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ON SPACES OF MATRICES WITH A BOUNDED NUMBER OF EIGENVALUES*

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Dedicated to Hans Schneider on the occasion of his seventieth birthday

Abstract. The following problem, originally proposed by Omladič and Šemrl [*Linear Algebra Appl.*, 249:29–46 (1996)], is considered. Let k and n be positive integers such that $k < n$. Let L be a subspace of $M_n(F)$, the space of $n \times n$ matrices over a field F, such that each $A \in L$ has at most k distinct eigenvalues (in the algebraic closure of F). Then, what is the maximal dimension of L. Omladič and Šemrl assumed that $F = \mathbb{C}$ and solved the problem for $k = 1$, $k = 2$ and n odd, and $k = n - 1$ (under a mild assumption). In this paper, their results for $k = 1$ and $k = n - 1$ are extended to any F such that $\mathrm{char}(F) = 0$, and a solution for $k = 2$ and any n, and for $k = 3$ is given.

Key words. Distinct eigenvalues; cornal ordering; irreducible polynomial.

AMS subject classifications. 15A42; 15A45.

1. Introduction. Let F be a field. Let $M_{m,n}(F)$ denote the space of all $m \times n$ matrices over F, and $S_n(F)$ denote the space of all $n \times n$ symmetric matrices over F. Let $M_n(F) = M_{n,n}(F)$. In recent years there have been many works considering spaces of matrices which satisfy certain properties. For example, given a positive integer k, what can be said about a subspace L of $M_{m,n}(F)$ (or $S_n(F)$) if every nonzero matrix in L has rank k. Or, what can be said if every matrix in L has rank bounded above by k. One can consider also spectral properties. Gerstenhaber [G] showed that every subspace L of $M_n(F)$ consisting of nilpotent matrices has dimension bounded by $\binom{n}{2}$ and if $|F| \geq n$ and $\dim L = \binom{n}{2}$, then L is conjugate to the algebra of strictly upper triangular matrices; see also [Se] and [BC].

In this paper we consider the following problem. Let k and n be positive integers such that $k < n$. Let L be a subspace of $M_n(F)$ such that every $A \in L$ has at most k distinct eigenvalues (in the algebraic closure of F). Then, what is the maximal dimension of L? This problem was proposed by Omladič and Šemrl [OS]. Atkinson [A] considered a subspace L of $M_n(F)$ with the property that every $A \in L$ has zero eigenvalue of (algebraic) multiplicity at least r, where $1 \leq r < n$; see Theorem 2.7. Clearly such L has the property that every $A \in L$ has at most $n - r + 1$ distinct eigenvalues. Omladič and Šemrl [OS] obtained the following results.

THEOREM 1.1. *(a) Let L be a subspace of $M_n(\mathbb{C})$ such that every $A \in L$ has only one eigenvalue. Then,* $\dim L \leq \binom{n}{2} + 1$.
(b) Let L be a subspace of $M_n(\mathbb{C})$ where n is odd and such that every $A \in L$ has at most 2 distinct eigenvalues. Then, $\dim L \leq \binom{n}{2} + 2$.
(c) Let L be a subspace of $M_n(\mathbb{C})$ such that every $A \in L$ has at most $n - 1$ distinct

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eigenvalues. Assume that there exists some $A \in L$ which has exactly $n-1$ distinct eigenvalues. Then,

$$\dim L \leq \binom{n}{2} + \binom{n-1}{2} + 1.$$

In each case Omladič and Šemrl also determined the structure of every L for which the maximum dimension is attained. Of course, (a) is closely related to the nilpotent case. It is our purpose to extend the results of Theorem 1.1 to any field such that $\mathrm{char}(F) = 0$, complete the case $k = 2$ for any n, and give a solution to the case $k = 3$. The results obtained seem to suggest the following conjecture.

CONJECTURE 1.2. *Let k and n be integers, $k < n$, and let F be a field such that $\mathrm{char}(F) = 0$. Let L be a subspace of $M_n(F)$ such that every $A \in L$ has at most k distinct eigenvalues. Then,*

$$\dim L \leq \binom{n}{2} + \binom{k}{2} + 1.$$

In Section 2 we bring some preliminary notations and results, while in Section 3 we give our main results of this paper.

We assume throughout (unless explicitly stated otherwise) that *F is a field such that* char $(F) = 0$ *and consider $\alpha = (\alpha_1, \ldots, \alpha_m)$ as an arbitrary point in F^m or as a vector of indeterminates*, according to our convenience.

