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INERTIALLY ARBITRARY NONZERO PATTERNS OF ORDER 4*

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Abstract. Inertially arbitrary nonzero patterns of order at most 4 are characterized. Some of these patterns are demonstrated to be inertially arbitrary but not spectrally arbitrary. The order 4 sign patterns which are inertially arbitrary and have a nonzero pattern that is not spectrally arbitrary are also described. There exists an irreducible nonzero pattern which is inertially arbitrary but has no signing that is inertially arbitrary. In fact, up to equivalence, this pattern is unique among the irreducible order 4 patterns with this property.

Key words. Sign pattern, Nonzero pattern, Inertia, Spectrum.

AMS subject classifications. 15A18, 15A29, 05C50.

1. Introduction: Definitions and Context. A *sign pattern* is a matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ with entries in $\{+,-,0\}$. The set of all real matrices with the same sign pattern as $\mathcal{A}$ is the qualitative class

$$Q(\mathcal{A}) = \{A = [a_{ij}] \in M_n(\mathbb{R}) : sign(a_{ij}) = \mathcal{A}_{ij} \text{ for all } i, j\}.$$

A *nonzero pattern* is a matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ with entries in $\{*, 0\}$ with

$$Q(\mathcal{A}) = \{A \in M_n(\mathbb{R}) : a_{ij} \neq 0 \Leftrightarrow \mathcal{A}_{ij} = * \text{ for all } i, j\}.$$

If a real matrix A is in $Q(\mathcal{A})$, then A is called a *matrix realization* of $\mathcal{A}$. The characteristic polynomial of A is denoted by $p_A(x)$ and a pattern $\mathcal{A}$ realizes a polynomial $p(x)$ if there is a matrix $A \in Q(\mathcal{A})$ such that $p_A(x) = p(x)$. A *signing* of a nonzero pattern $\mathcal{A}$ is a fixed sign pattern $\mathcal{B}$ such that $\mathcal{B}_{ij} = 0$ whenever $\mathcal{A}_{ij} = 0$ and $\mathcal{B}_{ij} \in \{+,-\}$ whenever $\mathcal{A}_{ij} = *$.

The *spectrum* of a sign (or nonzero) pattern $\mathcal{A}$ is the collection of all multisets U of n complex numbers such that U consists of the eigenvalues of some matrix $A \in Q(\mathcal{A})$. A pattern $\mathcal{A}$ is *spectrally arbitrary* if every multiset of n complex numbers, closed under complex conjugation, is in the spectrum of $\mathcal{A}$.

The *inertia* of a matrix A is an ordered triple $i(A) = (n_1, n_2, n_3)$ where n_1 is the number of eigenvalues of A with positive real part, n_2 is the number of eigenvalues with negative real part, and n_3 is the number of eigenvalues with zero real part. The inertia of a sign (or nonzero) pattern $\mathcal{A}$ is $i(\mathcal{A}) = \{i(A) | A \in Q(\mathcal{A})\}$. An n-by-n pattern $\mathcal{A}$ is *inertially arbitrary* if $i(\mathcal{A})$ contains every ordered triple (n_1, n_2, n_3) with $n_1 + n_2 + n_3 = n$. If a pattern is spectrally arbitrary it must also be inertially arbitrary.

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A sign pattern $\mathcal{P}$ is *signature similar* to pattern $\mathcal{A}$ if $\mathcal{P} = \mathcal{D}\mathcal{A}\mathcal{D}^T$, where $\mathcal{D}$ is a diagonal matrix with diagonal entries from $\{+,-\}$. If $\mathcal{A}$ is a spectrally or inertially arbitrary sign pattern, then so is any matrix obtained from $\mathcal{A}$ via a signature similarity. If $\mathcal{A}$ is a spectrally or inertially arbitrary sign pattern, then so is $-\mathcal{A}$. Likewise the property of being spectrally or inertially arbitrary is invariant under transposition, or permutation similarity for both sign and nonzero patterns. Thus we say a sign pattern $\mathcal{P}$ is *equivalent* to $\mathcal{A}$ if $\mathcal{A}$ can be obtained from $\mathcal{P}$ by a combination of signature similarity, negation, transposition and permutation similarity. Likwise a nonzero pattern $\mathcal{P}$ is *equivalent* to $\mathcal{A}$ if $\mathcal{A}$ can be obtained from $\mathcal{P}$ via transposition and/or permutation similarity. We use the notation such as $T(34)$ to represent a permutation (34) followed by a transposition.

We say $\mathcal{P}$ is a *subpattern* of an n-by-n pattern $\mathcal{A}$ if $\mathcal{P} = \mathcal{A}$ or $\mathcal{P}$ is obtained from $\mathcal{A}$ by replacing one or more nonzero entries by a zero. If $\mathcal{P}$ is a subpattern of $\mathcal{A}$, then we also say $\mathcal{A}$ is a *superpattern* of $\mathcal{P}$. A pattern which is spectrally (inertially) arbitrary is *minimal*, if no proper subpattern is spectrally (inertially) arbitrary.

Spectrally and inertially arbitrary sign patterns were introduced in [6]. Classes of inertially arbitrary sign patterns were derived by Gao and Shao [7] as well as Miao and Li [13]. In [1], Britz et. al. characterized the spectrally arbitrary sign patterns of order 3. In [4] the inertially arbitrary sign patterns of order 3 were characterized and were shown to be identical to the spectrally arbitrary sign patterns of order 3. There are other recent papers which explore classes of spectrally and inertially arbitrary sign patterns (see for example [3, 11, 12]). Each of these sign patterns induce an inertially arbitrary nonzero pattern.

