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LINEAR COMBINATIONS OF GRAPH EIGENVALUES*

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Abstract. Let $\mu_1(G) \geq \ldots \geq \mu_n(G)$ be the eigenvalues of the adjacency matrix of a graph G of order n, and $\overline{G}$ be the complement of G. Suppose $F(G)$ is a fixed linear combination of $\mu_i(G)$, $\mu_{n-i+1}(G)$, $\mu_i(\overline{G})$, and $\mu_{n-i+1}(\overline{G})$, $1 \leq i \leq k$. It is shown that the limit

$$\lim_{n\to\infty} \frac{1}{n} \max\{F(G) : v(G) = n\}$$

always exists. Moreover, the statement remains true if the maximum is taken over some restricted families like "K_r-free" or "r-partite" graphs. It is also shown that

$$\frac{29+\sqrt{329}}{42}n - 25 \leq \max_{v(G)=n} \mu_1(G) + \mu_2(G) \leq \frac{2}{\sqrt{3}}n.$$

This inequality answers in the negative a question of Gernert.

Key words. Extremal graph eigenvalues, Linear combination of eigenvalues, Multiplicative property.

AMS subject classifications. 15A42, 05C50.

1. Introduction. Our notation is standard (e.g., see [1], [3], and [8]); in particular, all graphs are defined on the vertex set $[n] = \{1, \ldots, n\}$ and $\overline{G}$ stands for the complement of G. We order the eigenvalues of the adjacency matrix of a graph G of order n as $\mu_1(G) \geq \ldots \geq \mu_n(G)$.

Suppose $k > 0$ is a fixed integer and $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_k$ are fixed reals. For any graph G of order at least k, let

$$F(G) = \sum_{i=1}^{k} \alpha_i\mu_i(G) + \beta_i\mu_{n-i+1}(G) + \gamma_i\mu_i(\overline{G}) + \delta_i\mu_{n-i+1}(\overline{G}).$$

For a given graph property $\mathcal{F}$, i.e., a family of graphs closed under isomorphism, it is natural to look for $\max\{F(G) : G \in \mathcal{F},\ v(G) = n\}$. Questions of this type have been studied; here is a partial list:

$\max\{\mu_1(G) + \mu_n(G) :\ G \text{ is } K_r\text{-free},\ v(G) = n\}$	Brandt [2];
$\max\{\mu_1(G) - \mu_n(G) : v(G) = n\}$	Gregory *et al.* [7];
$\max\{\mu_1(G) + \mu_2(G) : v(G) = n\}$	Gernert [5];
$\max\{\mu_1(G) + \mu_1(\overline{G}) : v(G) = n\}$	Nosal [11], Nikiforov [9];
$\max\{\mu_i(G) + \mu_i(\overline{G}) : v(G) = n\}$	Nikiforov [10].

One of the few sensible questions in such a general setup is the following one: does the limit

$$\lim_{n\to\infty} \frac{1}{n} \max\{F(G) : G \in \mathcal{F},\ v(G) = n\}$$

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exist? We show that, under some mild conditions on $\mathcal{F}$, this is always the case.

For any graph $G = (V, E)$ and integer $t \geq 1$, write $G^{(t)}$ for the graph obtained by replacing each vertex $u \in V$ by a set V_u of t independent vertices and joining $x \in V_u$ to $y \in V_v$ if and only if $uv \in E$.

Call a graph property $\mathcal{F}$ *multiplicative* if : *(a)* $\mathcal{F}$ is closed under adding isolated vertices; *(b)* $G \in \mathcal{F}$ implies $G^{(t)} \in \mathcal{F}$ for every $t \geq 1$. Note that "K_r-free", "r-partite", and "any graph" are multiplicative properties.

Theorem 1.1. *For any multiplicative property $\mathcal{F}$ the limit*

$$c = \lim_{n\to\infty} \frac{1}{n} \max \{F(G) : G \in \mathcal{F},\ v(G) = n\} \tag{1.1}$$

exists. Moreover,

$$c = \limsup \left\{ \frac{1}{|G|} F(G) : G \in \mathcal{F} \right\}.$$

Note that, since the α_i's, β_i's, γ_i's, and δ_i's may have any sign, Theorem 1.1 implies that

$$\lim_{n\to\infty} \frac{1}{n} \min \{F(G) : G \in \mathcal{F},\ v(G) = n\}$$

exists as well.