2. Preliminary Notations and Results. Let $A \in M_n(F)$. We denote by $\sigma(A)$ the set of all eigenvalues of A in the algebraic closure of F. We denote by $\#\Big(\sigma(A)\Big)$ the number of distinct eigenvalues of A. A subspace V of $M_n(F)$ is said to be a *$\overline{k}$-spect subspace* provided that $k = \max\Big\{\#\Big(\sigma(A)\Big): A \in V\Big\}$. Let $\{A_1, A_2, \ldots, A_m\}$ be a set of m linearly independent matrices in $M_n(F)$ and let $p(t, \alpha) = \det\,(tI - \sum_{i=1}^{m} \alpha_i A_i)$. We consider $p(t, \alpha)$ as a polynomial in t with coefficients in $F[\alpha_1, \ldots, \alpha_m]$, that is, a polynomial in $F[\alpha_1, \ldots, \alpha_m][t]$ which is a unique factorization domain. Then $p(t, \alpha)$ can be split in $F[\alpha_1, \ldots, \alpha_m][t]$ as follows:

$$p(t, \alpha) = p_1^{k_1}(t, \alpha) \cdots p_\ell^{k_\ell}(t, \alpha),$$

where $p_j(t, \alpha)$ are monic distinct irreducible polynomials.

Denote $n_j = \deg(p_j)$. Then $p_j(t, \alpha)$ is of the form

$$p_j(t, \alpha) = t^{n_j} + q_{j1} t^{n_j - 1} + \cdots + q_{j, n_j - 1} t + q_{j, n_j},$$

where $q_{j,r} = q_{j,r}(\alpha_1, \ldots, \alpha_m)$ is a homogeneous polynomial in $\alpha_1, \ldots, \alpha_m$ of degree r $(r = 1, 2, \ldots, n_j)$.

Clearly $\sum_{j=1}^{\ell} k_j n_j = n$.

Denote $k = \sum_{j=1}^{\ell} n_j$. If $k < n$ and V is spanned by $\{A_1, \ldots, A_m\}$, then for every $A \in V$, $\#\Big(\sigma(A)\Big) \leq k$. In the following lemma we show that there exists $A \in V$ such that $\#\Big(\sigma(A)\Big) = k$, that is V is a $\overline{k}$-spect subspace of $M_n(F)$.

LEMMA 2.1. *Let F be a field with* $\mathrm{char}(F) = 0$. *Let V be a subspace of $M_n(F)$ and let $\{A_1, \ldots, A_m\}$ be a basis of V. Let*

$$p(t, \alpha) = \det\,(tI_n - \sum_{i=1}^{m} \alpha_i A_i)$$

be split into monic distinct irreducible polynomials in $F[\alpha_1, \ldots, \alpha_m][t]$

$$p(t, \alpha) = p_1^{k_1}(t, \alpha) \cdots p_\ell^{k_\ell}(t, \alpha).$$

Denote $n_j = \deg p_j$ $(j = 1, 2, \ldots, \ell)$ and $k = \sum_{j=1}^{\ell} n_j$. Then there exists a nonzero polynomial $\varphi(\alpha)$ in $F[\alpha_1, \ldots, \alpha_m]$ such that any $\hat{\alpha} \in F^m$ for which $\varphi(\hat{\alpha}) \neq 0$ satisfies $\#\left(\sigma\left(\sum_{i=1}^{m} \hat{\alpha}_i A_i\right)\right) = k.$

Proof. We can assume $k_j = 1$ $(j = 1, 2, \ldots, \ell)$. We shall proceed by induction with respect to ℓ. If $\ell = 1$, then $p(t, \alpha) = p_1(t, \alpha)$ is irreducible. Thus $p_1(t, \alpha)$ and its derivative $p_1'(t, \alpha)$ with respect to t are relatively prime. Hence, there exist polynomials $q_1(t, \alpha)$ and $q_2(t, \alpha)$ in $F[\alpha_1, \ldots, \alpha_m][t]$ and a nonzero polynomial $\varphi(\alpha) \in F[\alpha_1, \ldots, \alpha_m]$ such that

$$q_1(t, \alpha)p_1(t, \alpha) + q_2(t, \alpha)p_1'(t, \alpha) = \varphi(\alpha).$$

Now, for any $\hat{\alpha} \in F^m$ for which $\varphi(\hat{\alpha}) \neq 0$, $p_1(t, \hat{\alpha})$ and $p_1'(t, \hat{\alpha})$ are relatively prime as polynomials in $F[t]$. Therefore, $p_1(t, \hat{\alpha})$ has $k = n_1$ distinct roots, which implies $\#\left(\sigma\left(\sum_{i=1}^{m} \hat{\alpha}_i A_i\right)\right) = k.$

Assume $\ell > 1$. As we have seen, there exists a nonzero polynomial $\varphi_1(\alpha) \in F[\alpha_1, \ldots, \alpha_m]$ such that any $\hat{\alpha} \in F^m$ for which $\varphi_1(\hat{\alpha}) \neq 0$ implies that $p_1(t, \hat{\alpha})$ has n_1 distinct roots. By the induction hypothesis there exists a nonzero polynomial $\varphi_2(\alpha) \in F[\alpha_1, \ldots, \alpha_m]$ such that for any $\hat{\alpha} \in F^m$ for which $\varphi_2(\hat{\alpha}) \neq 0$ the polynomial $p_2(t, \alpha) \cdots p_\ell(t, \alpha)$ has $\sum_{j=2}^{\ell} n_j$ distinct roots.