The spectrally arbitrary nonzero patterns of order at most 4 were recently characterized by Corpuz and McDonald [2]. We use their description and arguments in Section 2 to characterize the inertially arbitrary nonzero patterns of order at most 4.

A pattern $\mathcal{A}$ is *reducible* if there is a permutation matrix P such that

$$P^T \mathcal{A} P = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ O & \mathcal{A}_3 \end{bmatrix}$$

where $\mathcal{A}_1$ and $\mathcal{A}_3$ are square matrices (called *components* of $\mathcal{A}$) of order at least one. Kim et. al. [9] explored a class of reducible nonzero patterns which are inertially but not spectrally arbitrary. In Proposition 2.4 we demonstrate that there are order 4 irreducible nonzero patterns which are inertially but not spectrally arbitrary.

In Section 3, we explore inertially arbitrary signings of the nonzero patterns which are inertially but not spectrally arbitrary. It was demonstrated in [4] that there is an order 4 sign pattern which is inertially but not spectrally arbitrary: we provide more order 4 sign patterns with this property in Section 3. In [2] it was noted that it is yet unknown whether every spectrally arbitrary nonzero pattern has a signing which is spectrally arbitrary. Reducible inertially arbitrary nonzero patterns, which have no signing that is inertially arbitrary, were presented in [9]. We demonstrate in Section 3 that there is an irreducible inertially arbitrary nonzero pattern which has no signing that is inertially arbitrary.

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2. Inertially arbitrary nonzero patterns of order at most four. We say a pattern $\mathcal{A}$ contains a 2-*cycle* if both $\mathcal{A}_{ij}$ and $\mathcal{A}_{ji}$ are nonzero for some i, j with $1 \le i < j \le n$.

LEMMA 2.1. *If pattern $\mathcal{A}$ is an inertially arbitrary pattern of order n, then $\mathcal{A}$ must contain two nonzero entries on the diagonal, a 2-cycle and at least two nonzero transversals.*

Proof. That an inertially arbitrary pattern $\mathcal{A}$ needs a 2-cycle and two nonzero diagonal entries can be obtained by observing that the corresponding results in [4] do not depend on the signing of the entries. $\mathcal{A}$ must contain at least two nonzero transversals otherwise the determinant of $\mathcal{A}$ is either zero or signed, in which case $\mathcal{A}$ can not realize either inertia $(n, 0, 0)$ or $(n-1, 0, 1)$. $\square$

It follows from Lemma 2.1 that there is exactly one nonzero pattern of order 2 which is inertially arbitrary:

$$\mathcal{T}_2 = \begin{bmatrix} * & * \\ * & * \end{bmatrix}.$$

PROPOSITION 2.2. *If $\mathcal{A}$ is an inertially arbitrary nonzero pattern of order 3 then $\mathcal{A}$ is equivalent to a superpattern of*

$$\mathcal{D}_1 = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix} \quad \text{or} \quad \mathcal{D}_2 = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{bmatrix}.$$

Proof. A reducible order 3 pattern with two nonzero transversals would necessarily have a nonzero component of order 1 and hence would not realize inertia $(0, 0, 3)$. Thus, by Lemma 2.1, an order 3 inertially arbitrary pattern must be irreducible. An irreducible pattern of order three satisfying the conditions of Lemma 2.1 must have at least six nonzero entries.

Up to equivalence, the patterns $\mathcal{D}_1$ and $\mathcal{D}_2$ are the only irreducible patterns with six nonzero entries satisfying the conditions of Lemma 2.1. In particular, suppose $\mathcal{D}$ has two nonzero diagonal entries, a 2-cycle, two nonzero transversals, and exactly six nonzero entries. Up to permutation we may assume that $\mathcal{D}_{11}$ and $\mathcal{D}_{33}$ are nonzero. Up to the permutation (13), we may also assume $\mathcal{D}_{12}\mathcal{D}_{21} \ne 0$ or $\mathcal{D}_{13}\mathcal{D}_{31} \ne 0$ since $\mathcal{D}$ has a 2-cycle. If $\mathcal{D}_{13}\mathcal{D}_{31} \ne 0$, then any placement of the remaining two nonzeros would give a pattern which is either reducible or has at most one nonzero tansversal. Thus $\mathcal{D}_{12} \ne 0$ and $\mathcal{D}_{21} \ne 0$. If $\mathcal{D} \ne \mathcal{D}_2$, then up to transposition, $\mathcal{D} = \mathcal{D}_1$, since $\mathcal{D}$ is irreducible and has two nonzero transversals.

Any superpattern of $\mathcal{D}_1$ or $\mathcal{D}_2$ is spectrally arbitrary (see [2]) and hence inertially arbitrary. Any irreducible pattern with more than six nonzero entries and satisfying Lemma 2.1 is equivalent to a superpattern of $\mathcal{D}_1$ or $\mathcal{D}_2$. $\square$

PROPOSITION 2.3. *If $\mathcal{A}$ is a reducible inertially arbitrary nonzero pattern of order 4, then $\mathcal{A}$ is equivalent to a superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$.*

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Proof. Suppose $\mathcal{A}$ is an order 4 reducible inertially arbitrary pattern. Then each component of $\mathcal{A}$ must be of order 2 and must realize both the inertias $(2,0,0)$ and $(0,0,2)$. In order to realize inertia $(2,0,0)$, each component must have a nonzero element on the diagonal. But then to realize inertia $(0,0,2)$, each component must have two nonzeros on the diagonal. Since each component is irreducible, it follows that $\mathcal{A}$ is a superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$. $\square$

Let

$$\mathcal{N}_1^* = \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}, \quad \mathcal{N}_2^* = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix} \text{ and } \mathcal{N}_3^* = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & 0 \end{bmatrix}.$$

Pattern $\mathcal{N}_2^*$ was demonstrated to be inertially arbitrary and not spectrally arbitrary by Corpuz and McDonald in [2, Corollary 3.8]. We next demonstrate that there are two other irreducible nonzero patterns of order 4, namely $\mathcal{N}_1^*$ and $\mathcal{N}_3^*$, which are inertially but not spectrally arbitrary.

