r\begin{bmatrix} AB \\ M(BC)^{(1)}B \end{bmatrix} \\ &= \max_{(BC)^{(1)}} r\left(\begin{bmatrix} AB \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -M \end{bmatrix} (BC)^{(1)}B \right) \\ &= \min\left\{ r\begin{bmatrix} AB & 0 \\ 0 & -M \end{bmatrix}, \quad r\begin{bmatrix} AB \\ 0 \\ B \end{bmatrix}, \quad r\begin{bmatrix} BC & B \\ 0 & AB \\ -M & 0 \end{bmatrix} - r(BC) \right\} \\ &= \min\{\, r(AB) + r(M), \quad r(B), \quad r(B) + r(M) - r(BC) \,\} \\ &= \min\{\, r(AB) + r(M), \quad r(B) + r(M) - r(BC) \,\}. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) yields

$$\begin{aligned} &\max_{(AB)^{(1)},\,(BC)^{(1)}} r[\, M - M(BC)^{(1)}B(AB)^{(1)}M \,] \\ &= \min\left\{ r(M), \quad \max_{(BC)^{(1)}} r\begin{bmatrix} AB \\ M(BC)^{(1)}B \end{bmatrix} - r(AB) \right\} \\ &= \min\{\, r(M), \quad r(M) - r(AB) - r(BC) + r(B) \,\}. \end{aligned} \tag{2.3}$$

Let the right-hand side of (2.3) be zero. Then we obtain the result in (c). $\square$

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3. Relationships between $(BC)^\dagger B(AB)^\dagger$ and $\{i,\dots,j\}$-inverses of ABC. In this section, we investigate the three reverse-order laws in (1.7) and (1.8).

THEOREM 3.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and let $M = ABC$. Then the following statements are equivalent*:

(a) $(BC)^\dagger B(AB)^\dagger$ *is a* $\{1\}$*-inverse of* M.

(b) $r\left(\begin{bmatrix} (BC)^* \\ A \end{bmatrix} B[\,(AB)^*,\, C\,]\right) = r(AB) + r(BC) - r(M)$.

Proof. Applying (1.28) and (1.29) to the product $(BC)^\dagger B(AB)^\dagger$ yields

$$\begin{aligned}(BC)^\dagger B(AB)^\dagger &= -[\,0,\,(BC)^*\,]\begin{bmatrix} (BC)^*B(AB)^* & (BC)^*BC(BC)^* \\ (AB)^*AB(AB)^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ (AB)^* \end{bmatrix} \\ &\overset{\text{def}}{=} -PJ^\dagger Q,\end{aligned}$$

with $r(J) = r(AB) + r(BC)$, $\mathscr{R}(Q) \subseteq \mathscr{R}(J)$ and $\mathscr{R}(P^*) \subseteq \mathscr{R}(J^*)$. In this case, applying (1.16) to $M - M(BC)^\dagger B(AB)^\dagger M = M + MPJ^\dagger QM$ and simplifying by elementary block matrix operations yield

$$\begin{aligned}
(3.1)\quad & r[\,M - M(BC)^\dagger B(AB)^\dagger M\,] \\
&= r(\,M + MPJ^\dagger QM\,) \\
&= r\begin{bmatrix} J & QM \\ MP & -M \end{bmatrix} - r(J) \\
&= r\begin{bmatrix} J + QMP & 0 \\ 0 & -M \end{bmatrix} - r(J) \\
&= r(\,J + QMP\,) + r(M) - r(AB) - r(BC) \\
&= r\begin{bmatrix} (BC)^*B(AB)^* & (BC)^*(BC)(BC)^* \\ (AB)^*(AB)(AB)^* & (AB)^*M(BC)^* \end{bmatrix} + r(M) - r(AB) - r(BC) \\
&= r\begin{bmatrix} (BC)^*B(AB)^* & (BC)^*(BC) \\ (AB)(AB)^* & M \end{bmatrix} + r(M) - r(AB) - r(BC) \quad \text{(by (1.32))} \\
&= r\left(\begin{bmatrix} (BC)^* \\ A \end{bmatrix} B[\,(AB)^*,\, C\,]\right) + r(M) - r(AB) - r(BC).
\end{aligned}$$

Let the right-hand side of (3.1) be zero. Then we obtain the equivalence of (a) and (b). ∎

THEOREM 3.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and let $M = ABC$. Then the following statements are equivalent*:

(a) $(BC)^\dagger B(AB)^\dagger$ *is a* $\{1,3\}$*-inverse of* M.

(b) $r\left(\begin{bmatrix} (BC)^* \\ M^*A \end{bmatrix} B[\,(AB)^*,\, C\,]\right) = r(BC)$.