Since $p_1(t, \alpha)$ and $p_2(t, \alpha) \cdots p_\ell(t, \alpha)$ are relatively prime, there exist $\hat{q}_1(t, \alpha)$ and $\hat{q}_2(t, \alpha)$ in $F[\alpha_1, \ldots, \alpha_m][t]$ such that

$$\hat{q}_1(t, \alpha)p_1(t, \alpha) + \hat{q}_2(t, \alpha)p_2(t, \alpha) \cdots p_\ell(t, \alpha) = \varphi_3(\alpha),$$

where $\varphi_3(\alpha)$ is a nonzero polynomial in $F[\alpha_1, \ldots, \alpha_m]$.

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For any $\hat{\alpha} \in F^m$ for which $\varphi_3(\hat{\alpha}) \neq 0$, $p_1(t, \hat{\alpha})$ and $p_2(t, \hat{\alpha}) \cdots p_\ell(t, \hat{\alpha})$ are relatively prime as polynomials in $F[t]$. Hence, they have no common root.
Define $\varphi(\alpha) = \varphi_1(\alpha)\varphi_2(\alpha)\varphi_3(\alpha)$. For any $\hat{\alpha} \in F^m$ for which $\varphi(\hat{\alpha}) \neq 0$, the polynomial $p(t, \hat{\alpha})$ has k distinct roots. Thus, $A = \sum_{i=1}^{m} \hat{\alpha}_i A_i$ satisfies $\#\Big(\sigma(A)\Big) = k$. ☐

The next lemma allows us to obtain, under some condition, an $(m-1)$-dimensional $\overline{\ell}$-spect subspace from an m-dimensional $\overline{k}$-spect subspace where $\ell \leq k-1$.

Lemma 2.2. *Let F be a field with $\operatorname{char}(F) = 0$. Let V be a $\overline{k}$-spect subspace of $M_n(F)$ and let $\{A_1, \ldots, A_m\}$ be a basis of V. Suppose that in the splitting of*

$$p(t, \alpha) = \det\,(tI_n - \sum_{i=1}^{m} \alpha_i A_i)$$

into monic irreducible polynomials, there occur two distinct linear polynomials. Then, V contains an $\overline{\ell}$-spect subspace of dimension $m-1$, where $\ell \leq k-1$.

Proof. Let

$$p(t, \alpha) = p_1^{k_1}(t, \alpha) p_2^{k_2}(t, \alpha) \cdots p_r^{k_r}(t, \alpha),$$

where $p_j(t, \alpha)$ are distinct monic irreducible polynomials in $F[\alpha_1, \ldots, \alpha_m][t]$. Suppose p_1 and p_2 are linear. We can write

$$p_1(t, \alpha) = t - \mu_1(\alpha) \quad \text{and} \quad p_2(t, \alpha) = t - \mu_2(\alpha),$$

where

$$\mu_1(\alpha) = \sum_{i=1}^{m} a_i \alpha_i \quad \text{and} \quad \mu_2(\alpha) = \sum_{i=1}^{m} b_i \alpha_i.$$

Clearly a_i and b_i are eigenvalues of A_i $(i = 1, \ldots, m)$.

Since $p_1(t, \alpha)$ and $p_2(t, \alpha)$ are relatively prime, there exists i_0, $1 \leq i_0 \leq m$, such that $a_{i_0} \neq b_{i_0}$. For any $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0+1}, \ldots, \alpha_m) \in F^{m-1}$ there exists a unique $\hat{\alpha}_{i_0} \in F$ which satisfies

$$\sum_{\substack{i=1 \\ i \neq i_0}}^{m} \alpha_i (a_i - b_i) + \hat{\alpha}_{i_0}(a_{i_0} - b_{i_0}) = 0.$$

Denote $B_i = A_i - \frac{a_i - b_i}{a_{i_0} - b_{i_0}} A_{i_0}$, $i = 1, \ldots, m$, $i \neq i_0$ and $\hat{V} = \operatorname{span}\,\{B_i\}_{\substack{i=1 \\ i \neq i_0}}^{m}$. Clearly, $\hat{V}$ is a subspace of V of dimension $m-1$. For $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0+1}, \ldots, \alpha_m)$ in F^{m-1} we have

$$\sum_{\substack{i=1 \\ i \neq i_0}}^{m} \alpha_i B_i = \sum_{\substack{i=1 \\ i \neq i_0}}^{m} \alpha_i A_i - \frac{1}{a_{i_0} - b_{i_0}} \sum_{\substack{i=1 \\ i \neq i_0}}^{m} \alpha_i (a_i - b_i) A_{i_0} = \sum_{\substack{i=1 \\ i \neq i_0}}^{m} \alpha_i A_i + \hat{\alpha}_{i_0} A_{i_0}.$$

Hence, $\mu_1(\alpha) = \mu_2(\alpha)$ for all $\alpha = (\alpha_1, \ldots, \alpha_{i_0-1}, \hat{\alpha}_{i_0}, \alpha_{i_0+1}, \ldots, \alpha_m) \in F^m$.