Proposition 2.4. *The patterns $\mathcal{N}_1^*, \mathcal{N}_2^*$, and $\mathcal{N}_3^*$ are inertially but not spectrally arbitrary. Any proper superpattern of $\mathcal{N}_1^*, \mathcal{N}_2^*$ or $\mathcal{N}_3^*$ is spectrally (and hence inertially) arbitrary.*

Proof. It was demonstrated in [4] that there is a sign pattern with nonzero pattern $\mathcal{N}_1^*$ which is inertially arbitrary. Thus $\mathcal{N}_1^*$ is an inertially arbitrary nonzero pattern. But $\mathcal{N}_1^*$ is not spectrally arbitrary since it is equivalent to a proper subpattern of

$$\mathcal{B} = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \end{bmatrix},$$

and $\mathcal{B}$ is a minimal spectrally arbitrary pattern (see the first pattern in the fourth row of [2, Appendix B]).

The pattern $\mathcal{N}_2^*$ is shown to be inertially arbitrary and not spectrally arbitrary by Corpuz and McDonald in [2, Corollary 3.8].

The pattern $\mathcal{N}_3^*$ appears as the second matrix in the first row of [2, Appendix C] and hence is not spectrally arbitrary. The following are matrix examples of $A \in Q(\mathcal{N}_3^*)$ with inertia triples $(0,0,4)$, $(1,0,3)$, $(1,1,2)$, $(2,0,2)$, $(2,2,0)$, $(2,1,1)$, $(3,0,1)$, $(3,1,0)$, $(4,0,0)$ respectively.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Note that if $A \in Q(\mathcal{N}_3^*)$ then $-A \in Q(\mathcal{N}_3^*)$, and so if $(n_1, n_2, n_3) \in i(\mathcal{N}_3^*)$ then $(n_2, n_1, n_3) \in i(\mathcal{N}_3^*)$. Therefore $\mathcal{N}_3^*$ is inertially arbitrary.

We next claim that any superpattern of $\mathcal{N}_1^*$ with nine nonzero entries is spectrally arbitrary. Suppose $\mathcal{A}$ is a superpattern of $\mathcal{N}_1^*$ with nine nonzero entries. If any one of the entries $\mathcal{A}_{13}, \mathcal{A}_{21}, \mathcal{A}_{34}$ or $\mathcal{A}_{42}$ are nonzero, then $\mathcal{A}$ is equivalent to a superpattern of the second matrix in row three of [2, Appendix A] (using the transformations $T(324)$, (34), (3241), and $T(134)$ respectively on $\mathcal{A}$ to obtain each desired pattern). If either entry $\mathcal{A}_{14}$ or $\mathcal{A}_{41}$ is nonzero, then $\mathcal{A}$ is equivalent to a superpattern of the first matrix in the fifth row of [2, Appendix A] (via transformation (213) and $T(13)$ respectively). If either entry $\mathcal{A}_{22}$ or $\mathcal{A}_{33}$ is nonzero, then $\mathcal{A}$ is equivalent to the spectrally arbitrary pattern $\mathcal{B}$. In each case, these superpatterns of $\mathcal{N}_1^*$ are spectrally arbitrary and hence inertially arbitrary.

Finally we note that any superpattern of $\mathcal{N}_1^*$, $\mathcal{N}_2^*$ or $\mathcal{N}_3^*$ with ten nonzero entries will not appear in [2, Appendix D], since each pattern in [2, Appendix D] has at most one nonzero transversal. Therefore any superpattern of $\mathcal{N}_1^*$, $\mathcal{N}_2^*$ or $\mathcal{N}_3^*$ with ten nonzero entries will be spectrally arbitrary by [2, Theorem 3.6]. Also by [2, Theorem 3.6], any superpattern of $\mathcal{N}_1^*$, $\mathcal{N}_2^*$ or $\mathcal{N}_3^*$ with more than ten nonzero entries is spectrally arbitrary. □

Next we characterize the order 4 irreducible inertially arbitrary nonzero patterns. It is interesting to note that Proposition 2.3 demonstrates that the conditions of Lemma 2.1 are sufficient for an order 3 irreducible pattern to be inertially arbitrary. On the other hand, these conditions are not sufficient for order 4 irreducible patterns; a careful look at the proof Theorem 2.5 reveals that there are 14 (non-equivalent) irreducible patterns of order 4 satisfying Lemma 2.1 which are not inertially arbitrary.

All of the nonzero patterns in Appendix 1 are irreducible and are shown to be spectrally arbitrary by Corpuz and McDonald [2] and hence are inertially arbitrary. Note that the matrices in Appendix 1 consist of the matrices from [2, Appendix A] followed by the matrices from [2, Appendix B], except for $\mathcal{B}$ which is equivalent to a superpattern of $\mathcal{N}_1^*$.