Proof. From Lemma 1.3(a), $(BC)^\dagger B(AB)^\dagger$ is a $\{1,3\}$-inverse of M if and only if $M^*M(BC)^\dagger B(AB)^\dagger = M^*$. Also note that

$$[\,M^* - M^*M(BC)^\dagger B(AB)^\dagger\,](AB)(AB)^* = M^*(AB)(AB)^* - M^*M(BC)^\dagger B(AB)^*,$$

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and

$$[\, M^*(AB)(AB)^* - M^*M(BC)^\dagger B(AB)^* \,][(AB)^*]^\dagger (AB)^\dagger = M^* - M^*M(BC)^\dagger B(AB)^\dagger.$$

Hence we find by (1.24) that

$$\begin{aligned} r[\, M^* - M^*M(BC)^\dagger B(AB)^\dagger \,] &= r[\, M^*(AB)(AB)^* - M^*M(BC)^\dagger B(AB)^* \,] \\ &= r\begin{bmatrix} (BC)^*BC & (BC)^*B(AB)^* \\ M^*M & M^*(AB)(AB)^* \end{bmatrix} - r(BC) \\ &= r\begin{bmatrix} (BC)^*B(AB)^* & (BC)^*BC \\ M^*(AB)(AB)^* & M^*M \end{bmatrix} - r(BC) \\ &= r\left(\begin{bmatrix} (BC)^* \\ M^*A \end{bmatrix} B[\, (AB)^*, \, C \,] \right) - r(BC). \end{aligned} \tag{3.2}$$

Let the right-hand side of (3.2) be zero. Then we obtain the equivalence of (a) and (b). ◻

By a similar approach, we can also show that

$$r[\, M^* - (BC)^\dagger B(AB)^\dagger MM^* \,] = r\left(\begin{bmatrix} A \\ (BC)^* \end{bmatrix} B[\, (AB)^*, \, CM^* \,] \right) - r(AB). \tag{3.3}$$

Also note from Lemma 1.3(b) that $(BC)^\dagger B(AB)^\dagger$ is a $\{1,4\}$-inverse of M if and only if $(BC)^\dagger B(AB)^\dagger MM^* = M^*$. Let the right-hand side of (3.3) be zero, we obtain the following result.

Theorem 3.3. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and let $M = ABC$. Then the following statements are equivalent*:

(a) $(BC)^\dagger B(AB)^\dagger$ *is a* $\{1,4\}$*-inverse of* M.

(b) $r\left(\begin{bmatrix} A \\ (BC)^* \end{bmatrix} B[\, (AB)^*, \, CM^* \,] \right) = r(AB)$.

The rank formula associated with the reverse-order law $(ABC)^\dagger = (BC)^\dagger B(AB)^\dagger$ is

$$r[\, (ABC)^\dagger - (BC)^\dagger B(AB)^\dagger \,] = r\left(\begin{bmatrix} (BC)^* \\ (ABC)^*A \end{bmatrix} B[\, (AB)^*, \, C(ABC)^* \,] \right) - r(ABC),$$

see Tian [6, 7]. Hence,

$$(ABC)^\dagger = (BC)^\dagger B(AB)^\dagger \Leftrightarrow r\left(\begin{bmatrix} (BC)^* \\ (ABC)^*A \end{bmatrix} B[\, (AB)^*, \, C(ABC)^* \,] \right) = r(ABC). \tag{3.4}$$

4. Relationships between $(BC)^{(1,i)}B(AB)^{(1,i)}$ for $i = 3, 4$ and $\{i,\dots,j\}$-inverses of ABC. In this section, we investigate the reverse-order laws in (1.6) and (1.9).

Theorem 4.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and let $M = ABC$. Then*:

(a) *There exist $(AB)^{(1,3)}$ and $(BC)^{(1,3)}$ such that $(BC)^{(1,3)}B(AB)^{(1,3)}$ is a $\{1,3\}$-inverse of M if and only if*

$$r[\, (AB)^*M, \, B^*BC \,] = r[\, (AB)^*, \, B^*BC \,] + r(M) - r(AB).$$

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(b) *The set inclusion* $\{(BC)^{(1,3)}B(AB)^{(1,3)}\} \subseteq \{M^{(1,3)}\}$ *holds if and only if* $\mathscr{R}[\,(AB)^*M\,] \subseteq \mathscr{R}(B^*BC)$.