It follows from Lemma 2.1 that $\hat{V}$ is an $\overline{\ell}$-spect subspace, where $\ell \leq k-1$. □

We introduce now a linear ordering on the elements of $[n] \times [n]$, where $[n] = \{1, 2, \ldots, n\}$.

Definition 2.3. A linear ordering , on $[n] \times [n]$ is said to be a *cornal ordering* if it satisfies the following three conditions.
(I) $(i_1, 1) \underset{\Gamma}{<} (i_2, 1)$ iff $i_1 > i_2$.
(II) $(1, j_1) \underset{\Gamma}{<} (1, j_2)$ iff $j_1 < j_2$.
(III) $(p, 1) \underset{\Gamma}{<} (i, j) \underset{\Gamma}{<} (1, q)$ for all $i, j, p > 1$ and $q \geq 1$.

Definition 2.4. Let , be a linear ordering on $[n] \times [n]$ and let $A = (a_{ij})$ be a nonzero matrix in $M_n(F)$. We denote $d_\Gamma(A) = (p, q)$, where $(p, q) = \min\{(i, j),\ a_{ij} \neq 0\}$ and the minimum is taken with respect to , .

Let V be a subspace of $M_n(F)$ and let $\{A_1, \ldots, A_m\}$ be a basis of V. Let , be a linear ordering on $[n] \times [n]$. We may think of a matrix in $M_n(F)$ as an n^2-tuple taken in the order specified by , . Performing Gaussian elimination on $\{A_1, \ldots, A_m\}$ with respect to , , we get a basis $\{B_1, \ldots, B_m\}$ of V which satisfies $d_\Gamma(B_i) \underset{\Gamma}{<} d_\Gamma(B_j)$ for all $1 \leq i < j \leq m$ and each matrix B_i has an entry equal to 1 in the position $d_\Gamma(B_i)$. We call that 1 the *leading* 1 of B_i with respect to , . We may assume that for all $i, j = 1, \ldots, m$, $i \neq j$, B_j has a zero entry in the position of the leading 1 of B_i.

Definition 2.5. Let V be a subspace of $M_n(F)$ and let , be a linear ordering on $[n] \times [n]$. We say that the basis $\{B_1, \ldots, B_m\}$ of V is a , -ordered basis if it is obtained from some basis of V by Gaussian elimination with respect to the order , .

We use the technique of the leading one's described above to obtain the following useful lemma.

Lemma 2.6. *Let , be a cornal ordering on $[n] \times [n]$ and let A_r and A_s be matrices in $M_n(F)$ such that $d_\Gamma(A_r) = (\ell, 1)$ and $d_\Gamma(A_s) = (1, \ell)$ for some $2 \leq \ell \leq n$. Then the coefficient of t^{n-2} in*

$$p(t, \alpha_r, \alpha_s) = \det\left[tI_n - (\alpha_r A_r + \alpha_s A_s)\right]$$

must depend on α_s.

Proof. The coefficient of t^{n-2} in $p(t, \alpha_r, \alpha_s)$ equals $\sigma_2(\alpha_r A_r + \alpha_s A_s)$, where $\sigma_2(\alpha_r A_r + \alpha_s A_s)$ is the sum of the principal minors of order 2 of the matrix $\alpha_r A_r + \alpha_s A_s$. It follows from the choice of , that the term $-\alpha_r \cdot \alpha_s$ must appear in $\sigma_2(\alpha_r A_r + \alpha_s A_s)$. □

Finally, we quote a theorem due to Atkinson [A], and establish an immediate corollary.

Theorem 2.7. *Let F be a field, $|F| \geq n$, let r be an integer, $r < n$, and let V be a subspace of $M_n(F)$ with the property that every $A \in V$ has at least r zero eigenvalues. Then $\dim V \leq \frac{1}{2}r(r-1) + n(n-r)$.*

Remark 2.8. We notice that if $A \in M_n(F)$ has at least r zero eigenvalues, then $\#\Big(\sigma(A)\Big) \leq n - r + 1$. From [A] we can deduce that if equality holds in Theorem 2.7, then there exists $A \in V$ such that $\#\Big(\sigma(A)\Big) = n - r + 1$. Therefore V is $\overline{k}$-spect, where $k = n - r + 1$. For $k = n - r + 1$, straightforward computation shows that

$$\frac{1}{2}r(r-1) + n(n-r) = \binom{n}{2} + \binom{k}{2}.$$

Thus, if we adjoin the identity matrix to a subspace of the maximum dimension having that property, we get a $\overline{k}$-spect subspace of the maximum dimension in our conjecture.