Theorem 2.5. *Let $\mathcal{A}$ be a 4×4 irreducible nonzero pattern. Then $\mathcal{A}$ is inertially arbitrary if and only if up to equivalence, $\mathcal{A}$ is a superpattern of $\mathcal{N}_1^*, \mathcal{N}_2^*$, $\mathcal{N}_3^*$, or one of the nonzero patterns in Appendix* 1. *Further, the patterns in the Appendices are minimal inertially arbitrary patterns.*

Proof. For the sake of brevity, we refer extensively to the arguments by Corpuz and McDonald [2]. For reference purposes, we put the notation of [2] in the footnotes.

Suppose $\mathcal{A}$ is an irreducible inertially arbitrary pattern. Note by Lemma 2.1 we know that $\mathcal{A}$ has two nonzero diagonal entries. In fact, by the requirements of Lemma 2.1 and irreducibility, it follows that $\mathcal{A}$ must have at least seven nonzero entries.

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We next claim that $\mathcal{A}$ must have at least eight nonzero entries. In [2, Lemma 3.4], Corpuz and McDonald observe that an order 4 spectrally arbitrary pattern needs at least eight nonzero entries. Their proof can be reworked to apply to inertially arbitrary patterns. In particular, using results similar to Lemma 2.1, Corpuz and McDonald identify only one candidate [1] with seven nonzero entries satisfying Lemma 2.1:

$$\mathcal{P}_1 = \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix}.$$

We note that Kim et. al. [9, Theorem 3] have shown that $\mathcal{P}_1$ is not inertially arbitrary. Therefore $\mathcal{A}$ must contain at least eight nonzero entries.

Case 1. Suppose $\mathcal{A}$ has exactly eight nonzero entries. We carefully follow the proof of Theorem 3.5 from Corpuz and McDonald [2]. By considering the characteristic polynomial for a matrix realization, Corpuz and McDonald [2] showed that for some of their cases, if the coefficient of x^3 is set to equal zero then the coefficient of x will be nonzero. These nonzero patterns cannot obtain the inertia $(0, 0, 4)$ and thus are not inertially arbitrary. There are also patterns which were identified to fail the conditions of Lemma 2.1. Below we consider the remaining patterns.

Consider the pattern [2]

$$\mathcal{P}_2 = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & * & 0 \end{bmatrix}.$$

We claim that this nonzero pattern cannot obtain the inertia $(1, 0, 3)$, since the characteristic polynomial

$$p(x) = x^4 - qx^3 + px^2 - pqx$$

with $p \geq 0, q > 0$ cannot be realized. Otherwise, if $A \in Q(\mathcal{P}_2)$ and $p_A(x) = p(x)$, then

$$\begin{aligned} a_{11} + a_{22} &= q \\ a_{11}a_{22} - a_{12}a_{21} - a_{34}a_{43} &= p \qquad \text{and} \\ (a_{11} + a_{22})a_{34}a_{43} &= -pq. \end{aligned}$$

Thus $a_{34}a_{43} = -p$. But then $\det(A) = -a_{41}a_{12}a_{23}a_{34}$ contradicting the fact that the constant term in $p_A(x)$ is zero. Thus $\mathcal{P}_2$ is not inertially arbitrary.

Later, Corpuz and McDonald consider the pattern [3]

$$\mathcal{P}_3 = \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix}.$$

[1] In [2], $\mathcal{A}1^*$ with a_{33} and a_{44} nonzero.
[2] In [2], $\mathcal{A}1^*$ with a_{11}, a_{22}, and a_{43} nonzero.
[3] In [2], $\mathcal{A}1^*$ with a_{11}, a_{33}, and a_{44} nonzero.

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They give an argument showing that if the coefficient of x^3 and x are both zero, then the coefficient of x^2 is nonzero, in particular negative. This is enough to show that the inertia $(0,0,4)$ cannot be realized and hence $\mathcal{P}_3$ is not inertially arbitrary.

Finally, consider the pattern [4]

$$\mathcal{P}_4 = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix}.$$

We claim $\mathcal{P}_4$ cannot realize the inertia $(0,0,4)$. Let $A \in Q(\mathcal{P}_4)$. By positive diagonal similarity, we may assume $a_{12} = a_{23} = a_{34} = 1$. Now suppose $i(A) = (0,0,4)$. Thus $p_A(x) = x^4 + (p+q)x^2 + pq$ for some $p, q \geq 0$. Since the coefficient of x^3 is zero, $a_{22} = -a_{11}$. Since the coefficient of x is zero,

$$a_{11} = \frac{a_{42} + a_{41}a_{24}}{a_{24}a_{42}}$$

and thus $\det(A) = \frac{a_{42}}{a_{24}} \neq 0$. Since the constant term in $p_A(x)$ is nonnegative, a_{42} and a_{24} have the same sign. Thus the coefficient of x^2 is $-((a_{11})^2 + a_{42}a_{24}) < 0$, but this contradicts the fact that $p + q \geq 0$. Therefore $\mathcal{P}_4$ is not inertially arbitrary.

The pattern [5]

$$\begin{bmatrix} * & * & 0 & 0, \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \end{bmatrix}$$

is equivalent to $\mathcal{N}_1^*$ and is inertially arbitrary as noted in Proposition 2.4.

This completes the case where $\mathcal{A}$ has exactly eight nonzeros: $\mathcal{A}$ is either equivalent to $\mathcal{N}_1^*$ or is spectrally arbitrary and appears in [2, Appendix A].

