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NOTE ON DELETING A VERTEX AND WEAK INTERLACING OF THE LAPLACIAN SPECTRUM*

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Abstract. The question of what happens to the eigenvalues of the Laplacian of a graph when we delete a vertex is addressed. It is shown that

$$\lambda_i - 1 \leq \lambda_i^v \leq \lambda_{i+1},$$

where λ_i is the ith smallest eigenvalues of the Laplacian of the original graph and λ_i^v is the ith smallest eigenvalues of the Laplacian of the graph $G[V-v]$; i.e., the graph obtained after removing the vertex v. It is shown that the average number of leaves in a random spanning tree $\mathcal{F}(G) > \frac{2|E|e^{\frac{-1}{\alpha}}}{\lambda_n}$, if $\lambda_2 > \alpha n$.

Key words. Spectrum, Random spanning trees, Cayley formula, Laplacian, Number of leaves.

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1. Introduction. Given a graph $G = (V, E)$ with n vertices $V = \{1, ..., n\}$ and E edges, let A be the adjacency matrix of G, i.e. $a_{i,j} = 1$ if vertex $i \in V$ is adjacent to vertex $j \in V$ and $a_{i,j} = 0$ otherwise. The *Laplacian* matrix of graph G is $L = D - A$, where D is a diagonal matrix where $d_{i,i}$ is equal to the degree d_i of vertex i in G. The Laplacian of a graph is one of the basic matrices associated with a graph. The spectrum of the Laplacian fully characterizes the Laplacian (for more detail see [1]). Since L is symmetric and positive semidefinite, its eigenvalues are all nonnegative. We denote them by $\lambda_1 \leq ... \leq \lambda_n$. One of the elementary operations on a graph is deleting a vertex $v \in V$, we denote the graph obtained from deleting the node v by $G[V-v]$, and the Laplacian Matrix of $G[V-v]$ by L^v. Finally let $\lambda_1^v \leq ... \leq \lambda_{n-1}^v$ be the eigenvalues of L_i^v. A well known theorem in Algebraic Graph theory is the *interlacing* of Laplacian spectrum under addition/deletion of an edge; see for example [1, Thm. 13.6.2]) quoted next.

THEOREM 1.1. *Let X be a graph with n vertices and let Y be obtained from X by adding an edge joining distinct vertices of X then*

$$\lambda_{i-1}(L(Y)) \leq \lambda_i(L(X)) \leq \lambda_i(L(Y)),$$

for all $i = 1, ..., n$, (we assume that $\lambda_0 = -\infty$).

We remark that the eigenvalues of adjacency matrices $A(G)$ and $A(G[V-v])$ also interlace; see, for example, [1, Thm. 9.1.1]. A natural question is whether we get a similar behavior for the Laplacian when we add/delete a vertex. In this note we study this question.

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Related Work. This work uses two theorems from Matrix Analysis. The first is Cauchy's Interlacing theorem which states that the eigenvalues of a Hermitian matrix A of order n interlace the eigenvalues of the principal submatrix of order $n-1$, obtained by removing the ith row and the ith column for each $i \in \{1, ..., n\}$.

THEOREM 1.2. *Let A be a Hermitian matrix of order n and let B be a principal submatrix of A of order $n-1$. Then the eigenvalues of A and B are interlacing i.e. $\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_{n-1}(B) \leq \lambda_n(A)$.*

Proof of this theorem can be found in [2].

The second theorem we use is the Courant-Fischer Theorem. This theorem is an extremely useful characterization of the eigenvalues of symmetric matrices.

THEOREM 1.3. *Let L be a symmetric matrix. Then*

1. *the ith eigenvalue λ_i of L is given by*

$$\lambda_i = \min_U \max_{x \in U} \frac{x^t L x}{x^t x};$$

2. *the $(n-i+1)st$ eigenvalue λ_{n-i+1} of L is given by*

$$\lambda_{n-i+1} = \max_U \min_{x \in U} \frac{x^t L x}{x^t x},$$

where U ranges over all i dimensional subspaces.

Proof of this theorem can be found in [3, p. 186]. Let $v \in V$ be a vertex. Let P be the principal submatrix after we delete the row and column that correspond to the vertex v of the Laplacian. Denote the eigenvalues of P by $\rho_1 \leq \cdots \leq \rho_{n-1}$.