COROLLARY 2.9. *Let V be a subspace of $M_n(F)$, $\operatorname{char}(F) = 0$, that includes the identity matrix. Let $\{A_1, \ldots, A_m\}$ be a basis of V in which $A_m = I_n$. Denote*

$$p(t, \alpha) = \det\,(tI_n - \sum_{i=1}^{m} \alpha_i A_i).$$

Assume $p(t, \alpha)$ splits in $F[\alpha_1, \ldots, \alpha_m][t]$ in the form $p(t, \alpha) = Q(t, \alpha) \cdot q^r(t, \alpha)$, where $q(t, \alpha)$ is monic and linear. Then, $\dim V \leq \frac{1}{2}r(r-1) + n(n-r) + 1$.

Proof. Write $q(t, \alpha) = t - \mu(\alpha)$, where

$$\mu(\alpha) = \sum_{i=1}^{m} a_i \alpha_i.$$

Since a_i is an eigenvalue of A_i, we have $a_m = 1$.

Define $B_i = A_i - a_i I_n$, $i = 1, 2, \ldots, m$ and $\hat{V} = \operatorname{span}\,\{B_i\}_{i=1}^{m}$. Clearly $\dim \hat{V} = m - 1$. For every $\alpha_i \in F$, $i = 1, 2, \ldots, m-1$,

$$\sum_{i=1}^{m-1} \alpha_i B_i = \sum_{i=1}^{m-1} \alpha_i (A_i - a_i I_n) = \sum_{i=1}^{m-1} \alpha_i A_i - \sum_{i=1}^{m-1} \alpha_i a_i I_n.$$

Now, denote $\bar{p}(t, \bar{\alpha}) = \det\,(tI_n - \sum_{i=1}^{m-1} \alpha_i B_i)$, where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{m-1}) \in F^{m-1}$. Then $\bar{p}(t, \bar{\alpha})$ splits in the form

$$\bar{p}(t, \bar{\alpha}) = \bar{Q}(t, \bar{\alpha}) \cdot t^r.$$

Hence, every $A \in \hat{V}$ has at least r zero eigenvalues. By Theorem 2.7

$$\dim \hat{V} \leq \frac{1}{2}r(r-1) + n(n-r);$$

thus, $\dim V \leq \frac{1}{2}r(r-1) + n(n-r) + 1$. ∎

3. Main Results. The upper bound of $\overline{1}$-spect subspaces is achieved by the following result.

THEOREM 3.1. *Let F be a field with $\operatorname{char}(F) = 0$. Let V be a $\overline{1}$-spect subspace of $M_n(F)$. Then $\dim V \leq \frac{n(n-1)}{2} + 1$ and if equality holds then V is conjugate to the space of all upper triangular matrices having equal diagonal entries.*

Proof. We may assume that $I_n \in V$. Let $\{A_1, \ldots, A_m\}$ be a basis of V. Denote

$$p(t,\alpha) = \det\,(tI_n - \sum_{i=1}^{m} \alpha_i A_i).$$

By Lemma 2.1, $p(t,\alpha)$ splits in $F[\alpha_1, \ldots, \alpha_m][t]$ in the following form:

$$p(t,\alpha) = (t - \sum_{i=1}^{m} a_i \alpha_i)^n.$$

For $i = 1, \ldots, m$ define

$$B_i = A_i - a_i I_n.$$

Let $\hat{V} = \text{span}\,\{B_i\}_{i=1}^{m}$. Clearly $\hat{V}$ is a space of nilpotent matrices of dimension $m-1$. The assertion follows from Gerstenhaber's result. □

For $\overline{2}$-spect subspaces we have the following theorem.

THEOREM 3.2. *Let F be a field with* $\text{char}(F) = 0$, *and $n \geq 3$. Let V be a $\overline{2}$-spect subspace of $M_n(F)$. Then*

$$\dim V \leq \frac{n(n-1)}{2} + 2.$$

Proof. We may assume $I_n \in V$. Let , be a cornal ordering on $[n] \times [n]$ and let $\{A_1, \ldots, A_m\}$ be a , -ordered basis of V. Denote

$$p(t,\alpha) = \det\,(tI_n - \sum_{i=1}^{m} \alpha_i A_i).$$

By Lemma 2.1, $p(t,\alpha)$ splits in $F[\alpha_1, \ldots, \alpha_m][t]$ in one of the following two forms.