Case 2: Suppose $\mathcal{A}$ has nine nonzero entries. By Lemma 2.1, $\mathcal{A}$ is not a subpattern of a pattern in [2, Appendix D] since such subpatterns would have less than two nonzero transversals. The patterns $\mathcal{N}_3^*$ and $\mathcal{N}_2^*$ have been noted to be inertially arbitrary in Proposition 2.4; these appear as the second and fourth matrices in the first row of [2, Appendix C]. Also note that all the patterns in the third row, as well as the first and third patterns in the second row of [2, Appendix C] are not inertially arbitrary as they all have exactly one nonzero term in the determinant. It remains to show that the first and third patterns in row one and the second pattern in row two of [2, Appendix C] are not inertially arbitrary. But considering the characteristic polynomial for any matrix having one of these patterns, Corpuz and McDonald [2] showed that if the coefficient of x^3 is set to equal zero then the coefficient of x will be nonzero. Hence, these nonzero patterns cannot obtain the inertia $(0,0,4)$; thus they are not inertially arbitrary.

[4] In [2], $\mathcal{A}2^*$ with b_{11} and b_{22} nonzero.
[5] In [2], $\mathcal{A}2^*$ with b_{11} and b_{33} nonzero.

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Therefore, in this case, if $\mathcal{A}$ is not the pattern $\mathcal{N}_3^*$ or $\mathcal{N}_2^*$, then $\mathcal{A}$ appears in [2, Appendix B] or is a superpattern of a pattern in [2, Appendix A] and hence is spectrally arbitrary. (Recall that $\mathcal{B}$ is the first pattern in the fourth row of [2, Appendix B] and is equivalent to a proper superpattern of $\mathcal{N}_1^*$. Hence $\mathcal{B}$ is not a minimal inertially arbitrary pattern.)

Case 3: Suppose $\mathcal{A}$ has ten nonzero entries. Note that $\mathcal{A}$ does not appear in [2, Appendix D], because each pattern in [2, Appendix D] has at most one nonzero transversal and thus fails a condition of Lemma 2.1. Therefore $\mathcal{A}$ is spectrally arbitrary [2, Theorem 1.2 (iv)] and hence inertially arbitrary.

Case 4: Suppose $\mathcal{A}$ has at least eleven nonzero entries. In this case $\mathcal{A}$ is spectrally arbitrary [2, Theorem 1.2(v)] and hence inertially arbitrary.

Finally, we note that by the nature in which this proof discovers the inertially arbitrary patterns, starting with the ones with the fewest entries, (and the fact that no pattern in [2, Appendix B] is a superpattern of a pattern in [2, Appendix A]) the patterns in the appendices are all minimal inertially arbitrary patterns. □

In [1] it was demonstrated that any irreducible spectrally arbitrary sign pattern must have at least $2n-1$ nonzero entries and conjectured that no fewer than $2n$ nonzero entries are possible in any irreducible spectrally arbitrary pattern. In [2], this $2n$ conjecture was extended to include nonzero spectrally arbitrary patterns and demonstrated to be true for patterns of order at most four. In [5], the conjecture is confirmed for patterns up to order five. Reducible inertially arbitrary nonzero patterns were found in [9] which have less than $2n$ nonzero entries. As for irreducible patterns, Proposition 2.2 and Theorem 2.5 demonstrate that at least $2n$ nonzero entries are needed in an irreducible inertially arbitrary pattern for each order $n \leq 4$. We have yet to find any order $n > 4$ irreducible inertially arbitrary nonzero pattern with less than $2n$ nonzero entries.

In [2], the question was raised as to what is the maximum number of nonzero entries possible in an irreducible nonzero pattern of order n which is not spectrally arbitrary. The same question might be asked with respect to inertially arbitrary patterns. The example given in [2, Theorem 1.4] provides an example of a pattern with many nonzero entries which is not spectrally arbitrary; we note that the same example is not inertially arbitrary. In particular there is an irreducible pattern of order n with n^2-2n+2 nonzero entries which is not inertially arbitrary.

3. Some inertially arbitrary sign patterns that are not spectrally arbitrary. With Proposition 2.4 and Theorem 2.5, we have characterized the nonzero patterns of order 4 which are inertially but not spectrally arbitrary. ¿From these patterns, we now determine the signed patterns which are inertially arbitrary and have a nonzero pattern that is (inertially but) not spectrally arbitrary. It follows that these sign patterns (labelled $\mathcal{N}_1$, $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ below) are inertially but not spectrally arbitrary.

Each pattern in Appendix 1 has a signing which is spectrally arbitrary [2] and hence inertially arbitrary. We will see in Corollary 3.6 that, up to equivalence, $\mathcal{N}_3^*$ is the only order 4 inertially arbitrary pattern that has no signing which is inertially arbitrary. Some reducible nonzero patterns that are inertially arbitrary but have no

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signing which is inertially arbitrary may be found in [9].

Proposition 3.1. *Up to equivalence,*

$$\mathcal{N}_1 = \begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & + \\ - & - & 0 & 0 \\ 0 & 0 & - & - \end{bmatrix}$$

is the only inertially arbitrary signing of $\mathcal{N}_1^*$.

Proof. The sign pattern $\mathcal{N}_1$ was demonstrated to be inertially arbitrary in [4].

Let $\mathcal{A}$ be a signing of $\mathcal{N}_1^*$ which is inertially arbitrary. Since an inertially arbitrary sign pattern must contain both a positive and negative on the main diagonal, we may assume, up to equivalence that $\mathcal{A}_{11} = +$, $\mathcal{A}_{44} = -$. By signature similarity, we may also assume $\mathcal{A}_{12} = +$, $\mathcal{A}_{23} = +$ and $\mathcal{A}_{24} = +$. Further, since an inertially arbitrary pattern must have a negative 2-cycle [4, Lemma 5.1], $\mathcal{A}_{32} = -$. If both $\mathcal{A}_{43}$ and $\mathcal{A}_{31}$ are positive, then the determinant is signed. Thus the corresponding sign pattern is not inertially arbitrary. If both $\mathcal{A}_{43}$ and $\mathcal{A}_{31}$ are negative, then we get a sign pattern equivalent to $\mathcal{N}_1$.