2. Weak Interlace for the L, L^v. In this section we show a weak interlacing connection between the L and L^v. Since L is a symmetric matrix we can use Cauchy's interlacing theorem. The next corollary simply applies this theorem for L and P.

COROLLARY 2.1. $\lambda_1 \leq \rho_1 \leq \cdots \leq \rho_{n-1} \leq \lambda_n$.

The next lemma uses the Courant-Fischer Theorem in order to prove weak interlacing for L, P.

LEMMA 2.2. *For all $i = 1, ..., n-1$, $\rho_i \leq \lambda_i^v + 1$*

Proof. Let $I_v = P - L^v$. Note that I_v is a (0,1) diagonal matrix whose jth diagonal entry is 1 if and only if j is connected to v in G. Fix $i \in \{1, ..., n-1\}$. Using the Courant-Fischer Theorem it follows that

$$\rho_{n-i+1} = \max_U \min_{x \in U} \{\frac{x^t P x}{x^t x} : U \subseteq \mathbb{R}^n, \dim(U) = i, x \in U = span(U)\},$$

where x^t is the transpose of x. Substituting $L^v + I_v$ in P it follows that

$$\rho_{n-i+1} = \max_U \min_{x \in U} \{\frac{x^t (L^v + I_v) x}{x^t x} : U \subseteq \mathbb{R}^n, \dim(U) = i, x \in U = span(U)\}.$$

Using standard calculus we get

$$\rho_{n-i+1} \leq \max_U \min_{x \in U} \{\frac{x^t L^v x}{x^t x} : U \subseteq \mathbb{R}^n, \dim(U) = i, x \in U = span(U)\}$$

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$$+\max_U \min_{x\in U}\{\frac{x^t I_v x}{x^t x} : U \subseteq \mathbb{R}^n, \dim(U)=i, x\in U = span(U)\}$$
$$\le \lambda^v_{n-i+1} + 1. \square$$

We now use the previous lemma to get a lower bound on λ^v_i.

LEMMA 2.3. *For all $v = 1, ..., n$ and for all $i = 1, ..., n-1$,*

$$\lambda_i - 1 \le \lambda^v_i.$$

Proof. Fix $i \in \{1, ..., n-1\}$. From Lemma 2.2 it follows that $\rho_i \le \lambda^v_i + 1$. Now this lemma follows from substituting the conclusion of Corollary 2.1 into the previous inequality $\lambda_i \le \rho_i \le \lambda^v_i + 1$. $\square$

The next lemma provides an upper bound on λ^v_i.

LEMMA 2.4. *For all $v = 1, ..., n$ and for all $i = 1, ..., n-1$,*

$$\lambda^v_i \le \lambda_{i+1}.$$

Proof. We prove this lemma by induction on d_v, the degree of the node v. If the degree is $d_v = 0$, then by removing the node v we reduce the multiplicity of the small eigenvalues, which is 0. Formally $\lambda^v_i = \lambda_{i+1}$ for $i = 1, ..., n-1$. Therefore the lemma holds in this case. For the induction step, suppose that the statement holds for $d_v = k$ and consider the case $d_v = k+1$. Since $d_v > 0$ it follows that there exists an edge e connecting the vertex v to some other node u. Denote the graph obtained by removing the edge e from the graph G by X. Let $\sigma_1 \le .... \le \sigma_{n-1}$ be the eigenvalues of the Laplacian of the graph X. From Theorem 1.1 it follows that $\sigma_i \le \lambda_i$ for all $i = 1, ..., n$. Using induction we obtain that $\lambda^v_{i-1} \le \sigma_i \le \lambda_i$, for all $i = 2, ..., n$ $\square$

Now we present our main theorem.

THEOREM 2.5. *For all $v = 1, ..., n$ and for all $i = 1, ..., n-1$,*

$$\lambda_i - 1 \le \lambda^v_i \le \lambda_{i+1}.$$

Proof. The proof is a direct consequence of Lemmas 2.3 and 2.4. $\square$

We remark that both inequalities above are tight. To see that, we show there exist graphs such that $\lambda_i - 1 = \lambda^v_i$. Consider the graph K_n. It is well known that the eigenvalues of K_n are $0, n, ..., n$, where the multiplicity of the eigenvalue n is $n-1$ and 0 is a simple eigenvalue. Now removing a vertex from K_n produces the graph K_{n-1}. Again the eigenvalues of K_{n-1} are $0, n-1, ..., n-1$, where the multiplicity of the eigenvalue $n-1$ is $n-2$ and 0 is a simple eigenvalue. To see that there are graphs that satisfy $\lambda^v_i = \lambda_{i+1}$, consider the graph without any edges.