Case 1: $p(t,\alpha) = p_1^{k_1}(t,\alpha) \cdot p_2^{k_2}(t,\alpha)$, where $p_1(t,\alpha)$ and $p_2(t,\alpha)$ are linear and distinct. Hence, by Lemma 2.2, V contains a $\overline{1}$-spect subspace of dimension $m-1$. Now, the assertion follows from Theorem 3.1.

Case 2: $p(t,\alpha) = q^{n/2}(t,\alpha)$ (in this case n must be even), where $q(t,\alpha)$ is quadratic and irreducible of the form

$$q(t,\alpha) = t^2 + q_1(\alpha)t + q_2(\alpha),$$

where $q_1(\alpha)$ and $q_2(\alpha)$ are homogeneous in $\alpha_1, \ldots, \alpha_m$.

CLAIM: There do not exist $1 \leq r < s \leq m$ and $2 \leq \ell \leq n$ such that A_r and A_s have leading one's with respect to , , in the positions $(\ell, 1)$ and $(1, \ell)$ respectively.

Proof of claim: Suppose $d_\Gamma(A_r) = (\ell, 1)$ and $d_\Gamma(A_s) = (1, \ell)$ for some $2 \leq \ell \leq n$ and $1 \leq r < s \leq m$. Denote $\hat{\alpha} = \alpha_r e_r + \alpha_s e_s \in F^m$. Thus

$$q(t,\hat{\alpha}) = t^2 + q_1(\alpha_r, \alpha_s)t + q_2(\alpha_r, \alpha_s),$$

where $q_1(\alpha_r, \alpha_s)$ and $q_2(\alpha_r, \alpha_s)$ are of the form

$$\begin{aligned} q_1(\alpha_r, \alpha_s) &= a_1\alpha_r + b_1\alpha_s, \\ q_2(\alpha_r, \alpha_s) &= a_{11}\alpha_r^2 + a_{12}\alpha_r\alpha_s + a_{22}\alpha_s^2, \\ a_1, b_1, a_{11}, & a_{12}, a_{22} \in F. \end{aligned}$$

Since $q_1(0, \alpha_s) = q_2(0, \alpha_s) = 0$ for all α_s, the coefficients b_1 and a_{22} must vanish.

We have $\det(\alpha_r A_r + \alpha_s A_s) = [q_2(\alpha_r, \alpha_s)]^{n/2} = (a_{11}\alpha_r^2 + a_{12}\alpha_r\alpha_s)^{n/2}$, but on the other hand $\det(\alpha_r A_r + \alpha_s A_s)$ is either linear or independent of α_s. Hence, a_{12} must vanish also. It follows that $q_1(\alpha_r, \alpha_s)$ and $q_2(\alpha_r, \alpha_s)$ are independent of α_s which contradicts Lemma 2.6.

Now, denote

$$S = \{A_i\colon d_\Gamma(A_i) = (1, q) \text{ or } d_\Gamma(A_i) = (p, 1) \text{ for some } 2 \le p, q \le n\}.$$

By our claim $|S| \le n - 1$.

Define $V_1 = \text{span}\,(\{A_1, \ldots, A_m\} \setminus S)$. We have $\dim V_1 \ge \dim V - (n-1)$. Let $\hat{V}$ denote the subspace of $M_{n-1}(F)$ obtained from V_1 by deleting the first row and column of every matrix in V_1.

i) If there exists $1 \le j \le m$ such that $d_\Gamma(A_j) = (1, 1)$, then $\dim \hat{V} = \dim V_1 - 1$ and $\hat{V}$ is $\overline{1}$-spect subspace of $M_{n-1}(F)$. By Theorem 3.1

$$\dim \hat{V} \le \frac{(n-1)(n-2)}{2} + 1.$$

Hence, $\dim V \le \frac{(n-1)(n-2)}{2} + 1 + (n-1) + 1 = \frac{n(n-1)}{2} + 2$.

ii) If there is no $1 \le j \le m$ such that $d_\Gamma(A_j) = (1, 1)$, then $\dim \hat{V} = \dim V_1$. Since $\hat{V}$ is either $\overline{2}$-spect or $\overline{1}$-spect subspace of $M_{n-1}(F)$ and $n-1$ is odd, it follows from the proof of Case 1 and Theorem 3.1 that

$$\dim \hat{V} \le \frac{(n-1)(n-2)}{2} + 2,$$

which yields the assertion of the theorem. ☐

For the case $k = 3$ we have the following result.

THEOREM 3.3. *Let F be a field with* $\text{char}(F) = 0$ *and $n \ge 4$. Let V be a $\overline{3}$-spect subspace of $M_n(F)$. Then* $\dim V \le \frac{n(n-1)}{2} + 4$.

Proof. As in our proof of Theorem 3.2, we may assume that $I_n \in V$ and $\{A_1, \ldots, A_m\}$ is a , -ordered basis of V, where , is a cornal ordering. We proceed by induction with respect to n. Denote $p(t, \alpha) = \det\,(tI_n - \sum_{i=1}^{m} \alpha_i A_i)$. By Lemma 2.1 $p(t, \alpha)$ splits in $F[\alpha_1, \ldots, \alpha_m][t]$ into a product of powers of distinct monic irreducible polynomials and one of the following 3 possibilities occurs.