Finally, assume the product $\mathcal{A}_{31}\mathcal{A}_{43}$ is negative. If $A \in Q(\mathcal{A})$, then by a positive diagonal similarity, A is equivalent to

$$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ u & -b & 0 & 0 \\ 0 & 0 & v & -c \end{bmatrix},$$

where $a, b, c > 0$, and $uv < 0$. The characteristic polynomial of A is

$$x^4 + (c-a)x^3 + (b-ac)x^2 + (-u+bc+vb-ab)x - abc - vab - uc - uv.$$

In order to obtain the inertia $(0,0,4)$ the coefficients of x^3 and x must be zero. In this case $c = a$, and hence $u = vb$. This contradicts the fact that uv is negative. Therefore the corresponding sign pattern is not inertially arbitrary. □

Let

$$\mathcal{N}_{2,1} = \begin{bmatrix} + & + & + & 0 \\ - & - & - & 0 \\ 0 & 0 & 0 & + \\ - & - & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{N}_{2,2} = \begin{bmatrix} + & + & + & 0 \\ - & - & - & 0 \\ 0 & 0 & 0 & - \\ - & - & 0 & 0 \end{bmatrix}.$$

We will determine that $\mathcal{N}_{2,1}$, and $\mathcal{N}_{2,2}$ are inertially arbitrary sign patterns. The next argument uses a technique from [10, Lemma 5].

Lemma 3.2. *For any* $r_1, r_2, r_4 \in \mathbb{R}$ *and* $r_3 \neq 0$, *there exists a matrix* $A \in Q(\mathcal{N}_{2,1})$ *and* $B \in Q(\mathcal{N}_{2,2})$ *such that*

$$p_A(x) = p_B(x) = x^4 + r_1x^3 + r_2x^2 + r_3x + r_4.$$

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Proof. For $c > 0$, since $A \in Q(\mathcal{A})$ if and only if $cA \in Q(\mathcal{A})$, and since

$$p_{cA}(x) = x^4 + cr_1x^3 + c^2r_2x^2 + c^3r_3x + c^4r_4,$$

it suffices to show that the result holds for (r_1, r_2, r_3, r_4) arbitrarily close to $(0, 0, 0, 0)$ with $r_3 \neq 0$. Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 1 & 0 \\ -1 & -a_{22} & -1 & 0 \\ 0 & 0 & 0 & v \\ -a_{41} & -a_{42} & 0 & 0 \end{bmatrix} \tag{3.1}$$

with $a_{ij} > 0$ and $v \neq 0$. Then A has characteristic polynomial

$$x^4 + (a_{22} - a_{11})x^3 + (a_{12} - a_{11}a_{22})x^2 + v(a_{41} - a_{42})x + v[a_{41}(a_{22} - a_{12}) + a_{42}(a_{11} - 1)].$$

We will analyze the sign patterns $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ simultaneously by fixing either $v = 1$ or $v = -1$ accordingly. Fix $r_3 \neq 0$. Then fix positive numbers a_{41} and a_{42} such that $v(a_{41} - a_{42}) = r_3$. We seek positive numbers a_{11}, a_{22}, a_{12} such that

$$\begin{aligned} a_{22} - a_{11} - r_1 &= 0 \\ a_{12} - a_{11}a_{22} - r_2 &= 0 \qquad \text{and} \\ v[a_{41}(a_{22} - a_{12}) + a_{42}(a_{11} - 1)] - r_4 &= 0. \end{aligned}$$

If $r_1 = r_2 = r_4 = 0$, then a solution to the above system of equations is $a_{11} = a_{22} = a_{12} = 1$. Let $f_1 = a_{22} - a_{11}$, $f_2 = a_{12} - a_{11}a_{22}$, and $f_3 = v[a_{41}(a_{22} - a_{12}) + a_{42}(a_{11} - 1)]$. Then, using the Implicit Function Theorem, it is sufficient to show that $\frac{\partial(f_1, f_2, f_3)}{\partial(a_{11}, a_{12}, a_{22})}$ is nonzero when $(a_{11}, a_{12}, a_{22}) = (1, 1, 1)$ in order to complete the proof.

$$\left.\frac{\partial(f_1, f_2, f_3)}{\partial(a_{11}, a_{12}, a_{22})}\right|_{(1,1,1)} = \det \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & -1 \\ va_{42} & -va_{41} & va_{41} \end{bmatrix} = v(a_{41} - a_{42}) = r_3 \neq 0.$$

Thus, for any r_1, r_2, r_4 and any $r_3 \neq 0$ sufficiently close to 0, there exist positive values $a_{11}, a_{22}, a_{12}, a_{41}, a_{42}$ such that $p_A(x) = x^4 + r_1x^3 + r_2x^2 + r_3x + r_4$ with $r_3 \neq 0$. ∎

The following result mimics a result of Kim et. al. [10, Theorem 1] in which a sufficient condition for a pattern to be inertially arbitrary is found with respect to the set of polynomials that a pattern realizes. We use the fact that if A is a matrix of order n with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the coefficient r_k of x^{n-k} in the characteristic polynomial of A is described by an *elementary symmetric function* (see for example [8, p.41]):

$$r_k = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

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Lemma 3.3. *For $n \geq 4$, let $\mathcal{A}$ be a sign pattern of order n that realizes each polynomial $p(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_{n-1}x + r_n$ with $r_{n-1} \neq 0$. If n is odd, then $\mathcal{A}$ is an inertially arbitrary sign pattern. If n is even, then $\mathcal{A}$ can realize all inertias except possibly $(0, 0, n)$.*