Proof. From Lemma 1.3(a), $(BC)^{(1,3)}B(AB)^{(1,3)}$ is a $\{1,3\}$-inverse of M if and only if $M^*M(BC)^{(1,3)}B(AB)^{(1,3)} = M^*$. Also note $BC(BC)^{(1,3)} = BC(BC)^{\dagger}$. Applying (1.19) to $M^* - M^*M(BC)^{(1,3)}B(AB)^{(1,3)}$ gives

$$\begin{aligned}
(4.1)\qquad & \min_{(AB)^{(1,3)},(BC)^{(1,3)}} r[\,M^* - M^*M(BC)^{(1,3)}B(AB)^{(1,3)}\,] \\
&= \min_{(AB)^{(1,3)}} r[\,M^* - M^*M(BC)^{\dagger}B(AB)^{(1,3)}\,] \\
&= r\begin{bmatrix} (AB)^*AB & (AB)^* \\ M^*M(BC)^{\dagger}B & M^* \end{bmatrix} - r\begin{bmatrix} AB \\ M^*M(BC)^{\dagger}B \end{bmatrix}.
\end{aligned}$$

Simplifying the two block matrices by elementary block matrix operations, we obtain

$$\begin{aligned}
(4.2)\qquad & r\begin{bmatrix} (AB)^*AB & (AB)^* \\ M^*M(BC)^{\dagger}B & M^* \end{bmatrix} \\
&= r\begin{bmatrix} 0 & (AB)^* \\ M^*M(BC)^{\dagger}B - M^*AB & 0 \end{bmatrix} \\
&= r[\,M^*M(BC)^{\dagger}B - M^*AB\,] + r(AB) \\
&= r\begin{bmatrix} (BC)^*BC & (BC)^*B \\ M^*M & M^*AB \end{bmatrix} - r(BC) + r(AB) \quad \text{(by (1.24))} \\
&= r\begin{bmatrix} 0 & (BC)^*B \\ 0 & M^*AB \end{bmatrix} - r(BC) + r(AB) \\
&= r[\,(AB)^*M,\ B^*BC\,] - r(BC) + r(AB),
\end{aligned}$$

and

$$\begin{aligned}
(4.3)\qquad r\begin{bmatrix} AB \\ M^*M(BC)^{\dagger}B \end{bmatrix} &= r\begin{bmatrix} M(BC)^{\dagger}B \\ AB \end{bmatrix} \\
&= r\left(\begin{bmatrix} 0 \\ AB \end{bmatrix} - \begin{bmatrix} -A \\ 0 \end{bmatrix}(BC)(BC)^{\dagger}B\right) \\
&= r\begin{bmatrix} (BC)^*BC & (BC)^*B \\ -ABC & 0 \\ 0 & AB \end{bmatrix} - r(BC) \quad \text{(by (1.24))} \\
&= r\begin{bmatrix} 0 & (BC)^*B \\ ABC & 0 \\ 0 & AB \end{bmatrix} - r(BC) \\
&= r[\,(AB)^*,\ B^*BC\,] + r(M) - r(BC).
\end{aligned}$$

Substituting (4.2) and (4.3) into (4.1) yields

$$\begin{aligned}
(4.4)\qquad & \min_{(AB)^{(1,3)},(BC)^{(1,3)}} r[\,M^* - M^*M(BC)^{(1,3)}B(AB)^{(1,3)}\,] \\
&= r[\,(AB)^*M,\ B^*BC\,] - r[\,(AB)^*,\ B^*BC\,] - r(M) + r(AB).
\end{aligned}$$

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The result in (a) is a direct consequence of (4.4).

Also from (1.20),

$$
\begin{aligned}
(4.5)\qquad & \max_{(AB)^{(1,3)},(BC)^{(1,3)}} r[\, M^* - M^*M(BC)^{(1,3)}B(AB)^{(1,3)}\,] \\
&= \max_{(AB)^{(1,3)}} r[\, M^* - M^*M(BC)^{\dagger}B(AB)^{(1,3)}\,] \\
&= \min\left\{ r\begin{bmatrix} (AB)^*AB & (AB)^* \\ M^*M(BC)^{\dagger}B & M^* \end{bmatrix} - r(AB), \quad n \right\} \\
&= \min\{\, r[\,(AB)^*M,\, B^*BC\,] - r(BC), \quad n\,\} \quad \text{(by (4.2))} \\
&= r[\,(AB)^*M,\, B^*BC\,] - r(BC) \\
&= r[\,(AB)^*M,\, B^*BC\,] - r(B^*BC) \quad \text{(by (1.32))}.
\end{aligned}
$$

Let the right-hand side of (4.5) be zero, we see that $\{(BC)^{(1,3)}B(AB)^{(1,3)}\} \subseteq \{M^{(1,3)}\}$ holds if and only if $r[\,(AB)^*M,\, B^*BC\,] = r(B^*BC)$, which is also equivalent to $\mathscr{R}[\,(AB)^*M\,] \subseteq \mathscr{R}(B^*BC)$ by (1.33), as required for (b). ∎