Gernert [5] (see also Stevanović [12]) has proved that the inequality

$$\mu_1(G) + \mu_2(G) \leq v(G)$$

holds if the graph G has fewer than 10 vertices or is one of the following types: regular, triangle-free, toroidal, or planar; he consequently asked whether this inequality holds for any graph G. We answer this question in the negative by showing that

$$1.122n - 25 < \frac{29 + \sqrt{329}}{42} n - 25 \leq \max_{v(G)=n} \mu_1(G) + \mu_2(G) \leq \frac{2}{\sqrt{3}} n < 1.155n. \tag{1.2}$$

2. Proofs. Given a graph G and an integer $t > 0$, set $G^{[t]} = \overline{\overline{G}^{(t)}}$, i.e., $G^{[t]}$ is obtained from $G^{(t)}$ by joining all vertices within V_u for every $u \in V$. The following two facts are derived by straightforward methods.

(i) The eigenvalues of $G^{(t)}$ are $t\mu_1(G), \ldots, t\mu_n(G)$ together with $n(t-1)$ additional 0's.

(ii) The eigenvalues of $G^{[t]}$ are $t\mu_1(G) + t - 1, \ldots, t\mu_n(G) + t - 1$ together with $n(t-1)$ additional (-1)'s.

We shall show that the extremal k eigenvalues of $G^{(t)}$ and $G^{[t]}$ are roughly proportional to the corresponding eigenvalues of G.

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LEMMA 2.1. *Let $1 \leq k < n$, $t \geq 2$. Then for every $s \in [k]$,*

$$0 \leq \mu_s\left(G^{(t)}\right) - t\mu_s(G) < \frac{tn}{\sqrt{n-k}}, \tag{2.1}$$

$$0 \geq \mu_{n-s+1}\left(G^{(t)}\right) - t\mu_{n-s+1}(G) > -\frac{tn}{\sqrt{n-k}}, \tag{2.2}$$

$$0 \leq \mu_s\left(G^{[t]}\right) - t\mu_s(G) < t + \frac{tn}{\sqrt{n-k}}, \tag{2.3}$$

$$0 \geq \mu_{n-s+1}\left(G^{[t]}\right) - t\mu_{n-s+1}(G) > -t - \frac{tn}{\sqrt{n-k}}. \tag{2.4}$$

Proof. We shall prove (2.1) first. Fix some $s \in [k]$ and note that *(i)* implies that $G^{(t)}$ and G have the same number of positive eigenvalues. In particular, $G^{(t)}$ has at most $n-1$ negative eigenvalues, and so $\mu_s\left(G^{(t)}\right) \geq 0$. If $\mu_s\left(G^{(t)}\right) > 0$, then $\mu_s(G) > 0$ and $\mu_s\left(G^{(t)}\right) = t\mu_s(G)$, so (2.1) holds. If $\mu_s\left(G^{(t)}\right) = 0$, then

$$0 \geq \mu_s(G) \geq \ldots \geq \mu_n(G),$$

and inequality (2.1) follows from

$$(n-k)\,\mu_s^2(G) \leq (n-s+1)\,\mu_s^2(G) \leq \sum_{i=s}^{n} \mu_i^2(G) < n^2.$$

Next we shall prove (2.3). Note that *(ii)* implies that $G^{[t]}$ and G have the same number of eigenvalues that are greater than -1. Since $G^{[t]}$ has at most $n-1$ eigenvalues that are less than -1, it follows that $\mu_s\left(G^{[t]}\right) \geq -1$. If $\mu_s\left(G^{[t]}\right) > -1$, then $\mu_s(G) > -1$ and $\mu_s\left(G^{[t]}\right) = t\mu_s(G) + t - 1$; thus, (2.3) holds. If $\mu_s\left(G^{[t]}\right) = -1$, then

$$-1 \geq \mu_s(G) \geq \ldots \geq \mu_n(G),$$

and inequality (2.3) follows from

$$(n-k)\,\mu_s^2(G) < (n-s+1)\,\mu_s^2(G) \leq \sum_{i=s}^{n} \mu_i^2(G) < n^2.$$

Inequalities (2.2) and (2.4) follow likewise with proper changes of signs. ∎

We also need the following lemma.