Proof. Suppose $n_1 + n_2 + n_3 = n$. Let A be a matrix having eigenvalues $1, -n, 0, i$ and $-i$ with algebraic multiplicities

$$n_1,\ n_2,\ \left\lceil \frac{n_3}{2} \right\rceil - \left\lfloor \frac{n_3}{2} \right\rfloor,\ \left\lfloor \frac{n_3}{2} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{n_3}{2} \right\rfloor$$

respectively. Then $i(A) = (n_1, n_2, n_3)$. Let $p_A(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_{n-1}x + r_n$. Noting that r_{n-1} can be described by an elementary symmetric function, $r_{n-1} = \pm(n)^{n_2-1}(n_2 - n \cdot n_1)$ if n_3 is even, and $r_{n-1} = \pm(n)^{n_2}$ if n_3 is odd. Thus, unless $n_1 = n_2 = 0$, $r_{n-1} \neq 0$ when n is even. Further $r_{n-1} \neq 0$ whenever n is odd.

Let $\mathcal{A}$ be an n-by-n sign pattern that realizes each polynomial $p(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_{n-1}x + r_n$ with $r_{n-1} \neq 0$. Then there is a matrix $A \in Q(\mathcal{A})$ which realizes each spectrum above giving $r_{n-1} \neq 0$. Hence, $\mathcal{A}$ is inertially arbitrary whenever n is odd. If n is even then $\mathcal{A}$ can attain all inertias except possibly $(0, 0, n)$. □

Proposition 3.4. *The sign patterns $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ are inertially arbitrary, and up to equivalence, are the only inertially arbitrary signings of $\mathcal{N}_2^*$.*

Proof. By Lemmas 3.2 and 3.3, $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ can realize all possible inertias except possibly $(0, 0, 4)$. If $A \in Q(\mathcal{N}_{2,1})$ or $A \in Q(\mathcal{N}_{2,2})$ and if each nonzero entry of A has magnitude 1, then A (see equation (3.1)) has characteristic polynomial x^4 and hence inertia (0,0,4). Therefore $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$ are inertially arbitrary.

Let $\mathcal{A}$ be a signing of $\mathcal{N}_2^*$ which is inertially arbitrary. Since an inertially arbitrary sign pattern must contain both a positive and negative on the main diagonal, we may assume, up to equivalence that $\mathcal{A}_{11} = +$, $\mathcal{A}_{22} = -$. By signature similarity, we may also assume $\mathcal{A}_{12} = +$, $\mathcal{A}_{13} = +$ and $\mathcal{A}_{41} = -$. Further, since an inertially arbitrary pattern must have a negative 2-cycle [4, Lemma 5.1], $\mathcal{A}_{21} = -$. If $\mathcal{A}_{42}$, $\mathcal{A}_{23}$ are both positive then the determinant is signed, hence the corresponding sign pattern is not inertially arbitrary. If $\mathcal{A}_{42}$, $\mathcal{A}_{23}$ are both negative, then this corresponds to the patterns $\mathcal{N}_{2,1}$ and $\mathcal{N}_{2,2}$, and thus are inertially arbitrary. The coefficient of x is signed if the product $\mathcal{A}_{42}\mathcal{A}_{23}$ is negative, and hence the inertia $(0, 0, 4)$ cannot be obtained in this case. □

We next demonstrate that there exists an irreducible nonzero pattern which is inertially arbitrary but has no signing that is inertially arbitrary.

Proposition 3.5. *There are no inertially arbitrary signings of $\mathcal{N}_3^*$.*

Proof. Let $\mathcal{A}$ be a signing of $\mathcal{N}_3^*$ which is inertially arbitrary. Since an inertially arbitrary sign pattern must contain both a positive and negative on the main diagonal, we may assume, up to equivalence that $\mathcal{A}_{11} = +$, $\mathcal{A}_{22} = -$. By signature similarity, we may also assume $\mathcal{A}_{12} = +$, $\mathcal{A}_{13} = +$ and $\mathcal{A}_{34} = +$. If $A \in Q(\mathcal{A})$, then by positive

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diagonal similarity, A is equivalent to

$$\begin{bmatrix} a & 1 & 1 & 0 \\ u & -b & v & 0 \\ w & 0 & 0 & 1 \\ y & 0 & 0 & 0 \end{bmatrix}$$

with $a, b > 0$ and u, v, w, y nonzero. A has characteristic polynomial

$$x^4 + (b-a)x^3 + (-u-w-ab)x^2 + (-wv-wb-y)x - y(b+v).$$

Now consider the inertias $(0,1,3)$ and $(1,0,3)$ (which require a real zero eigenvalue). In order to obtain these inertias we need $v = -b$ for the determinant to be zero. But then the coefficient of x is $-y$ which will either be positive or negative depending on the sign of y. Fixing the sign of y results in not being able to attain one of the two inertias $(0,1,3)$ or $(1,0,3)$. Therefore there is no signing of $\mathcal{N}_3^*$ which is an inertially arbitrary sign pattern. $\square$

Corollary 3.6. *The nonzero pattern $\mathcal{N}_3^*$ is the only inertially arbitrary nonzero pattern of order 4 which has no signing that is inertially arbitrary.*

Proof. If $\mathcal{A}$ is a reducible inertially arbitrary pattern of order 4 then by Proposition 2.3 $\mathcal{A}$ is a superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$. Since the sign pattern

$$\mathcal{T} = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$$

is spectrally arbitrary [6], an inertially arbitrary signing of $\mathcal{A}$ can be obtained by constructing a signed superpattern of $\mathcal{T} \oplus \mathcal{T}$.