Theorem 4.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and let $M = ABC$. Then:*

(a) *There exist $(AB)^{(1,4)}$ and $(BC)^{(1,4)}$ such that $(BC)^{(1,4)}B(AB)^{(1,4)}$ is a $\{1,4\}$-inverse of M if and only if $r\begin{bmatrix} M(BC)^* \\ ABB^* \end{bmatrix} = r\begin{bmatrix} (BC)^* \\ ABB^* \end{bmatrix} + r(M) - r(BC)$.*

(b) *The set inclusion $\{(BC)^{(1,4)}B(AB)^{(1,4)}\} \subseteq \{M^{(1,4)}\}$ holds if and only if $\mathscr{R}(\,BCM^*\,) \subseteq \mathscr{R}(BB^*A^*)$.*

Proof. It is easy to show by (1.21) and (1.22) that

$$
\min_{(AB)^{(1,4)},\,(BC)^{(1,4)}} r[\,M^* - (BC)^{(1,4)}B(AB)^{(1,4)}MM^*\,] = r\begin{bmatrix} M(BC)^* \\ ABB^* \end{bmatrix} - r\begin{bmatrix} (BC)^* \\ ABB^* \end{bmatrix} - r(M) + r(BC),
$$

$$
\max_{(AB)^{(1,4)},\,(BC)^{(1,4)}} r[\,M^* - (BC)^{(1,4)}B(AB)^{(1,4)}MM^*\,] = r\begin{bmatrix} M(BC)^* \\ ABB^* \end{bmatrix} - r(AB).
$$

The details are omitted. Let the right-hand sides of these two rank equalities be zero, we obtain the results in (a) and (b). ∎

Theorem 4.3. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and let $M = ABC$. Then:*

(a) *There always exist $(AB)^{(1,3)}$ and $(BC)^{(1,3)}$ such that $(BC)^{(1,3)}B(AB)^{(1,3)}$ is a $\{1\}$-inverse of M.*

(b) *There always exist $(AB)^{(1,4)}$ and $(BC)^{(1,4)}$ such that $(BC)^{(1,4)}B(AB)^{(1,4)}$ is a $\{1\}$-inverse of M.*

(c) *The set inclusion $\{(BC)^{(1,3)}B(AB)^{(1,3)}\} \subseteq \{M^{(1)}\}$ holds if and only if*

$$
r\begin{bmatrix} AB \\ (BC)^*B \end{bmatrix} = r(AB) + (BC) - r(M).
$$

(d) *The set inclusion $\{(BC)^{(1,4)}B(AB)^{(1,4)}\} \subseteq \{M^{(1)}\}$ holds if and only if*

$$
r[\,B(AB)^*,\, BC\,] = r(AB) + (BC) - r(M).
$$

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Proof. From the definition of $\{1\}$-inverse, $(BC)^{(1,3)}B(AB)^{(1,3)}$ is a $\{1\}$-inverse of M if and only if $M(BC)^{(1,3)}B(AB)^{(1,3)}M = M$. Applying (1.19) and $BC(BC)^{(1,3)} = BC(BC)^{\dagger}$ to $M - M(BC)^{(1,3)}B(AB)^{(1,3)}M$ and simplifying by elementary block matrix operations, we have

$$\begin{aligned}(4.6)\qquad &\min_{(AB)^{(1,3)},(BC)^{(1,3)}} r[\, M - M(BC)^{(1,3)}B(AB)^{(1,3)}M\,] \\ &= \min_{(AB)^{(1,3)}} r[\, M - M(BC)^{\dagger}B(AB)^{(1,3)}M\,] \\ &= r\begin{bmatrix} (AB)^*AB & (AB)^*M \\ M(BC)^{\dagger}B & M \end{bmatrix} + r\begin{bmatrix} M \\ M \end{bmatrix} - r\begin{bmatrix} AB & 0 \\ 0 & M \\ M(BC)^{\dagger}B & M \end{bmatrix} \\ &= r\begin{bmatrix} (AB)^*AB & (AB)^*M \\ M(BC)^{\dagger}B & M \end{bmatrix} - r\begin{bmatrix} AB \\ M(BC)^{\dagger}B \end{bmatrix} \\ &= r\begin{bmatrix} (AB)^*AB & 0 \\ M(BC)^{\dagger}B & 0 \end{bmatrix} - r\begin{bmatrix} AB \\ M(BC)^{\dagger}B \end{bmatrix} = 0 \quad \text{(by (1.32))}.\end{aligned}$$