LEMMA 2.2. *Let G be a graph of order n and H be an induced subgraph of G of order $n-1$. Then for every $1 \leq s \leq 3n/4$,*

$$0 \leq \mu_s(G) - \mu_s(H) < 3\sqrt{n}, \tag{2.5}$$

$$0 \geq \mu_{n-s+1}(G) - \mu_{n-s}(H) > -3\sqrt{n}. \tag{2.6}$$

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Proof. We shall assume that $V(G) = \{1, \ldots, n\}$ and $V(H) = \{1, \ldots, n-1\}$. Let A be the adjacency matrix of G and let A_1 be the $n \times n$ symmetric matrix obtained from A by zeroing its nth row and column. Since the adjacency matrix of H is the principal submatrix of A in the first $n-1$ columns and rows, the eigenvalues of A_1 are $\mu_1(H), \ldots, \mu_{n-1}(H)$ together with an additional 0. This implies that, for every $s \in [n-1]$,

$$\mu_s(A_1) = \begin{cases} \mu_s(H), & \text{if } \mu_s(A_1) > 0 \\ \mu_{s-1}(H) & \text{if } \mu_s(A_1) \leq 0. \end{cases} \tag{2.7}$$

We first show that, for every $s \in [n-1]$,

$$\mu_s(A_1) - \mu_s(H) \leq \frac{n}{\sqrt{n-s}}. \tag{2.8}$$

In view of (2.7), this is obvious if $\mu_s(A_1) > 0$. If $\mu_s(A_1) \leq 0$, then we have

$$\mu_s(A_1) - \mu_s(H) = \mu_{s-1}(H) - \mu_s(H) \leq |\mu_s(H)|.$$

Inequality (2.8) follows now from

$$(n-s)\mu_s^2(H) \leq (n-s+1)\mu_s^2(H) \leq \sum_{i=s}^{n} \mu_i^2(H) < n^2.$$

Likewise, with proper changes of signs, we can show that, for every $s \in [n-1]$,

$$\mu_{n-s+1}(A_1) - \mu_{n-s}(H) \geq -\frac{n}{\sqrt{n-s}}.$$

Having proved (2.8), we turn to the proof of (2.5) and (2.6). Note that the first inequalities in both (2.5) and (2.6) follow by Cauchy interlacing theorem. On the other hand, Weyl's inequalities imply that

$$\mu_n(A - A_1) \leq \mu_s(A) - \mu_s(A_1) \leq \mu_1(A - A_1).$$

Obviously, $\mu_1(A - A_1)$ is maximal when the off-diagonal entries of the nth row and column of A are 1's. Thus, $\mu_1(A - A_1) \leq \sqrt{n-1}$ and $\mu_n(A - A_1) = -\mu_1(A - A_1) \geq -\sqrt{n-1}$. Hence,

$$\mu_s(G) - \mu_s(H) = \mu_s(A) - \mu_s(A_1) + \mu_s(A_1) - \mu_s(H) \leq \sqrt{n-1} + \frac{n}{\sqrt{n-s}} < 3\sqrt{n}.$$

Likewise,

$$\begin{aligned} \mu_{n-s+1}(G) - \mu_{n-s}(H) &= \mu_{n-s+1}(A) - \mu_{n-s+1}(A_1) + \mu_{n-s+1}(A_1) - \mu_{n-s}(H) \\ &\geq -\sqrt{n-1} - \frac{n}{\sqrt{n-s}} > -3\sqrt{n}, \end{aligned}$$

completing the proof of Lemma 2.2. $\square$

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COROLLARY 2.3. *Let G_1 be a graph of order n and G_2 be an induced subgraph of G_1 of order $n-l$. Then, for every $1 \leq s \leq 3(n-l)/4$,*

$$|\mu_s(G_1) - \mu_s(G_2)| < 3l\sqrt{n},$$
$$|\mu_{n-s+1}(G_1) - \mu_{n-l-s+1}(G_2)| < 3l\sqrt{n}.$$