Case 1: $p(t, \alpha) = p_1^{k_1}(t, \alpha) \cdot p_2^{k_2}(t, \alpha) \cdot p_3^{k_3}(t, \alpha)$, where $p_i(t, \alpha)$ $(i = 1, 2, 3)$ are linear. By Lemma 2.2, V includes an $\overline{\ell}$-spect subspace $\hat{V}$ of dimension $m - 1$, where $\ell = 1$ or 2. By theorems 3.1 and 3.2, $\dim \hat{V} \le \frac{n(n-1)}{2} + 2$ and the assertion follows.

Case 2: $p(t,\alpha) = p_1^{k_1}(t,\alpha)\cdot p_2^{k_2}(t,\alpha)$, where $p_1(t,\alpha)$ is quadratic and $p_2(t,\alpha)$ is linear.

If $k_1 = 1$, then $k_2 = n-2$, and by Corollary 2.9 (taking $r = n-2$) we have

$$\dim V \leq \frac{1}{2}(n-2)(n-3) + n\cdot 2 + 1 = \frac{n(n-1)}{2} + 4.$$

Now the theorem follows for $n = 4$. Suppose $n > 4$ and $k_1 > 1$. We have a similar claim as in the proof of Theorem 3.2.

Claim: There do not exist $1 \leq r < s \leq m$ and $2 \leq \ell \leq n$ such that $d_\Gamma(A_r) = (\ell, 1)$ and $d_\Gamma(A_s) = (1,\ell)$.

Proof of claim: Denote $\hat{\alpha} = \alpha_r e_r + \alpha_s e_s \in F^m$.

We have $p_1(t,\hat{\alpha}) = t^2 + q_1(\alpha_r,\alpha_s)t + q_2(\alpha_r,\alpha_s)$, $p_2(t,\hat{\alpha}) = t + \mu(\alpha_r,\alpha_s)$, where $q_1(\alpha_r,\alpha_s)$, $q_2(\alpha_r,\alpha_s)$ and $\mu(\alpha_r,\alpha_s)$ are of the form

$$\begin{aligned} q_1(\alpha_r,\alpha_s) &= a_1\alpha_r + a_2\alpha_s,\\ q_2(\alpha_r,\alpha_s) &= a_{11}\alpha_r^2 + a_{12}\alpha_r\alpha_s + a_{22}\alpha_s^2,\\ \mu(\alpha_r,\alpha_s) &= b_1\alpha_r + b_2\alpha_s.\end{aligned}$$

Clearly a_2, a_{22} and b_2 must vanish.

We have $\det(\alpha_r A_r + \alpha_s A_s) = \pm(a_{11}\alpha_r^2 + a_{12}\alpha_r\alpha_s)^{k_1}(b_1\alpha_r)^{k_2}$. Suppose $b_1 \neq 0$. Since $\det(\alpha_r A_r + \alpha_s A_s)$ is either linear or independent of α_s, a_{12} must vanish. Thus $q_1(\alpha_r,\alpha_s)$, $q_2(\alpha_r,\alpha_s)$ and $\mu(\alpha_r,\alpha_s)$ are independent of α_s which contradicts Lemma 2.6.

Suppose $b_1 = 0$. Then $q_1(\alpha_r,\alpha_s) = a_1\alpha_r$, $q_2(\alpha_r,\alpha_s) = a_{11}\alpha_r^2 + a_{12}\alpha_r\alpha_s$ and $\mu(\alpha_r,\alpha_s) = 0$. Thus $p(t,\hat{\alpha}) = (t^2 + a_1\alpha_r t + a_{11}\alpha_r^2 + a_{12}\alpha_r\alpha_s)^{k_1}\cdot t^{k_2}$. The coefficient of t^{n-4} in $p(t,\hat{\alpha})$ equals $\sigma_4(\alpha_r A_r + \alpha_s A_s)$, where $\sigma_4(\alpha_r A_r + \alpha_s A_s)$ is the sum of the principal minors of order 4 of the matrix $\alpha_r A_r + \alpha_s A_s$, which must be either linear or independent of α_s. Thus a_{12} must vanish. Again we contradict Lemma 2.6.