Result (a) follows from (4.6). Also by (1.20), $BC(BC)^{(1,3)} = BC(BC)^{\dagger}$ and elementary block matrix operations,

$$\begin{aligned}(4.7)\qquad &\max_{(AB)^{(1,3)},(BC)^{(1,3)}} r[\, M - M(BC)^{(1,3)}B(AB)^{(1,3)}M\,] \\ &= \max_{(AB)^{(1,3)}} r[\, M - M(BC)^{\dagger}B(AB)^{(1,3)}M\,] \\ &= \min\left\{ r\begin{bmatrix} (AB)^*AB & (AB)^*M \\ M(BC)^{\dagger}B & M \end{bmatrix} - r(A), \quad r\begin{bmatrix} M \\ M \end{bmatrix} \right\} \\ &= \min\left\{ r\begin{bmatrix} AB \\ M(BC)^{\dagger}B \end{bmatrix} - r(AB), \quad r(M) \right\} \\ &= r\begin{bmatrix} AB \\ (BC)^*B \end{bmatrix} + r(M) - r(AB) - r(BC) \quad \text{(by (4.3))}.\end{aligned}$$

Result (c) follows from (4.7). Similarly, we can show that

$$(4.8)\qquad \min_{(AB)^{(1,4)},(BC)^{(1,4)}} r[\, M - M(BC)^{(1,4)}B(AB)^{(1,4)}M\,] = 0,$$

$$\begin{aligned}(4.9)\qquad \max_{(AB)^{(1,4)},\,(BC)^{(1,4)}} r[\, M - M(BC)^{(1,4)}B(AB)^{(1,4)}M\,] &= r\begin{bmatrix} ABB^* \\ (BC)^* \end{bmatrix} + r(M) \\ &\quad - r(AB) - r(BC).\end{aligned}$$

Results (b) and (d) are direct consequences of (4.8) and (4.9). $\square$

Rewriting ABC as $ABC = (ABB^{\dagger})(BC)$ and applying (1.26) to it, we obtain

$$\begin{aligned}(4.10)\qquad &r[\, ABC - ABC(BC)^{\dagger}(ABB^{\dagger})^{\dagger}ABC\,] \\ &= r[\, (ABB^{\dagger})^*,\, BC\,] + r(ABC) - r(ABB^{\dagger}) - r(BC) \\ &= r[\, BB^{\dagger}A^*,\, BC\,] + r(ABC) - r(AB) - r(BC)\end{aligned}$$

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$$= r[\, B^*A^*,\, B^*BC \,] + r(ABC) - r(AB) - r(BC) \quad \text{(by (1.32))}$$
$$= r\begin{bmatrix} AB \\ (BC)^*B \end{bmatrix} + r(ABC) - r(AB) - r(BC).$$

Similarly, we have

$$\begin{aligned} &r[\, ABC - ABC(B^\dagger BC)^\dagger (AB)^\dagger ABC \,] \\ &\quad = r[\, B(AB)^*,\, BC \,] + r(ABC) - r(AB) - r(BC). \end{aligned} \tag{4.11}$$

Comparing (4.10) and (4.11) with Theorem 4.3(c) and (d), we obtain the following two equivalences

$$\{(BC)^{(1,3)}B(AB)^{(1,3)}\} \subseteq \{(ABC)^{(1)}\} \Leftrightarrow (BC)^\dagger (ABB^\dagger)^\dagger \in \{(ABC)^{(1)}\},$$
$$\{(BC)^{(1,4)}B(AB)^{(1,4)}\} \subseteq \{(ABC)^{(1)}\} \Leftrightarrow (B^\dagger BC)^\dagger (AB)^\dagger \in \{(ABC)^{(1)}\}.$$

5. The reverse-order laws $(ABC)^\dagger = (BC)^\dagger B(AB)^{(1,3)}$ and $(ABC)^\dagger = (BC)^{(1,4)}B(AB)^\dagger$. In this section, we investigate the two reverse-order laws in (1.10) and (1.11).

THEOREM 5.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and let $M = ABC$. Then:*
(a) *There exists a $(AB)^{(1,3)}$ such that $M^\dagger = (BC)^\dagger B(AB)^{(1,3)}$ if and only if*

$$r\begin{bmatrix} M^*AB \\ (BC)^*B \end{bmatrix} = r\begin{bmatrix} AB \\ (BC)^*B \end{bmatrix} + r(M) - r(AB).$$