Proof. Let $\{v_1,\ldots,v_l\} = V(G_1)\backslash V(G_2)$. Set $H_0 = G_1$; for every $i \in [l]$, let H_i be the subgraph of G_1 induced by the set $V(G_1)\setminus\{v_1,\ldots,v_i\}$; clearly, $H_l = G_2$. Since H_{i+1} is an induced subgraph of H_i with $|H_{i+1}| = |H_i| - 1$, Lemma 2.2 implies that for every $1 \leq s \leq 3(n-l)/4$,

$$|\mu_s(G_1) - \mu_s(G_2)| \leq \sum_{i=0}^{l-1} |\mu_s(H_i) - \mu_s(H_{i+1})| \leq \sum_{i=0}^{l-1} 3\sqrt{n-i} < 3l\sqrt{n},$$

$$\begin{aligned}|\mu_{n-s+1}(G_1) - \mu_{n-l-s+1}(G_2)| &\leq \sum_{i=0}^{l-1} |\mu_{n-i+s+1}(H_i) - \mu_{n-i-1-s+1}(H_{i+1})| \\ &\leq \sum_{i=0}^{l-1} 3\sqrt{n-i} < 3l\sqrt{n},\end{aligned}$$

completing the proof of the corollary. □

Proof of Theorem 1.1 Set

$$\varphi(n) = \frac{1}{n}\max\{F(G) : G \in \mathcal{F},\, v(G) = n\}$$

Let $M = \sum_{i=1}^{k} |\alpha_i| + |\beta_i| + |\gamma_i| + |\delta_i|$ and set

$$c = \lim_{n\to\infty} \sup \varphi(n).$$

Since $|F(G)| \leq Mn$, the value c is defined. We shall prove that, in fact, c satisfies (1.1).

Note first if $t \geq 2$, $n > 4k/3$, and G is a graph of order n, then for any $i \in [k]$, Lemma 2.1 implies that

$$F\left(G^{(t)}\right) - tF(G) \geq -M\left(t + \frac{tn}{\sqrt{n-k}}\right) \geq -M\left(t + 2t\sqrt{n}\right) \geq -3Mt\sqrt{n}. \tag{2.9}$$

Select $\varepsilon > 0$ and let $G \in \mathcal{F}$ be a graph of order $n > (3M/\varepsilon)^2$ such that

$$c + \varepsilon \geq \varphi(n) = \frac{F(G)}{n} \geq c - \varepsilon.$$

Suppose $N \geq n\left\lceil n\max\left\{2, (|c|/\varepsilon + 1), (3M/\varepsilon)^2\right\}\right\rceil$; therefore the value $t = \lfloor N/n \rfloor$ satisfies $t \geq n\max\left\{2, (|c|/\varepsilon + 1), (3M/\varepsilon)^2\right\}$. We shall show that $\varphi(N) \geq c - 4\varepsilon$, which implies the assertion.

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Let G_1 be the union of $G^{(t)}$ and $N - tn$ isolated vertices. Clearly $v(G_1) = N$ and, since $\mathcal{F}$ is multiplicative, $G_1 \in \mathcal{F}$. In view of $N - tn < n$, Corollary 2.3 implies that

$$F(G_1) \geq F\left(G^{(t)}\right) - 3Mn\sqrt{N}.$$

Therefore, in view of $\varphi(N) \geq F(G_1)/N$ and (2.9),

$$\varphi(N) \geq \frac{F\left(G^{(t)}\right) - 3Mn\sqrt{N}}{N} \geq \frac{tF(G) - 3Mt\sqrt{n} - 3Mn\sqrt{N}}{N}.$$

We find that

$$\begin{aligned}
\varphi(N) &\geq \frac{F(G)}{n} - \frac{F(G)(N-tn)}{nN} - \frac{3Mt\sqrt{n} + 3Mn\sqrt{N}}{N} \\
&\geq \varphi(n) - \frac{|\varphi(n)|\, n}{N} - \frac{3Mt\sqrt{n} + 3Mn\sqrt{N}}{N} \\
&\geq \varphi(n) - \frac{n^2(|c| + |\varepsilon|) + 3Mt\sqrt{n} + 3Mn\sqrt{N}}{N} \\
&\geq \varphi(n) - \frac{n^2(|c| + |\varepsilon|) + 3Mt\sqrt{n}}{nt} - \frac{3Mn}{\sqrt{nt}} \\
&= \varphi(n) - \frac{n(|c| + |\varepsilon|)}{t} - \frac{3M}{\sqrt{n}} - 3M\sqrt{\frac{n}{t}} \geq c - 4\varepsilon,
\end{aligned}$$

completing the proof of Theorem 1.1. □

We turn now to the proof of inequality (1.2); we present it in two propositions.