We note that Corpuz and McDonald [2] showed that every superpattern of a minimal spectrally arbitrary nonzero pattern of order $n \leq 4$ has a signing which is spectrally arbitrary. Hence each such sign pattern is inertially arbitrary. In the proof of Proposition 2.4, we noted that any proper superpattern of $\mathcal{N}_1^*, \mathcal{N}_2^*$, or $\mathcal{N}_3^*$ is a superpattern of a minimal spectrally arbitrary pattern described in [2]. Hence each such pattern is inertially arbitrary. For irreducible patterns, the result then follows from Theorem 2.5, Proposition 3.1, Proposition 3.4, Proposition 3.5 and the fact that every superpattern of a pattern in Appendix 1 is spectrally arbitrary. $\square$

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Appendix 1. Irreducible Minimal Inertially Arbitrary Patterns That Are Spectrally Arbitrary.

(The first 13 patterns have eight nonzero entries, the next 17 have nine nonzeros each.)

$$\begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & \ast \\ 0 & 0 & \ast & \ast \\ \ast & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ \ast & \ast & 0 & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ \ast & 0 & \ast & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ \ast & 0 & 0 & \ast \\ \ast & 0 & 0 & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & 0 & \ast \\ \ast & \ast & 0 & \ast \end{bmatrix}$$

$$\begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & 0 & \ast \\ \ast & 0 & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & \ast \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ 0 & \ast & 0 & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & \ast \\ 0 & 0 & \ast & \ast \\ 0 & \ast & 0 & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ 0 & \ast & \ast & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ 0 & \ast & 0 & \ast \end{bmatrix}$$

$$\begin{bmatrix} 0 & \ast & 0 & \ast \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ 0 & \ast & 0 & \ast \end{bmatrix} \begin{bmatrix} \ast & 0 & 0 & \ast \\ \ast & 0 & \ast & \ast \\ 0 & 0 & \ast & \ast \\ 0 & \ast & 0 & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & \ast & 0 & \ast \\ 0 & 0 & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & \ast \\ \ast & \ast & \ast & 0 \\ 0 & 0 & \ast & \ast \\ \ast & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & \ast & \ast & 0 \\ 0 & 0 & \ast & \ast \\ \ast & 0 & 0 & \ast \end{bmatrix}$$

$$\begin{bmatrix} \ast & \ast & \ast & 0 \\ \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast \\ \ast & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ast & \ast & \ast & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ \ast & 0 & 0 & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ 0 & \ast & \ast & \ast \\ \ast & 0 & 0 & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & \ast \\ 0 & \ast & 0 & \ast \\ \ast & 0 & 0 & \ast \end{bmatrix} \begin{bmatrix} 0 & \ast & \ast & 0 \\ \ast & 0 & \ast & 0 \\ \ast & 0 & \ast & \ast \\ \ast & 0 & 0 & \ast \end{bmatrix}$$

$$\begin{bmatrix} 0 & \ast & \ast & 0 \\ \ast & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \\ \ast & \ast & 0 & \ast \end{bmatrix} \begin{bmatrix} 0 & \ast & 0 & 0 \\ \ast & 0 & \ast & \ast \\ \ast & 0 & \ast & \ast \\ \ast & 0 & 0 & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ 0 & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast \\ \ast & \ast & 0 & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ 0 & \ast & \ast & \ast \\ \ast & 0 & 0 & \ast \\ \ast & \ast & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \ast & 0 & 0 \\ 0 & \ast & \ast & \ast \\ \ast & 0 & 0 & \ast \\ \ast & \ast & 0 & \ast \end{bmatrix}$$

$$\begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast \\ 0 & \ast & \ast & 0 \end{bmatrix} \begin{bmatrix} 0 & \ast & 0 & 0 \\ \ast & \ast & \ast & 0 \\ 0 & 0 & \ast & \ast \\ 0 & \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} 0 & \ast & 0 & 0 \\ \ast & 0 & \ast & \ast \\ 0 & 0 & \ast & \ast \\ 0 & \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & 0 & 0 \\ \ast & \ast & \ast & 0 \\ 0 & \ast & \ast & \ast \\ 0 & 0 & \ast & 0 \end{bmatrix} \begin{bmatrix} 0 & \ast & \ast & \ast \\ \ast & \ast & 0 & 0 \\ \ast & 0 & \ast & 0 \\ \ast & 0 & 0 & \ast \end{bmatrix}$$

Appendix 2. Irreducible Minimal Inertially Arbitrary Patterns That Are Not Spectrally Arbitrary

$$\mathcal{N}_1^* = \begin{bmatrix} \ast & \ast & 0 & 0 \\ 0 & 0 & \ast & \ast \\ \ast & \ast & 0 & 0 \\ 0 & 0 & \ast & \ast \end{bmatrix} \; \mathcal{N}_2^* = \begin{bmatrix} \ast & \ast & \ast & 0 \\ \ast & \ast & \ast & 0 \\ 0 & 0 & 0 & \ast \\ \ast & \ast & 0 & 0 \end{bmatrix} \; \mathcal{N}_3^* = \begin{bmatrix} \ast & \ast & \ast & 0 \\ \ast & \ast & \ast & 0 \\ \ast & 0 & 0 & \ast \\ \ast & 0 & 0 & 0 \end{bmatrix}$$

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