(b) *There exists a $(BC)^{(1,4)}$ such that $M^\dagger = (BC)^{(1,4)}B(AB)^\dagger$ if and only if*

$$r[\, BCM^*,\, B(AB)^* \,] = r[\, BC,\, B(AB)^* \,] + r(M) - r(BC).$$

Proof. Applying (1.19) to $M^\dagger - (BC)^\dagger B(AB)^{(1,3)}$ and simplifying by elementary block matrix operations, we obtain

$$\begin{aligned} &\min_{(AB)^{(1,3)}} r[\, M^\dagger - (BC)^\dagger B(AB)^{(1,3)} \,] \\ &= r\begin{bmatrix} (AB)^*AB & (AB)^* \\ (BC)^\dagger B & M^\dagger \end{bmatrix} - r\begin{bmatrix} AB \\ (BC)^\dagger B \end{bmatrix} \\ &= r\begin{bmatrix} 0 & (AB)^* \\ (BC)^\dagger B - M^\dagger AB & 0 \end{bmatrix} - r\begin{bmatrix} AB \\ (BC)^*B \end{bmatrix} \\ &= r[\, (BC)^\dagger B - M^\dagger AB \,] + r(AB) - r[\, (AB)^*,\, B^*BC \,]. \end{aligned} \tag{5.1}$$

Note that $(BC)^\dagger BC[\, (BC)^\dagger B - M^\dagger AB \,] = (BC)^\dagger B - M^\dagger AB$. Hence

$$r[\, (BC)^\dagger B - M^\dagger AB \,] = r[\, C(BC)^\dagger B - CM^\dagger AB \,]. \tag{5.2}$$

It is easy to verify that $[C(BC)^\dagger B]^2 = C(BC)^\dagger B$ and $(CM^\dagger AB)^2 = CM^\dagger AB$, and from (1.31) that

$$\mathscr{R}[C(BC)^\dagger B] = \mathscr{R}[C(BC)^*], \quad \mathscr{R}(CM^\dagger AB) = \mathscr{R}(CM^*), \tag{5.3}$$
$$\mathscr{R}\{[C(BC)^\dagger B]^*\} = \mathscr{R}(B^*BC), \quad \mathscr{R}[(CM^\dagger AB)^*] = \mathscr{R}[(AB)^*M]. \tag{5.4}$$

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In this case, applying (1.25) to the right-hand side of (5.2) and simplifying by (5.3), (5.4) and (1.36) yields

$$
\begin{aligned}
(5.5)\quad & r[\,C(BC)^{\dagger}B - CM^{\dagger}AB\,] \\
&= r\begin{bmatrix} C(BC)^{\dagger}B \\ CM^{\dagger}AB \end{bmatrix} + r[\,C(BC)^{\dagger}B,\ CM^{\dagger}AB\,] - r[\,C(BC)^{\dagger}B\,] - r(CM^{\dagger}AB) \\
&= r\begin{bmatrix} (BC)^{*}B \\ M^{*}AB \end{bmatrix} + r[\,C(BC)^{*},\ CM^{*}\,] - r(BC) - r(M) \\
&= r\begin{bmatrix} M^{*}AB \\ (BC)^{*}B \end{bmatrix} - r(M).
\end{aligned}
$$

Substituting (5.5) into (5.2), and then (5.2) into (5.1) gives

$$
(5.6)\qquad \min_{(AB)^{(1,3)}} r[\,M^{\dagger} - (BC)^{\dagger}B(AB)^{(1,3)}\,] = r\begin{bmatrix} M^{*}AB \\ (BC)^{*}B \end{bmatrix} - r\begin{bmatrix} AB \\ (BC)^{*}B \end{bmatrix} - r(M) + r(AB).
$$

Let the right-hand side of (5.6) be zero, we obtain the result in (a). Similarly, we can show by (1.21) that

$$
(5.7)\quad \min_{(BC)^{(1,4)}} r[\,M^{\dagger} - (BC)^{(1,4)}B(AB)^{\dagger}\,] = r[\,BCM^{*},\ B(AB)^{*}\,] - r[\,BC,\ B(AB)^{*}\,] - r(M) + r(BC).
$$

Result (b) is a direct consequence of (5.7). ◻

6. The reverse-order law $(ABC)^{\dagger} = (BC)^{(1,2,4)}B(AB)^{(1,2,3)}$. For the product AB, Wibker, Howe and Gilbert [14] showed that there exist $A^{(1,2,3)}$ and $B^{(1,2,4)}$ such that the Moore-Penrose inverse of AB can be expressed as $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,2,3)}$. In this section, we extend this result to the Moore-Penrose inverse of ABC.