As in the proof of Theorem 3.2, we define $V_1 = \text{span}\,(\{A_1,\ldots,A_m\}\setminus S)$, where

$$S = \{A_i : d_\Gamma(A_i) = (1,q) \text{ or } d_\Gamma(A_i) = (p,1) \text{ for some } 2 \leq p,q \leq n\}$$

and $\hat{V}$ is the subspace of $M_{n-1}(F)$ obtained from V_1 by deleting the first row and column of every matrix of V_1.

i) If there exists $1 \leq j \leq m$ such that $d_\Gamma(A_j) = (1,1)$, then there exists an $\overline{\ell}$-spect subspace $\hat{V}$ of $M_{n-1}(F)$, where $\ell = 1$ or 2 such that $\dim\hat{V} \geq \dim V - n$. By theorems 3.1 and 3.2 $\dim\hat{V} \leq \frac{(n-1)(n-2)}{2} + 2$, hence $\dim V \leq \frac{n(n-1)}{2} + 3$.

ii) If there is no $1 \leq j \leq m$ such that $d_\Gamma(A_j) = (1,1)$, then there exists an $\overline{\ell}$-spect subspace $\hat{V}$ of $M_{n-1}(F)$, where $\ell = 1, 2$ or 3 such that $\dim\hat{V} \geq \dim V - (n-1)$. If $\ell = 3$ then by our induction hypothesis $\dim\hat{V} \leq \frac{(n-1)(n-2)}{2} + 4$, hence $\dim V \leq \frac{n(n-1)}{2} + 4$. If $\ell = 1, 2$ then the conclusion follows using theorems 3.1 and 3.2.

Case 3: $p(t,\alpha) = q^k(t,\alpha)$, where $q(t,\alpha)$ is cubic. Here $n \equiv 0 \pmod 3$ (so $n \geq 6$). We have the same claim as in the previous case. In this case,

$$q(t,\hat{\alpha}) = t^3 + q_1(\alpha_r,\alpha_s)t^2 + q_2(\alpha_r,\alpha_s)t + q_3(\alpha_r,\alpha_s),$$

where

$$\begin{aligned} q_1(\alpha_r,\alpha_s) &= a_1\alpha_r + a_2\alpha_s,\\ q_2(\alpha_r,\alpha_s) &= a_{11}\alpha_r^2 + a_{12}\alpha_r\alpha_s + a_{22}\alpha_s^2,\\ q_3(\alpha_r,\alpha_s) &= b_1\alpha_r^3 + b_2\alpha_r^2\alpha_s + b_3\alpha_r\alpha_s^2 + b_4\alpha_s^3. \end{aligned}$$

By similar reasoning explained before we conclude that a_2, a_{22}, b_2, b_3, b_4 must vanish.

Using $\sigma_4(\alpha_r A_r + \alpha_s A_s)$ we imply that a_{12} vanishes, which yields a contradiction to Lemma 2.6. As in the previous case the result follows immediately if there is $1 \leq j \leq m$ such that $d_\Gamma(A_j) = (1,1)$. Suppose that there is no such j; then there exists a $\overline{\ell}$-spect subspace V of $M_{n-1}(F)$, where $\ell = 1, 2$ or 3 such that $\dim \hat{V} \geq \dim V - (n-1)$. Suppose $\ell = 3$. Since $(n-1) \not\equiv 0 \pmod 3$, then V belongs to either case 1 or case 2. Thus $\dim \hat{V} \leq \frac{(n-1)(n-2)}{2} + 4$ and the conclusion follows. If $\ell = 1$ or 2 the conclusion follows using theorems 3.1 and 3.2. □

Finally, we give the following simple proof for the case $k = n-1$.

Theorem 3.4. *Let F be a field with* $\operatorname{char}(F) = 0$ *and $n \geq 5$. Let V be a $\overline{(n-1)}$-spect subspace of $M_n(F)$. Then* $\dim V \leq \binom{n}{2} + \binom{n-1}{2} + 1$.

Proof. We can assume $I_n \in V$. Let $\{A_1, \ldots, A_m\}$ be some basis of V in which $A_m = I_n$. Denote $p(t,\alpha) = \det\,(tI_n - \sum_{i=1}^{m} \alpha_i A_i)$.

By Lemma 2.1, $p(t,\alpha)$ splits into the following product

$$p(t,\alpha) = q_1^2(t,\alpha)\cdot q_2(t,\alpha)\cdots q_r(t,\alpha),$$

where $q_1, \ldots, q_r$ are distinct irreducible polynomials in $F[\alpha_1,\ldots,\alpha_m][t]$ and q_1 is monic and linear. Now the assertion follows from Corollary 2.9 taking $r = 2$. □

Remark 3.5. Let V be a subspace of $M_n(F)$ consisting of all matrices of the form

$$\left[\begin{array}{ccc|c} a & & * & \\ & \ddots & & * \\ O & & a & \\ \hline & O & & * \end{array}\right],$$

where a and stars are arbitrary elements of F, and the block in the lower right corner has order $k-1$. Clearly V is a $\overline{k}$-spect subspace of dimension $\binom{n}{2} + \binom{k}{2} + 1$.

This shows that the upper bounds given in theorems 3.2, 3.3 and 3.4 are sharp (for the appropriate k).

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