Proposition 2.4. *If G is a graph of order n, then $\mu_1(G) + \mu_2(G) \leq \left(2/\sqrt{3}\right)n$.*

Proof. Setting $m = e(G)$, we see that

$$\mu_1^2(G) + \mu_2^2(G) \leq \mu_1^2(G) + \ldots + \mu_n^2(G) = 2m. \tag{2.10}$$

If $m \leq n^2/4$, the result follows from

$$\mu_1(G) + \mu_2(G) \leq \sqrt{2\left(\mu_1^2(G) + \mu_2^2(G)\right)} \leq 2\sqrt{m} \leq n,$$

so we shall assume that $m > n^2/4$. From (2.10), we clearly have

$$\mu_1(G) + \mu_2(G) \leq \sqrt{2m - \mu_2^2(G)} + \mu_2(G).$$

The value $\sqrt{2m - x^2} + x$ is increasing in x for $x \leq \sqrt{m}$. On the other hand, Weyl's inequalities imply that

$$\mu_2(G) + \mu_n\left(\overline{G}\right) \leq \mu_2(K_n) = -1.$$

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Hence, if $G \neq K_n$, we have $\mu_2(G) \geq 0$ and so, $\mu_2^2(G) < \mu_n^2\left(\overline{G}\right)$; if $G = K_n$, then $\mu_2^2(G) = \mu_n^2\left(\overline{G}\right) + 1$; thus we always have

$$\mu_2^2(G) \leq \mu_n^2\left(\overline{G}\right) + 1.$$

From

$$\mu_2^2(G) \leq \mu_n^2\left(\overline{G}\right) + 1 \leq e\left(\overline{G}\right) + 1 \leq \frac{n(n-1)}{2} + 1 - m \leq \frac{n^2}{2} - m < m,$$

we see that

$$\mu_1(G) + \mu_2(G) \leq \sqrt{3m - n^2/2} + \sqrt{n^2/2 - m}.$$

The right-hand side of this inequality is maximal for $m = 5n^2/12$ and the result follows. ∎

Proposition 2.5. *For every $n \geq 21$ there exists a graph of order n with*

$$\mu_1(G) + \mu_2(G) > \frac{29 + \sqrt{329}}{42} n - 25.$$

Proof. Suppose $n \geq 21$; set $k = \lfloor n/21 \rfloor$; let G_1 be the union of two copies of K_{8k} and G_2 be the join of K_{5k} and G_1; clearly $v(G_2) = 21k$. Add $n - 21k$ isolated vertices to G_2 and write G for the resulting graph. By Cauchy's interlacing theorem, we have

$$\begin{aligned} &\mu_1(G) \geq \mu_1(G_2), \\ &\mu_2(G) \geq \mu_2(G_2) \geq \mu_2(G_1) = 8k - 1. \end{aligned}$$

Since the graphs K_{5k} and G_1 are regular, a theorem of Finck and Grohmann [6] (see also [3], Theorem 2.8) implies that $\mu_1(G_2)$ is the positive root of the equation

$$(x - 5k + 1)(x - 8k + 1) - 80k^2 = 0,$$

that is to say,

$$\mu_1(G_2) = \frac{13k - 2 + k\sqrt{329}}{2}.$$

Alternatively, applying Theorem 9.3.3. of [4], we see that

$$\mu_1(G_2) \geq \frac{13k - 2 + k\sqrt{329}}{2}.$$

Hence,

$$\begin{aligned} \mu_1(G) + \mu_2(G) &\geq \frac{(29k - 4) + k\sqrt{329}}{2} > \frac{(29(n-20) - 84) + (n-20)\sqrt{329}}{42} \\ &> \frac{29 + \sqrt{329}}{42} n - 25, \end{aligned}$$

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completing the proof. □

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