THEOREM 6.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{p\times q}$. Then there exist $(AB)^{(1,2,3)}$ and $(BC)^{(1,2,4)}$ such that $(ABC)^{\dagger} = (BC)^{(1,2,4)}B(AB)^{(1,2,3)}$ holds.*

Proof. It is well known that the general expressions of $A^{(1,2,3)}$ and $A^{(1,2,4)}$ can be written as $A^{(1,2,3)} = A^{\dagger} + F_AVAA^{\dagger}$ and $A^{(1,2,4)} = A^{\dagger} + A^{\dagger}AWE_A$, where V and W are two arbitrary matrices; see, e.g., [2]. Hence, the general expressions of $(AB)^{(1,2,3)}$ and $(BC)^{(1,2,4)}$ can be written as

$$
(AB)^{(1,2,3)} = (AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger},\quad (BC)^{(1,2,4)} = (BC)^{\dagger} + (BC)^{\dagger}BCWE_{BC},
$$

where V and W are two arbitrary matrices. By elementary block matrix operations, we first obtain

$$
\begin{aligned}
(6.1)\quad & \\
& r[\,M^{\dagger} - (BC)^{(1,2,4)}B(AB)^{(1,2,3)}\,] \\
&= r\{\,M^{\dagger} - [\,(BC)^{\dagger} + (BC)^{\dagger}BCWE_{BC}\,]B[\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,]\,\} \\
&= r\begin{bmatrix} M^{\dagger} & [\,(BC)^{\dagger} + (BC)^{\dagger}BCWE_{BC}\,]B \\ B[\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,] & B \end{bmatrix} - r(B)
\end{aligned}
$$

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$$= r\left(\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B \\ B(AB)^{\dagger} & B \end{bmatrix} + \begin{bmatrix} 0 \\ BF_{AB} \end{bmatrix} V[\, AB(AB)^{\dagger},\, 0\,] + \begin{bmatrix} (BC)^{\dagger}BC \\ 0 \end{bmatrix} W[\,0,\, E_{BC}B\,]\right) - r(B).$$

Further by (1.27),

$$\min_{V,\,W} r\left(\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B \\ B(AB)^{\dagger} & B \end{bmatrix} + \begin{bmatrix} 0 \\ BF_{AB} \end{bmatrix} V[\, AB(AB)^{\dagger},\, 0\,] + \begin{bmatrix} (BC)^{\dagger}BC \\ 0 \end{bmatrix} W[\,0,\, E_{BC}B\,]\right)$$
$$= r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B \\ B(AB)^{\dagger} & B \\ AB(AB)^{\dagger} & 0 \\ 0 & E_{BC}B \end{bmatrix} + r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B & 0 & (BC)^{\dagger}BC \\ B(AB)^{\dagger} & B & BF_{AB} & 0 \end{bmatrix} + \max\{\, s_1,\ \ s_2\,\}, \tag{6.2}$$

where

$$s_1 = r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B & 0 \\ B(AB)^{\dagger} & B & BF_{AB} \\ 0 & E_{BC}B & 0 \end{bmatrix} - r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B & 0 & (BC)^{\dagger}BC \\ B(AB)^{\dagger} & B & BF_{AB} & 0 \\ 0 & E_{BC}B & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B & 0 \\ B(AB)^{\dagger} & B & BF_{AB} \\ 0 & E_{BC}B & 0 \\ AB(AB)^{\dagger} & 0 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B & (BC)^{\dagger}BC \\ B(AB)^{\dagger} & B & 0 \\ AB(AB)^{\dagger} & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B & (BC)^{\dagger}BC & 0 \\ B(AB)^{\dagger} & B & 0 & BF_{AB} \\ AB(AB)^{\dagger} & 0 & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} M^{\dagger} & (BC)^{\dagger}B & (BC)^{\dagger}BC \\ B(AB)^{\dagger} & B & 0 \\ AB(AB)^{\dagger} & 0 & 0 \\ 0 & E_{BC}B & 0 \end{bmatrix}.$$

Simplifying the block matrices in (6.2) by (1.13), (1.14), (1.15) and elementary block matrix operations, and substituting (6.2) into (6.1) yield

$$\min_{V,\,W} r\{\, M^{\dagger} - [\,(BC)^{\dagger} + (BC)^{\dagger}BCWE_{BC}\,]B[\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,]\} = \max\{\,0,\ \ r(AB) + r(BC) - r(B) - r(ABC)\,\}. \tag{6.3}$$

The manipulations are omitted. Also by the Frobenius rank inequality $r(ABC) \geq r(AB) + r(BC) - r(B)$, the right-hand side of (6.3) becomes zero. Hence the result of the theorem is true. $\square$

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7. $\{i,\ldots,j\}$-inverses of sums of matrices. Applying the results in the previous sections to (1.3) and (1.4) may produce a variety of results on $\{i,\ldots,j\}$-inverses of $A+B$, some of which are given in the following three theorems.