3. Application to average leafy trees. In this section we use the weak interlacing Theorem 2.5 to obtain a bound on the average number of leaves in a random spanning tree $\mathcal{F}(G)$. Our bound is useful when $\lambda_2 > \alpha n$, for fixed $\alpha > 0$ and $|E| = O(n^2)$. We call such a graph a *dense expander*; in this case we show that the bound is linear in the number of vertices.

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It is well known that the smallest eigenvalue of L is 0 and that its corresponding eigenvector is $(1,1,\ldots,1)$. If G is connected, all other eigenvalues are greater than 0. Let P^v denote the submatrix of L obtained by deleting the vth row and vth column. Then, by the Matrix Tree Theorem, for each vertex $v \in V$ we have $t(G) = |\det(P^v)|$, where $t(G)$ is the number of spanning trees of G. One can rephrase the Matrix Tree Theorem in terms of the spectrum of the Laplacian matrix. The next theorem appears in [1, p. 284]; it connects the eigenvalues of the Laplacian of G and $t(G)$.

Theorem 3.1. *Let G be a graph on n vertices and let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of the Laplacian of G. Then the number of labeled spanning trees in G is $\frac{1}{n}\prod_{i=2}^{n}\lambda_i$.*

Let G be a graph. Using the previous theorem it is possible to define the following probability space: $\Omega(G) = \{T : T \text{ is a spanning tree in } G\}$. On this set we take a spanning tree in a uniform probability. We are interested in finding the average number of leaves in a random spanning tree. Let T be a random spanning tree taken from $\Omega(G)$ with the uniform distribution. Denote by $\mathcal{F}(G)$ the expected number of leaves in T. Using the matrix theorem we can get a formula to compute the average number of leaves in a random spanning tree.

Lemma 3.2.

$$\mathcal{F}(G) = \sum_{v\in V}\frac{nd_v\prod_{i=2}^{n-1}\lambda_i^v}{(n-1)\prod_{i=2}^{n}\lambda_i}.$$

Proof. The number of trees that have vertex v as a leaf is $\frac{d_i\prod_{i=2}^{n-1}\lambda_i^v}{n-1}$. The lemma follows by summing over all vertices and dividing by the total number of trees. ∎

The weak interlacing theorem enables us to bound the average number of leaves in a dense expander graph. More precisely, we show that $\mathcal{F}(G) = O(n)$.

Theorem 3.3. *Let G be a graph. If $\lambda_2 > \alpha n$, then the average number of leaves in T is bigger than $\frac{2|E|e^{\frac{-1}{\alpha}}}{\lambda_n}$.*

Proof.

$$\begin{aligned}
\mathcal{F}(G) &= \sum_{v\in V}\frac{nd_v\prod_{i=2}^{n-1}\lambda_i^v}{(n-1)\prod_{i=2}^{n}\lambda_i}\\
&\ge \sum_{v\in V}\frac{nd_v\prod_{i=2}^{n-1}(\lambda_i-1)}{(n-1)\prod_{i=2}^{n}\lambda_i}\\
&= \sum_{v\in V}\frac{nd_v\prod_{i=2}^{n-1}\frac{\lambda_i-1}{\lambda_i}}{(n-1)\lambda_n}\\
&= \sum_{v\in V}\frac{nd_v\prod_{i=2}^{n-1}(1-\frac{1}{\lambda_i})}{(n-1)\lambda_n}\\
&\ge \sum_{k\in V}\frac{nd_k(1-\frac{1}{\lambda_2})^n}{(n-1)\lambda_n}
\end{aligned}$$

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$$\geq \frac{2|E|e^{\frac{-n}{\lambda_2}}}{\lambda_n}$$
$$\geq \frac{2|E|e^{\frac{-1}{\alpha}}}{\lambda_n}. \quad \square$$

COROLLARY 3.4. *For any constant $\alpha > 0$, if $\lambda_2 > \alpha n$, and $|E| = O(n^2)$, then the average number of leaves in T is $O(n)$.*

Conclusion. In this paper we proved a weak interlacing theorem for the Laplacian. Using this theorem we showed that in a dense expander the average number of leaves is $O(n)$. A natural open question is to show that the average number of leaves in a random tree is an approximation to the maximal spanning leafy tree.

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