THEOREM 7.1. *Let $A,\, B \in \mathbb{C}^{m\times n}$. Then:*

(a) *The following statements are equivalent:*

(i) $\begin{bmatrix} A \\ B \end{bmatrix}^{\dagger} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{\dagger}$ *is a $\{1\}$-inverse of $A+B$.*

(ii) $r\begin{bmatrix} A+B & AA^*+BB^* \\ A^*A+B^*B & A^*AA^*+B^*BB^* \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix} + r[\,A,\, B\,] - r(\,A+B\,).$

(b) *The following statements are equivalent:*

(i) $\begin{bmatrix} A \\ B \end{bmatrix}^{\dagger} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{\dagger}$ *is a $\{1,3\}$-inverse of $A+B$.*

(ii) $r\left(\begin{bmatrix} A & B \\ I_n & I_n \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = r\begin{bmatrix} A \\ B \end{bmatrix}.$

(c) *The following statements are equivalent:*

(i) $\begin{bmatrix} A \\ B \end{bmatrix}^{\dagger} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{\dagger}$ *is a $\{1,4\}$-inverse of $A+B$.*

(ii) $r\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} A & I_m \\ B & I_m \end{bmatrix}\right) = r[\,A,\, B\,].$

(d) *The following statements are equivalent:*

(i) $(\,A+B\,)^{\dagger} = \begin{bmatrix} A \\ B \end{bmatrix}^{\dagger} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{\dagger}.$

(ii) $r\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = r(\,A+B\,).$

Proof. It follows from Theorems 3.1, 3.2 and 3.3, and (3.4). $\square$

THEOREM 7.2. *Let $A,\, B \in \mathbb{C}^{m\times n}$. Then:*

(a) *There exist $[\,A,\, B\,]^{(1,3)}$ and $\begin{bmatrix} A \\ B \end{bmatrix}^{(1,3)}$ such that*

$$\begin{bmatrix} A \\ B \end{bmatrix}^{(1,3)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{(1,3)} \in \{(\,A+B\,)^{(1,3)}\}$$

if and only if $r\begin{bmatrix} B^*A & A^*B \\ A^*A & B^*B \end{bmatrix} = r\begin{bmatrix} A & B \\ A^*A & B^*B \end{bmatrix} + r(\,A+B\,) - r[\,A,\, B\,].$

(b) *The set inclusion* $\left\{ \begin{bmatrix} A \\ B \end{bmatrix}^{(1,3)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{(1,3)} \right\} \subseteq \{(\,A+B\,)^{(1,3)}\}$ *holds if and only if* $\mathscr{R}\begin{bmatrix} A^*B \\ B^*A \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A^*A \\ B^*B \end{bmatrix}.$

(c) *There exist $[\,A,\, B\,]^{(1,4)}$ and $\begin{bmatrix} A \\ B \end{bmatrix}^{(1,4)}$ such that*

$$\begin{bmatrix} A \\ B \end{bmatrix}^{(1,4)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{(1,4)} \in \{(\,A+B\,)^{(1,4)}\}$$

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if and only if $r\begin{bmatrix} AB^* & AA^* \\ BA^* & BB^* \end{bmatrix} = r\begin{bmatrix} A & AA^* \\ B & BB^* \end{bmatrix} + r(\,A+B\,) - r\begin{bmatrix} A \\ B \end{bmatrix}$.

(d) *The set inclusion* $\left\{ \begin{bmatrix} A \\ B \end{bmatrix}^{(1,4)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\ B\,]^{(1,4)} \right\} \subseteq \{(\,A+B\,)^{(1,4)}\}$ *holds if and only if* $\mathscr{R}\begin{bmatrix} AB^* \\ BA^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} AA^* \\ BB^* \end{bmatrix}$.

Proof. It follows from Theorems 4.1 and 4.2. □

THEOREM 7.3. *Let* $A,\ B \in \mathbb{C}^{m\times n}$. *Then there exist* $[\,A,\ B\,]^{(1,2,3)}$ *and* $\begin{bmatrix} A \\ B \end{bmatrix}^{(1,2,4)}$ *such that*

$$(\,A+B\,)^{\dagger} = \begin{bmatrix} A \\ B \end{bmatrix}^{(1,2,4)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\ B\,]^{(1,2,3)}.$$

Proof. It follows from Theorem 6.1. □

The results in Theorems 7.1, 7.2 and 7.3 can be extended to the sum of k matrices. In fact, the sum $A_1 + \cdots + A_k$ of matrices $A_1, \ldots, A_k \in \mathbb{C}^{m\times n}$ can be rewritten as the product

$$A_1 + \cdots + A_k = [\,I, \ldots, I\,] \begin{bmatrix} A_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \overset{\text{def}}{=} PNQ.$$

Hence, a group of results on $\{i, \ldots, j\}$-inverses of $PNQ = A_1 + \cdots + A_k$ can trivially be derived from the theorems in the previous sections.

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